A.V. EFIMOV

LINEAR ALGEBRA AND FUNDAMENTALS OF MATHEMATICAL ANALYSIS

PREFACE TO THE ENGLISH EDITION

The idea of designing this three-part problem book belongs to Professor B. Demidovich, but his early death prevented him from putting it into practice. The book was prepared by a group of us some of whom have had long pedagogical experience lecturing higher mathematics to engineering students. We have tried to do our best to develop and realize the ideas and plans of the late Professor Demidovich.

The problems contained in the book cover all the branches of higher mathematics studied by engineering students. In addition, our problem book includes a number of problems which will enable the reader to repeat the main topics of mathematical analysis and vector algebra from his school course, studying them more carefully.

The present book (Part 1) entitled "Linear Algebra and Fundamentals of Mathematical Analysis" consists of seven chapters (1 to 7) and includes the topics which, as a rule, are studied during the first two semesters of the course. Among them are: vector algebra with elements of analytical geometry, linear algebra, and also differential calculus of functions of one and several variables, and integral calculus of functions of one variable.

Each chapter is supplied with answers to all the computational problems. The answers are given at the end of each chapter. Those problems marked with an * are provided with hints and the ones with **, with solutions to be found with answers.

Each section in all chapters is supplied with a brief introduction containing both the relevant theoretical material (definitions, formulas, and theorems) and a plenty of thoroughly worked examples.

Part 2, entitled "Advanced Topics of Mathematical Analysis", consists of six chapters (8 to 13) and includes the topics to be studied during the third and fourth semesters: multiple integrals, differential equations, vector analysis, theory of functions of a complex variable, series and their applications, and calculus.

Part 3, entitled "Concluding Chapters", consists of five chapters (14 to 18) and includes the following: theory of probability, mathematical statistics, optimization methods, equations of mathematical physics, and integral equations.

In our work we were fortunate to have had the aid of Professors A.I. Prilepko, V.A. Trenogin, and S.I. Pokhozhaev, who brought to it both their profound knowledge of the subject and lecturers' experience, for which we are very grateful. We should like to express our appreciation to Dr. L.V. Lapenko and S.A. Fomina for their help in preparing the present book for print. Special recognition is given to the translator of our book L.E. Levant for his numerous worthwhile suggestions.

It is the fond hope of the authors that both the instructors and students who use this book will enjoy it as much as the authors have enjoyed its preparation.

The Authors

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LINEAR ALGEBRA AND FUNDAMENTALS OF MATHEMATICAL ANALYSIS

HIGHER MATHEMATICS
FOR ENGINEERING STUDENTS

CONTENTS

Preface to the English Edition 5

CHAPTER 1. Introduction to Analysis

Sec. 1.1. Real Numbers. Sets. Logic Symbolism 9
Sec. 1.2. Functions of a Real Variable 20
Sec. 1.3. Limit of a Sequence of Real Numbers 29
Sec. 1.4. Limit of a Function. Continuity 33
Sec. 1.5. Complex Numbers 44
Answers 57

CHAPTER 2. Vector Algebra and Analytical Geometry

Sec. 2.1. Vector Algebra 69
Sec. 2.2. Linear Geometrical Objects 90
Sec. 2.3. Plane Curves 105
Sec. 2.4. Surfaces and Curves in Space 129
Answers 142

CHAPTER 3. Determinants and Matrices. Systems of Linear Equations

Sec. 3.1. Determinants 154
Sec. 3.2. Matrices 166
Sec. 3.3. Space of Arithmetical Vectors. Rank of a Matrix 174
Sec. 3.4. Systems of Linear Equations 183
Sec. 3.5. Some Computational Problems of Linear Algebra 195
Answers 202

CHAPTER 4. Elements of Linear Algebra

Sec. 4.1. Linear Vector Spaces and Spaces with a Scalar Product 209
Sec. 4.2. Linear Operators 224
Sec. 4.3. Bilinear and Quadratic Forms 241
Answers 251

CHAPTER 5. Differential Calculus: Functions of One Variable

Sec. 5.1. Derivatives 264
Sec. 5.2. The Differential 284
Sec. 5.3. Theorems on Differentiable Functions. Taylor's Formula 287

Sec. 5.4. Investigation of Behaviour of Functions and Construction of Graphs 298
Sec. 5.5. Vector and Complex Functions of Real Variable 311
Sec. 5.6. Numerical Methods of Functions of One Variable 324
Answers 339

CHAPTER 6. Integral Calculus: Functions of One Variable

Sec. 6.1. Basic Methods of Computing the Indefinite Integral 359
Sec. 6.2. Integration of the Basic Classes of Elementary Functions 370
Sec. 6.3. Mixed Problems on Integration 384
Sec. 6.4. The Definite Integral and Methods of Its Computation 386
Sec. 6.5. Improper Integrals 399
Sec. 6.6. Geometrical Applications of the Definite Integral 405
Sec. 6.7. Applications of the Definite Integral to Solution of Some Problems of Mechanics and Physics 421
Sec. 6.8. Numerical Integration of Functions of One Variable 429
Answers 435

CHAPTER 7. Differential Calculus: Functions of Several Variables

Sec. 7.1. Basic Concepts 452
Sec. 7.2. Differentiation of Composite and Implicit Functions 465
Sec. 7.3. Applications of Partial Derivatives 480
Sec. 7.4. Approximate Numbers and Operations on Them 497
Answers 502

Chapter 1

INTRODUCTION TO ANALYSIS

SEC. 1.1.
REAL NUMBERS. SETS.
LOGIC SYMBOLISM

1. The Notion of a Real Number. As is known, every nonnegative *real number* x is represented by a nonterminating decimal

$$[x].x_1x_2\ldots, \tag{1}$$

where $[x]$ is the greatest whole number not exceeding x and called the *integral part* of the number x, $x_n \in \{0, 1, 2, \ldots, 9\}$ for any $n \in \mathbb{N}$.

It should be noted that the decimals in which $x_n = 9$ for all $n \geqslant n_0$ (n_0 some natural number) are usually not considered here due to the following equalities:

$$[x].999\ldots = [x] + 1,$$

$$[x].x_1x_2\ldots x_{n_0-1}999\ldots = [x].x_1x_2\ldots(x_{n_0-1}+1)\ (n_0 > 1,\ x_{n_0-1}\neq 9).$$

A real number x is *rational*, i.e. representable in the form of the ratio $\frac{m}{n}$, $m,\ n \in \mathbb{Z}$, if and only if decimal (1) is periodic. Otherwise the number x is *irrational*.

The *absolute value* or the *modulus* of a real number x is defined as the nonnegative number

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{if } x < 0. \end{cases}$$

It is supposed that the rules for comparing real numbers as also the arithmetical operations on them are already known to the reader.

1.1. Prove that the number

$$0.1010010001\ldots\underbrace{10\ldots01}_{n}\ldots$$

is irrational. Write the first three terms of either of the sequences of finite (or terminating) decimals approximating this number with deficit or excess.

1.2. Represent the following numbers in the form of proper rational fractions:

(a) 1.(2); (b) 3.00(3); (c) 0.110(25).

1.3. Prove that the number $\log 5$ is irrational.

Let us assume that log 5 is a rational number, i.e.

$$\log 5=\frac{m}{n}\ ;\quad m,\ n\in\mathbb{Z}.$$

Then

$$10^{\frac{m}{n}}=5,$$
$$10^m=5^n,$$
$$2^m\cdot 5^m=5^n.$$

But the last equality is impossible, since the number 2 enters into the factorization of the left-hand member in simple factors, but does not enter into a similar factorization of the right-hand member which contradicts the uniqueness of presenting whole numbers in the form of prime factors. Therefore our assumption is false and, consequently, the number log 5 is irrational.

In problems 1.4 to 1.9 prove that the given numbers are irrational:

1.4. $\sqrt{3}$. **1.5.** $\sqrt[n]{p}$, p is a prime number, $n>1$.

1.6. $2+\sqrt{3}$. **1.7.** $\sqrt{2}+\sqrt{3}$.

1.8. $\log_3 p$, p is a prime number.

1.9. $\frac{\pi}{2}+\pi n$, $n\in\mathbb{Z}$, if it is known that π is irrational.

In Problems 1.10 to 1.13 compare the given numbers.

1.10. $\sqrt{2}-\sqrt{5}$ and $\sqrt{3}-2$.

Suppose the below inequality is correct:

$$\sqrt{2}-\sqrt{5}<\sqrt{3}-2. \tag{2}$$

Then:

$$\sqrt{2}+2<\sqrt{5}+\sqrt{3},$$
$$6+4\sqrt{2}<8+2\sqrt{15},$$
$$2\sqrt{2}<1+\sqrt{15},$$
$$8<16+2\sqrt{15}.$$

Since the last inequality is true, by virtue of the equivalence of the transformations performed, the initial inequality (2) is also true.

1.11. $\log_{1/2}\frac{1}{3}$ and $\log_{1/3}\frac{1}{2}$.

1.12. $\left(\frac{1}{5}\right)^{\log\frac{1}{7}}$ and $\left(\frac{1}{7}\right)^{\log\frac{1}{5}}$.

1.13. $\log_{\log_3 2}\frac{1}{2}$ and 1.

In Problems 1.14 to 1.16 prove the indicated numerical inequalities without using tables.

1.14. $\log_3 10 + 4 \log 3 > 4$.

1.15. $\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} > 2$.

1.16. $\log_4 26 > \log_6 17$.

1.17. Prove that the modulus of a real number possesses the following properties:

(a) $|x| = \max\{x, -x\}$;

(b) $|x \cdot y| = |x| \cdot |y|$ and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$;

(c) $|x + y| \leqslant |x| + |y|$ and $|x - y| \geqslant ||x| - |y||$ (the *triangle inequalities*);

(d) $\sqrt{x^2} = |x|$.

In Problems 1.18 to 1.22 solve the given equations:

1.18. $|3x - 4| = \frac{1}{2}$. **1.19.** $\sqrt{x^2} + x^3 = 0$.

1.20. $|-x^2 + 2x - 3| = 1$. **1.21.** $\left|\frac{2x-1}{x+1}\right| = 1$.

1.22. $\sqrt{(x-2)^2} = -x + 2$.

In Problems 1.23 to 1.27 solve the given inequalities:

1.23. $|x - 2| \geqslant 1$. **1.24.** $|x^2 - 7x + 12| > x^2 - 7x + 12$.

1.25. $x^2 + 2\sqrt{(x+3)^2} - 10 \leqslant 0$. **1.26.** $\frac{1}{|x-1|} < 4 - x$.

1.27. $\sqrt{(x+1)^2} \leqslant -x - 1$.

2. Sets and Set Operations. A *set* is understood as any well-defined collection of objects called the elements or members of the set.

The notation $a \in A$ means that the object a is an element of the set A (belongs to the set A), otherwise we write $a \notin A$. A special set that plays an important part in set theory is the *empty* set, sometimes called *null* set, which contains no elements. The notation used to denote the empty set is the symbol $\varnothing$. The notation $A \subset B$ (read: "A is contained in B", or, equivalently, "B contains A") means that every element in a set A is also an element of a set B, in this case A is called a *subset* of B. The sets A and B are *equal* (written $A = B$) if $A \subset B$ and $B \subset A$.

There are two basic ways of defining (describing) a set:

(a) The set A is defined by a direct enumeration of all of its elements $a_1, a_2, \ldots, a_n$, i.e. written in the form

$$A = \{a_1, a_2, \ldots, a_n\}.$$

(b) The set A is defined as the totality of those and only those elements belonging to a certain basic set T which possess the general property α. In this case we use a shorter notation

$$A = \{x \in T \mid \alpha(x)\},$$

where $\alpha(x)$ means that the element x possesses the property α.

Example 1. Describe the set

$$A = \{x \in \mathbb{Z} \mid (x-3)(x^2-1) = 0 \text{ and } x \geqslant 0\}$$

by enumerating (or listing) its elements.

A is the set of all integral nonnegative roots of the equation $(x-3)(x^2-1) = 0$. Consequently, $A = \{1, 3\}$.

The *union* of sets A and B is defined as the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

The *intersection* of the sets A and B is defined as the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The *difference* between the sets A and B is understood as the set

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

In particular, if A is a subset of some universal set T, then the difference set $T \setminus A$ is denoted by the symbol $\overline{A}$ and is called the *complement* of the set A (relative to T).

1.28. Which of the two given notations is correct:

(a) $\{1, 2\} \in \{1, 2, \{1, 2, 3\}\}$ or $\{1, 2\} \subset \{1, 2, \{1, 2, 3\}\}$;

(b) $\{1, 2\} \in \{1, 2, \{1, 2\}\}$ or $\{1, 2\} \subset \{1, 2, \{1, 2\}\}$.

In Problems 1.29 to 1.34 describe the given set by listing all of its elements.

1.29. $A = \{x \in \mathbb{R} \mid x^3 - 3x^2 + 2x = 0\}$.

1.30. $A = \left\{x \in \mathbb{R} \mid x + \frac{1}{x} \leqslant 2 \text{ and } x > 0\right\}$.

1.31. $A = \{x \in \mathbb{N} \mid x^2 - 3x - 4 \leqslant 0\}$.

1.32. $A = \left\{x \in \mathbb{Z} \mid \frac{1}{4} \leqslant 2^x < 5\right\}$.

1.33. $A = \left\{x \in \mathbb{N} \mid \log_{1/2} \frac{1}{x} < 2\right\}$.

1.34. $A = \{x \in \mathbb{R} \mid \cos^2 2x = 1 \text{ and } 0 < x \leqslant 2\pi\}$.

In Problems 1.35 to 1.42 represent the indicated sets on the coordinate plane.

1.35. $\{(x, y) \in \mathbb{R}^2 \mid x + y - 2 = 0\}$.

1.36. $\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 > 0\}$.

1.37. $\{(x, y) \in \mathbb{R}^2 \mid (x^2 - 1)(y + 2) = 0\}$.

1.38. $\{(x, y) \in \mathbb{R}^2 \mid y > \sqrt{2x + 1}$ and $2x + 1 \geqslant 0\}$.

1.39. $\{(x, y) \in \mathbb{R}^2 \mid y^2 > 2x + 1\}$.

1.40. $\{(x, y) \in \mathbb{R}^2 \mid 2^{x+1} = y^2 + 4$ and $2^{x-1} \leqslant y\}$.

1.41. $\{(x, y) \in \mathbb{R}^2 \mid \cos 2x = \cos 2y\}$.

1.42. $\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \frac{1}{x} > \frac{1}{y},\ x \neq 0,\ y \neq 0\right\}$.

1.43. Describe by enumerating all the elements the sets $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$ if

$$A = \{x \in \mathbb{R} \mid x^2 + x - 20 = 0\},$$

$$B = \{x \in \mathbb{R} \mid x^2 - x + 12 = 0\}.$$

The notation $m \mid n$, where $m, n \in \mathbb{Z}$, means that the number m is the divisor of the number n. Describe the following sets (Problems 1.44 to 1.47).

1.44. $\{x \in \mathbb{N} \mid x \mid 8$ and $x \neq 1\}$. **1.45.** $\{x \in \mathbb{Z} \mid 8 \mid x\}$.

1.46. $\{x \in \mathbb{N} \mid x \mid 12\} \cap \{x \in \mathbb{N} \mid x \mid 8\}$.

1.47. $\{x \in \mathbb{N} \mid 12 \mid x\} \cap \{x \in \mathbb{N} \mid 8 \mid x\}$.

1.48. Prove that:

(a) the equality $A \cap B = B$ is true if and only if $B \subset A$;

(b) the equality $A \cup B = B$ is correct if and only if $A \subset B$.

1.49. Let $A = (-1, 2]$ and $B = [1, 4)$. Find the sets $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$ and represent them on the number axis.

Taking the line segment $T = [0, 1]$ for a universal set, find and represent on the number axis the complements of the following sets (Problems 1.50 to 1.53):

1.50. $\{0, 1\}$. **1.51.** $(1/4, 1/2)$. **1.52.** $(0, 1/2]$.

1.53. $\{1/4\} \cup [3/4, 1)$.

1.54. Prove that the operation of taking a complement possesses the *reflexivity* property

$$\overline{(\overline{A})} = A,$$

and is also connected with the inclusive relation $\subset$ and

operations $\cup$ and $\cap$ by the following *duality laws*:

$$\text{if } A \subset B, \text{ then } \bar{A} \supset \bar{B},$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B} \text{ and } \overline{A \cap B} = \bar{A} \cup \bar{B}.$$

1.55. Prove that the operations $\cup$ and $\cap$ are related by the *distributivity laws*:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Using the results obtained in Problems 1.54 and 1.55, prove the following equalities (Problems 1.56 to 1.59):

1.56. $\overline{A \setminus B} \cap (\bar{A} \cup \bar{B}) = \bar{A}$.

Since $\bar{A} \cup \bar{B} = \overline{A \cap B}$ the left-hand member of the equality in question takes the form

$$(\overline{A \setminus B}) \cap (\overline{A \cap B}) = (\overline{A \setminus B}) \cup (\overline{A \cap B}) = \bar{A}.$$

1.57. $A \setminus B = A \cap \bar{B}$. **1.58.** $\overline{A \setminus B} = \bar{A} \cup B$.

1.59. $A \cap (\overline{A \setminus B}) = A \cap B$.

The operations $\cup$ and $\cap$ are generalized in a natural way for the case of an arbitrary (finite or infinite) family of sets. Let, for example, there be given a family of sets A_n, $n \in \mathbb{N}$. The *union* of sets belonging to this family is denoted by the symbol $\bigcup\limits_{n \in \mathbb{N}} A_n$ and is defined as the set of all those elements each of which belongs at least to one of the sets of A_n. The *intersection* $\bigcap\limits_{n \in \mathbb{N}} A_n$ is defined as the set of all the elements belonging to each of the sets of A_n.

In Problems 1.60 to 1.62 find $\bigcup\limits_{n \in \mathbb{N}} A_n$ and $\bigcap\limits_{n \in \mathbb{N}} A_n$ for the given families A_n, $n \in \mathbb{N}$:

1.60. $A_n = \{x \in \mathbb{Z} \mid -n \leqslant x \leqslant n\}$.

1.61. $A_n = \{3n - 2, 3n - 1\}$.

1.62. $A_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right\}$.

1.63. Let A be the set of all the points in the plane forming the sides of a triangle inscribed in a given circle. Describe the union and intersection of all such sets if

(a) triangles are arbitrary;

(b) triangles are regular;

(c) triangles are right-angled.

A set X is said to be *countable* if there can be established a one-to-one correspondence between the elements of this set and those of the set $\mathbb{N}$ of all natural numbers.

Example 2. Show that the set $\mathbb{Z}$ of all integers is countable.

Let us establish one-to-one correspondence between the elements of this set and natural numbers, for instance, by ordering the set $\mathbb{Z}$ in the following way:

$$0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ \ldots,$$

and then associating each integer with its ordinal number in this sequence.

In Problems 1.64 to 1.66 prove that the given sets are countable.

1.64. $\{n \in \mathbb{N} \mid n = 2k,\ k \in \mathbb{N}\}$.

1.65. $\{n \in \mathbb{N} \mid n = k^2,\ k \in \mathbb{N}\}$.

1.66. $\{n \in \mathbb{N} \mid n = 2^k,\ k \in \mathbb{N}\}$.

1.67. Prove that if the set X is countable and $A \subset X$ is its infinite subset, then the set A is also countable.

Using the obtained result, prove that the set

$$\{n \in \mathbb{Z} \mid n = k^2 - k + 1,\ k \in \mathbb{N}\}$$

is also countable.

1.68. Let $X_1, X_2, \ldots, X_n$ be countable sets. Prove that their union $\bigcup\limits_{n \in \mathbb{N}} X_n$ is a countable set.

Let $X_n = \{x_{n,1}, x_{n,2}, \ldots, x_{n,l}, \ldots\}$. Then the elements of the set $\bigcup\limits_{n \in \mathbb{N}} X_n$ can be tabulated as follows:

$$\begin{array}{l} x_{1,1},\ x_{1,2},\ \ldots,\ x_{1,l},\ \ldots, \\ x_{2,1},\ x_{2,2},\ \ldots,\ x_{2,l},\ \ldots, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{n,1},\ x_{n,2},\ \ldots,\ x_{n,l},\ \ldots, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

To prove the countability of the set $\bigcup\limits_{n \in \mathbb{N}} X_n$ it now suffices to number all elements of this table in an arbitrary way.

Taking advantage of the result of Problem 1.68, prove that the following sets are countable:

1.69. $\mathbb{Q} = \{x \in \mathbb{R} \mid x = \frac{m}{n}$ for some $m,\ n \neq 0$ from $\mathbb{Z}\}$ is the set of all rational numbers.

1.70. The set of all points in the plane having rational coordinates.

1.71. The set of all polynomials with rational coefficients.

3. Upper and Lower Bounds. Let X be an arbitrary non-empty set of real numbers. The number $M = \max X$ is called the *greatest (maximum)* element of the set X if $M \in X$ and for any $x \in X$ the inequality $x \leqslant M$ is fulfilled. The notion of the *least (minimum)* element $m = \min X$ of the set X is defined in a similar way.

The set X is said to be *bounded above* if there is a real number a such that $x \leqslant a$ for all $x \in X$. Any number possessing this property is termed the *upper bound* of the set X. For a given bounded above set X the set of all of its upper bounds has the least (minimum) element which is called the *least upper bound* or supremum and is denoted by the symbol $\sup X$.

The notions of a set *bounded below*, *lower bound* and the *greatest lower bound* or infimum are defined in an analogous manner. The latter is denoted by the symbol $\inf X$.

A set X bounded above and below is said to be *bounded*.

Example 3. Find the supremum and infimum of the set $[0, 1)$. This set has no greatest element, since for any $x \in [0, 1)$ there is $y \in [0, 1)$ such that $y > x$. The set of upper bounds for the half-interval $[0, 1)$ is the set $[1, \infty)$ with the least element equal to 1. Therefore

$$\sup [0, 1) = 1,$$

where $1 \notin [0, 1)$.

On the other hand, the least element of the set $[0, 1)$ exists and is equal to zero. The set of lower bounds is the set $(-\infty, 0]$ with the greatest element equal to zero which is the infimum of the half-interval $[0, 1)$. Thus,

$$\min [0, 1) = \inf [0, 1) = 0,$$

and $0 \in [0, 1)$.

1.72. Prove that the above formulated definition of the supremum is equivalent to the following:

The number M is the supremum of the set X if and only if:

(1) $x \leqslant M$ for all $x \in X$;

(2) for any $\varepsilon > 0$ there is an element $x \in X$ such that $x > M - \varepsilon$.

1.73. Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$.

(a) Indicate the least and greatest elements of this set (if any).

(b) What are the sets of upper and lower bounds for the set X? Find $\sup X$ and $\inf X$.

In Problems 1.74 to 1.78 find $\max X$, $\min X$, $\sup X$, and $\inf X$ (if any) for the indicated sets:

1.74. $X = \left\{x \in \mathbb{R} \mid x = \frac{1}{2^n},\ n \in \mathbb{N}\right\}$. **1.75.** $X = [-1, 1]$.

1.76. $X = \{x \in \mathbb{Z} \mid -5 \leqslant x < 0\}$. **1.77.** $X = \{x \in \mathbb{R} \mid x < 0\}$.

1.78. $X = \left\{x \in \mathbb{R} \mid x = \frac{m}{n};\ m,\ n \in \mathbb{N} \text{ and } m < n\right\}$.

1.79. Let X be the set of all the rational numbers satisfying the condition $r^2 < 2$. Show that the set X has no greatest element. Find $\sup X$.

1.80. Let $X \subset \mathbb{R}$ be an arbitrary bounded set. Prove that the set $-X = \{x \mid -x \in X\}$ is also bounded and the following equalities hold true:

$$\sup(-X) = -\inf X, \quad \inf(-X) = -\sup X.$$

1.81. Let X, $Y \subset \mathbb{R}$ be arbitrary bounded sets. Prove that the set

$$X + Y = \{z \in \mathbb{R} \mid z = x + y,\ x \in X,\ y \in Y\}$$

is bounded above and

$$\sup(X + Y) = \sup X + \sup Y.$$

1.82. Let $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$ be sets bounded above and below, respectively. Prove that the set

$$X - Y = \{z \in \mathbb{R} \mid z = x - y,\ x \in X,\ y \in Y\}$$

is bounded above and

$$\sup(X - Y) = \sup X - \inf Y.$$

4. Logic Symbolism. When presenting mathematical considerations it is expedient to use a short notation with specific symbols utilized in logic. Here we shall introduce only several simplest and most widely used symbols.

Let α, β, ... be some *sentences* or *statements*, i.e. narrative sentences each of which can be identified as true or false.

The notation $\overline{\alpha}$ means "not α", i.e. the *negation* of the statement α.

The notation $\alpha \Rightarrow \beta$ means: "the statement α implies the statement β" ($\Rightarrow$ is the *implication* symbol).

The notation $\alpha \Leftrightarrow \beta$ means: "the statement α is equivalent to the statement β", i.e. "if α, then β, and if β, then α" ($\Leftrightarrow$ is the *equivalence* symbol).

The symbolic notation $\alpha \wedge \beta$ means "α and β" ($\wedge$ is the symbol of *conjunction*).

The notation $\alpha \vee \beta$ means "α or β" ($\vee$ is the symbol of *disjunction*).
The logic notation

$$\forall\ x \in X\ \alpha\ (x)$$

means: "for any element $x \in X$ the statement $\alpha\ (x)$ is true" ($\forall$ is the *generality* or *universal quantifier*).

The symbolic notation

$$\exists\ x \in X\ \alpha\ (x)$$

means: "there exists an element $x \in X$ such that the statement $\alpha\ (x)$ is true for it" ($\exists$ is the *existential quantifier*).

If an element $x \in X$, for which the statement $\alpha\ (x)$ is true, not only exists, but is also unique, then we write

$$\exists!\ x \in X\ \alpha\ (x).$$

Example 4. Using the logic symbols, write the following statement: "The number M is the supremum of the set X".

The statement $M = \sup x$ implies that the following two conditions are fulfilled:

(a) $\forall\ x \in X\ (x \leqslant M)$ (i.e. M is the upper bound of the set X);

(b) $\forall\ A \in \mathbb{R}\ (\forall\ x \in X\ (x \leqslant A) \Rightarrow A \geqslant M)$ (i.e. M is the least among the upper bounds of the set X).

Condition (b) can also be written in the following equivalent form (see Problem 1.72):

$$\forall\ \varepsilon > 0\ \exists\ x \in X\ (x > M - \varepsilon).$$

Example 5. Using logic symbolism, formulate the principle of mathematical induction.

Let α be some statement making sense for all $n \in \mathbb{N}$. Let us introduce the set

$$A = \{n \in \mathbb{N} \mid \alpha\ (n)\},$$

i.e. the set of all those natural numbers for which the statement α is true. Then the principle of mathematical induction can be formulated in the following way:

$$((1 \in A) \wedge (n \in A \Rightarrow (n + 1) \in A)) \Rightarrow A = \mathbb{N}\ . \qquad (3)$$

Since the notation $\alpha\ (n)$ means that the statement α is true for the number $n \in \mathbb{N}$ statement (3) can be written differently:

$$(\alpha (1) \wedge \alpha\ (n) \Rightarrow \alpha\ (n + 1)) \Rightarrow \forall\ n \in \mathbb{N}\ \alpha\ (n).$$

Example 6. Write the negations of the sentences: $\forall\ x \in X\ \alpha\ (x)$ and $\exists\ x \in X\ \alpha\ (x)$.

The negation of the sentence $\forall\ x \in X\ \alpha\ (x)$ has the form $\exists\ x \in X\ \overline{\alpha\ (x)}$ (there is an element $x \in X$ such that the statement $\alpha\ (x)$ is false). In other words, for any statement α the following sen-

tence is true:

$$\overline{\forall x \in X\, \alpha(x)} \Leftrightarrow \exists x \in X\, \overline{\alpha(x)}.$$

Analogously,

$$\overline{\exists x \in X\, \alpha(x)} \Leftrightarrow \forall x \in X\, \overline{\alpha(x)}.$$

Example 7. Using the logic symbols, write the following statement: "The function f: $X \to \mathbb{R}$, $X \subset \mathbb{R}$, is continuous at a point $a \in X$", and its negation.

The given statement:

$$\forall \varepsilon > 0\, \exists \delta\, \forall x \in X\, (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

Its negation:

$$\exists \varepsilon > 0\, \forall \delta\, \exists x \in X\, (|x - a| < \delta \wedge |f(x) - f(a)| \geqslant \varepsilon)$$

(there is $\varepsilon > 0$ such that for any δ there can be found a number $x \in X$ satisfying the conditions $|x - a| < \delta$ and $|f(x) - f(a)| \geqslant \varepsilon$).

In Problems 1.83 to 1.91 read the given sentences, clarify their sense and find out whether they are true or false (the symbols x, y, z, a, b, c denote real numbers).

1.83. (a) $\forall x \exists y\ (x + y = 3)$; (b) $\exists y \forall x\ (x + y = 3)$; (c) $\exists x,\ y\ (x + y = 3)$; (d) $\forall x,\ y\ (x + y = 3)$.

1.84. $\exists x,\ y\ (x > y > 0 \wedge x + y = 0)$.

1.85. $\forall x,\ y\ (x < y) \Leftrightarrow \exists z\ (x < z < y)$.

1.86. $\forall x,\ y\ (x^2 \neq 2y^2)$.

1.87. $\forall x\ (x^2 > x \Leftrightarrow x > 1 \vee x < 0)$.

1.88. $\forall x\ (x > 2 \wedge \overline{x > 3} \Leftrightarrow 2 < x \leqslant 3)$.

1.89. $\exists x\ (\sqrt{x^2} < x)$.

1.90. (a) $\forall a, b, c\ (\exists x\ (ax^2 + bx + c = 0) \Leftrightarrow b^2 - 4ac \geqslant 0)$;

(b) $\forall a, b, c\ (\forall x\ (ax^2 + bx + c > 0) \Leftrightarrow b^2 - 4ac < 0 \wedge a > 0)$.

1.91. (a) $\forall b\, \exists a\, \forall x\ (x^2 + ax + b > 0)$;

(b) $\exists b\, \forall a\, \exists x\ (x^2 + ax + b = 0)$;

(c) $\exists a\, \forall b\, \exists x\ (x^2 + ax + b = 0)$.

In Problems 1.92 to 1.94 determine the exact meaning of the given sentences and write them using the logic symbols. Formulate and write the negations of these sentences.

1.92. (a) The number x_0 is a solution of the equation $f(x) = 0$.

(b) The number x_0 is the unique solution of the equation $f(x) = 0$.

(c) The equation $f(x) = 0$ has a unique real solution.

1.93. (a) The set $X \subset \mathbb{R}$ is bounded above.

(b) The number m is the least element of the set X.

(c) The set X has a minimum element.

1.94. (a) The number $m \in \mathbb{Z}$ is the divisor of the number $n \in \mathbb{Z}$, or in the shorter notation: $m \mid n$.

(b) If the number $n \in \mathbb{Z}$ is divisible by 2 and by 3, then it is divisible by 6.

(c) $p \in \mathbb{N}$ is a prime number.

SEC. 1.2.
FUNCTIONS OF A REAL VARIABLE

1. The Notion of Function. Let D be an arbitrary set of real numbers. If each number $x \in D$ is associated with exactly one definite real number $f(x)$, then we say that a *numerical function* f is defined on the set D. The set D is called the *domain of definition* and the set

$$E = \{y \in \mathbb{R} \mid y = f(x),\ x \in D\}$$

the *set of values* of the numerical function f, the symbolical notation being: $f\colon D \to E$ or $y = f(x)$.

The most widespread method of specifying a function is its analytical representation. The analytical representation of a function is given by a formula which shows how the value of the function $y = f(x)$ corresponding to any given value of the *independent variable* x can be determined. When a function is specified analytically its domain (of definition) is usually understood (provided that there are no additional conditions) as the maximum set of values of x for which the formula representing the function makes sense. This means that the application of the formula to these values of x should result in definite real values of y (*natural* domain of definition of function).

Example 1. Find the domain (of definition) and the set of values of the function $f(x) = \dfrac{1}{\sqrt{1-x^2}}$.

The natural domain of definition of this function is the set $D = \{x \mid |x| < 1\} = (-1, 1)$, and the set of values is the set $E = \{y \mid y \geqslant 1\} = [1, \infty)$.

Let the function $f\colon D \to E$ be such that for any $x_1, x_2 \in D$ from the condition $x_1 \neq x_2$ it follows that $f(x_1) \neq f(x_2)$. In this case any number $y \in E$ can be associated with quite a definite number $x \in D$ such that $f(x) = y$; in this way a new function is defined $f^{-1}\colon E \to D$ called the *inverse* of the given function f.

Let there be given two functions: $f\colon X \to Y$ and $g\colon Y \to Z$. Their *composition* (or the *composite function* obtained by combining the

functions f and g) is defined as the function $h = g \circ f\colon X \to Z$, specified by the equality

$$h(x) = g(f(x)), \quad x \in X.$$

2.1. Find the functional dependence of the radius R of a cylinder on its altitude H for the given volume $V = 1$.

2.2. Write the expression for the volume V of a cone as a function of its lateral surface S for the given generatrix $l = 2$.

2.3. Write the expression for the area S of an isosceles trapezoid with bases $a = 2$ and $b = 1$ as a function of the angle α at the base a.

2.4. Beginning with the instant of rest t_0 a body moves with a constant acceleration a. Find the dependence of its velocity and the path covered on the time of motion. How are the path and velocity related at the time t?

2.5. In an isosceles trapezoid $ABCD$ (Fig. 1) with bases a and b and altitude h a straight line MN is drawn perpendicular to the bases at a distance $|AM| = x$ from the vertex A. Express the area S of the figure $ABNM$ as a function of the variable x.

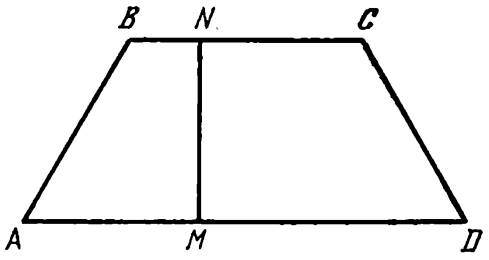

Fig. 1.

2.6. A cylinder is inscribed into a sphere of radius R. Write the functional dependence of the volume V of the cylinder on its altitude H. Find the domain of definition of this function.

2.7. A right circular cone is inscribed into a sphere of radius R. Write the functional dependence of the area of the lateral surface S of the cone on:

(a) its generatrix l;

(b) the angle α at the vertex of the cone in its axial section;

(c) the angle β at the base of the cone.

Find the domain of definition for each of the functions thus obtained.

2.8. Find $f(-1)$, $f(-0.001)$, $f(100)$, if $f(x) = \log x^2$.

2.9. Find $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$, if

$$f(x)=\begin{cases} 1+x, & -\infty<x\leqslant 0, \\ 2^x, & 0<x<+\infty. \end{cases}$$

2.10. Find $f(1)$, $f(a)$, $f(a+1)$, $f(a-1)$, $2f(2a)$, if $f(x)=x^3-1$.

2.11. Find $f(0)$, $f(-x)$, $f(x+1)$, $f(x)+1$, $f\left(\frac{1}{x}\right)$, $\frac{1}{f(x)}$, if $f(x)=\frac{1-x}{1+x}$.

In Problems 2.12 to 2.21 find the natural domain of definition D and the set of values E for the given function.

2.12. $y=\ln(x+3)$. **2.13.** $y=\sqrt{5-2x}$.

2.14. $y=\sqrt{\sin\sqrt{x}}$. **2.15.** $y=\arccos\frac{1-2x}{4}$.

2.16. $y=\ln(1-2\cos x)$. **2.17.** $y=\sqrt{1-|x|}$.

2.18. $y=\log(5x-x^2-6)$.

2.19. $y=\arcsin\sqrt{\frac{1-x^2}{2}}$.

2.20. $y=2^{\arccos(1-x)}$. **2.21.** $y=e^{x^2-2}$.

In Problems 2.22 to 2.27 find the set G onto which the set F is mapped by the given function.

2.22. $y=x^2$, $F=[-1, 2]$.

2.23. $y=|x|$, $F=\{x \mid 1\leqslant|x|\leqslant 2\}$.

2.24. $y=\frac{x}{2x-1}$, $F=(0, 1)$.

2.25. $y=\sqrt{x-x^2}$, $F=(0, 1)$.

2.26. $y=\log_3 x$, $F=(3, 27)$.

2.27. $y=\sin\frac{\pi x}{2}$, $F=[0, 1/2)$.

In Problems 2.28 to 2.31 find the *set of zeros* $D_0=\{x \mid f(x)=0\}$, the *domain of positiveness* $D_+=\{x \mid f(x)>0\}$ and the *domain of negativeness* $D_-=\{x \mid f(x)<0\}$ for the given function.

2.28. $f(x)=1+x$. **2.29.** $f(x)=2+x-x^2$.

2.30. $f(x)=\sin\frac{\pi}{x}$. **2.31.** $f(x)=1-e^{\frac{1}{x}-1}$.

In Problems 2.32 to 2.35 show that the function $y = f(x)$ satisfies the appropriate functional equation:

2.32. $f(x+2) - 2f(x+1) + f(x) = 0, f(x) = kx + b.$

2.33. $f(x) + f(x+1) = f(x(x+1)), f(x) = \log_a x.$

2.34. $f(x_1) f(x_2) = f(x_1 + x_2), f(x) = a^x.$

2.35. $f(x_1) + f(x_2) = f\left(\frac{x_1+x_2}{1+x_1x_2}\right), \quad f(x) = \log \frac{1+x}{1-x}.$

In Problems 2.36 to 2.39 determine the function $y = f(x)$ satisfying the given condition.

2.36. $f(x+1) = x^2 - 3x + 2.$

Let $x + 1 = t$, i.e. $x = t - 1$. Then $x^2 - 3x + 2 = t^2 - 5t + 6$, therefore

$$f(t) = f(x+1) = x^2 - 3x + 2 = t^2 - 5t + 6.$$

2.37. $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}, \quad x \neq 0.$

2.38. $f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}, \quad x > 0.$

2.39. $f(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2.$

A function $f(x)$ is said to be *even (odd)* if its domain is symmetric about the point $x = 0$ and $f(-x) = f(x)$ ($f(-x) = -f(x)$).

Which of the functions indicated in Problems 2.40 to 2.45 are even, which are odd, and which are neither even nor odd?

2.40. $f(x) = x^4 + 5x^2.$ **2.41.** $f(x) = x^2 + x.$

2.42. $f(x) = \frac{x}{2^x - 1}.$ **2.43.** $f(x) = \frac{e^x+1}{e^x-1}.$

2.44. $f(x) = \sin x - \cos x.$ **2.45.** $f(x) = \log \frac{1+x}{1-x}.$

2.46. Prove that the product of two even or two odd functions is an even function, while the product of an even function by an odd one is an odd function.

A function $f(x)$ is called *periodic* if there is a positive number T (period of the function) such that $\forall\, x \in D\ (f(x+T) = f(x))$.

Which of the functions given below are periodic? Determine the least period T for each function (Problems 2.47 to 2.52):

2.47. $f(x) = 5 \cos 7x.$ **2.48.** $f(x) = \cos^2 2x.$

2.49. $f(x) = x \sin x$. **2.50.** $f(x) = \cos x + \sin(\sqrt{3}x)$.

2.51. $f(x) = \sin x^2$. **2.52.** $f(x) = \tan \frac{x}{2} - 2 \tan \frac{x}{3}$.

Which of the functions indicated below have inverses? Find the appropriate inverses and the domains of their definition (Problems 2.53 to 2.59):

2.53. $y = ax + b$. **2.54.** $y = (x - 1)^3$.

2.55. $y = \cos 2x$. **2.56.** $y = \ln 2x$. **2.57.** $y = 2^{\frac{x}{2}}$.

2.58. $y = \frac{1-x}{1+x}$. **2.59.** $y = x^2 + 1$.

For the function $y = x^2 + 1$ the natural domain of definition is the entire number line $D = (-\infty, +\infty)$ and the set of its values is the ray $E = [1, +\infty)$. Since for any $a \in E$ the equation $x^2 + 1 = a$ has two distinct solutions $x_1(a) = \sqrt{a-1}$ and $x_2(a) = -\sqrt{a-1}$, the given function has no inverse.

But each of the functions

$$y_1 = x^2 + 1, \quad D_1 = [0, +\infty),$$

and

$$y_2 = x^2 + 1, \quad D_2 = (-\infty, 0],$$

has an inverse equal respectively to

$$x_1(y) = \sqrt{y-1}$$

and

$$x_2(y) = -\sqrt{y-1}.$$

In Problems 2.60 to 2.63 find the inverse function and domain of its definition if the given function is defined on the indicated interval:

2.60. $y = x^2 - 1$: (a) $x \in (-\infty, -1/2]$; (b) $x \in [1/2, +\infty)$.

2.61. $y = \sin x$: (a) $x \in [-\pi/2, \pi/2]$; (b) $x \in [\pi/2, 3\pi/2]$.

2.62. $y = \begin{cases} x, & x \in (-\infty, 0], \\ 2x, & x \in (0, +\infty). \end{cases}$

2.63. $y = \cos^2 x$: (a) $x \in [0, \pi/2]$; (b) $x \in [\pi/2, \pi]$; (c) $x \in [\pi, 3\pi/2]$.

In Problems 2.64 to 2.68 find the compositions $f \circ g$ and $g \circ f$ of the indicated functions:

2.64. $f(x) = x^2$, $g(x) = \sqrt{x}$.

We have:

$$f \circ g\,(x) = f\,(g\,(x)) = f\,(\sqrt{x}) = (\sqrt{x})^2 = x$$

and

$$g \circ f\,(x) = g\,(f\,(x)) = g\,(x^2) = \sqrt{x^2} = |\,x\,|.$$

2.65. $f\,(x) = 1 - x$, $g\,(x) = x^2$.

2.66. $f\,(x) = e^x$, $g\,(x) = \ln x$.

2.67. $f\,(x) = \sin x$, $x \in [-\pi, \pi]$, $g\,(x) = \arcsin x$.

2.68. $f\,(x) = \begin{cases} 0, & x \in (-\infty, 0], \\ x, & x \in (0, +\infty), \end{cases}$

$$g\,(x) = \begin{cases} 0, & x \in (-\infty, 0], \\ -x^2, & x \in (0, +\infty). \end{cases}$$

2.69. Find $f \circ f \circ f$, if:

(a) $f\,(x) = \dfrac{1}{1-x}$; (b) $f\,(x) = \dfrac{x}{\sqrt{1+x^2}}$.

2. Elementary Functions and Their Graphs. Listed below are the *basic elementary* functions:

1. *Power* function: $y = x^a$, $a \in \mathbb{R}$.
2. *Exponential* function: $y = a^x$, $a > 0$, $a \neq 1$.
3. *Logarithmic* function: $y = \log_a x$, $a > 0$, $a \neq 1$.
4. *Trigonometric* functions: $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$.
5. *Inverse trigonometric (circular)* functions: $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$, $y = \operatorname{arccot} x$.

Any function which can be constructed of a finite number of basic elementary functions with the aid of arithmetical operations and operations of forming a function of a function is called an *elementary* function.

The *graph* of the function $y = f\,(x)$ is defined as the set

$$\Gamma = \{(x,\ y) \in \mathbb{R}^2 \mid x \in D,\ y = f\,(x)\},$$

where $\mathbb{R}^2$ is the set of points in the plane.

On a plane with a rectangular Cartesian system of coordinates Oxy the graph of a function is represented by a set of points $M\,(x,\ y)$ whose coordinates satisfy the relation $y = f\,(x)$ (graphical representation of a function).

In constructing graphs the following simple geometrical considerations are frequently used. If Γ is the graph of the function $y = f\,(x)$, then:

(1) the graph of the function $y_1 = -f\,(x)$ is a mirror image of Γ relative to the x-axis;

(2) the graph of the function $y_2 = f\,(-x)$ is a mirror image of Γ relative to the y-axis;

(3) the graph of the function $y_3 = f\,(x - a)$ is a displacement of Γ along the x-axis over the quantity a;

(4) the graph of the function $y_4 = b + f(x)$ is a displacement of Γ along the y-axis over the quantity b;

(5) the graph of the function $y_5 = f(ax)$, $a > 0$, $a \neq 1$, is a-times contraction (for $a > 1$) or $1/a$ elongation (for $a < 1$) of Γ along the x-axis;

(6) the graph of the function $y_6 = bf(x)$, $b > 0$, $b \neq 1$, is b-times elongation (for $b > 1$) or $1/b$ contraction (for $b < 1$) of Γ along the y-axis.

In some cases for constructing the graph of a function it is advisable to break its domain of definition into several nonoverlapping intervals and construct the required graph for each of them in succession.

Example 2. Construct the graph of the function $y = |x| + |x^2 - 1|$.

Let us rewrite the given function as follows:

$$y = \begin{cases} x^2 - x - 1, & x \in (-\infty, -1], \\ -x^2 - x + 1, & x \in (-1, 0], \\ -x^2 + x + 1, & x \in (0, 1], \\ x^2 + x - 1, & x \in (1, +\infty). \end{cases}$$

The graph of the given function is a union of the graphs (parabolas) representing this function on each of the four intervals (see Fig. 2).

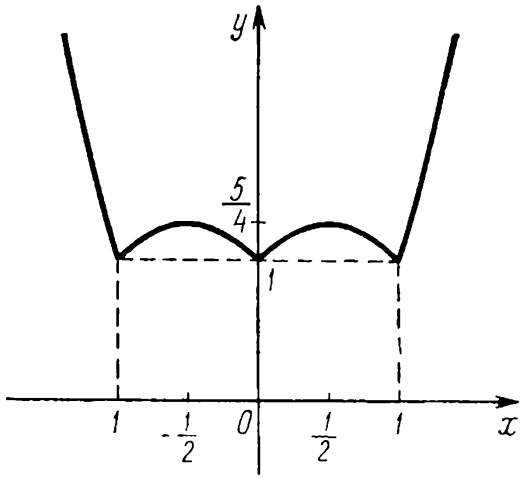

Fig. 2.

In Problems 2.70 to 2.75 write the given elementary function in the form of a composition of basic elementary functions:

2.70. $f(x) = |x|$.

2.71. $f(x) = \sin(\cos \sqrt{x})$.

2.72. $f(x) = 2^{\sin x^2}$.

2.73. $f(x) = \arcsin(e^{\sqrt[3]{x}})$.

2.74. $f(x) = \sin(2^{x^2})$.

2.75. $f(x) = \dfrac{1}{\sqrt[3]{\tan^2 \log_3 x}}$.

In Problems 2.76 to 2.80 find the graph of the indicated function:

2.76. $y = \sqrt{\ln \sin x}$.

The natural domain of definition of the given function is the set

$$D = \{x \mid \sin x = 1\} = \left\{\frac{\pi}{2} + 2\pi k \mid k \in \mathbb{Z}\right\},$$

therefore

$$\Gamma = \left\{\left(\frac{\pi}{2} + 2\pi k,\ 0\right) \middle| \ k \in \mathbb{Z}\right\}.$$

2.77. $y = x + \sqrt{1 - |\csc x|}$.

2.78. $y = \sqrt{-|x^2 - 1|} + 2$.

2.79. $y = \sqrt{\cos x - 1} + \dfrac{x}{2}$.

2.80. $y = 1 + \sqrt{\sin x} + \sqrt{-\sin x}$.

In Problems 2.81 to 2.112 construct the graphs of the given elementary functions:

2.81. $y = kx + b$, if:

(a) $k = 2$, $b = 0$; (b) $k = 0$, $b = -2$;

(c) $k = -1$, $b = -\dfrac{1}{3}$.

2.82. $y = y_0 + a(x - x_0)^2$, if:

(a) $a = 1$, $x_0 = 0$, $y_0 = -1$;

(b) $a = 2$, $x_0 = 1$, $y_0 = 0$;

(c) $a = -\dfrac{1}{2}$, $x_0 = -2$, $y_0 = \dfrac{3}{2}$.

2.83. $y = y_0 + \dfrac{k}{x - x_0}$, if:

(a) $k = 1$, $x_0 = 1$, $y_0 = -1$;

(b) $k = -2$, $x_0 = -1$, $y_0 = -\dfrac{1}{2}$.

2.84. $y = a \sin (kx + \alpha)$, if:
(a) $a = 1$, $k = 2$, $\alpha = \pi/3$;
(b) $a = -2$, $k = 1/2$, $\alpha = -\pi/3$.
2.85. $y = a \tan (kx + \alpha)$, if:
(a) $a = 3$, $k = 1/3$, $\alpha = \pi/4$;
(b) $a = -1/2$, $k = 2$, $\alpha = 3\pi/2$.
2.86. $y = p \arcsin (x + q)$, if:
(a) $p = 4$, $q = -1$; (b) $p = -2/3$, $q = 1/2$.
2.87. $y = p \arctan (x + q)$, if:
(a) $p = -3$, $q = 5/2$; (b) $p = 2/5$, $q = -6$.
2.88. $y = a^{kx+b}$, if:
(a) $a = 2$, $k = -1$, $b = 1$; (b) $a = 1/2$, $k = 2$, $b = -2$.
2.89. $y = \log_a (kx + b)$, if:
(a) $a = 10$, $k = 10$, $b = -1$;
(b) $a = 1/10$, $k = 1/2$, $b = 2$.
2.90. $y = |2 - x| + |2 + x|$.
2.91. $y = x^2 + x - |x|$. **2.92.** $y = x^2 - 6|x| + 9$.
2.93. $y = |6x^2 + x| - 1$.
2.94. $y = (x^2 + 2x)\frac{|x-1|}{x-1}$. **2.95.** $y = x - 1 - \sqrt{(x-1)^2}$.
2.96. $y = \left|\frac{2x-3}{x+2}\right|$. **2.97.** $y = \frac{|x|-1}{|x+2|}$.
2.98. $y = \operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$
2.99. $y = [x]$, where $[x]$ is the integral part of x.
2.100. $y = \{x\}$, where $\{x\} = x - [x]$ is the fractional part of x.
2.101. $y = 2^{|x|} - 1$. **2.102.** $y = \left(\frac{1}{3}\right)^{|x+1|} + 2$.
2.103. $y = \log_{1/2} |x - 3|$. **2.104.** $y = |\log_2 (x + 1)|$.
2.105. $y = \arcsin\left(\sin\left(x + \frac{\pi}{4}\right)\right)$.
2.106. $y = \arccos (\cos 3x)$.
2.107. $y = \cos x + |\sin x|$. **2.108.** $y = |\arctan (x - 1)|$.

2.109. $y = x \operatorname{sgn}(\cos x)$. **2.110.** $y = \left| \cot\left(x + \frac{\pi}{4}\right) \right|$.

2.111. $y = \sin^2 \frac{x}{2}$. **2.112.** $y = \sin\left(\arcsin \frac{x+2}{5}\right)$.

In Problems 2.113 to 2.118 on the (xy)-plane represent the sets of points whose coordinates satisfy the given conditions.

2.113. $xy = 0$. **2.114.** $|y| = |x^2 - 2|x| - 3|$.

2.115. $|x| + |y| = 1$. **2.116.** $|x + y| + |x - y| = 1$.

2.117. $||x| - |y|| = 1$.

2.118. $|2y - 1| + |2y + 1| + \frac{4}{\sqrt{3}}|x| = 4$.

SEC. 1.3.
LIMIT OF A SEQUENCE OF REAL NUMBERS

1. The Concept of a Sequence. The *sequence* of real numbers is understood as a function $f: \mathbb{N} \to \mathbb{R}$ defined on the set of all natural numbers. The number $f(n)$ is called the nth term of the sequence and is denoted by the symbol x_n, and $x_n = f(n)$ is called the *formula of the general term* of the sequence $(x_n)_{n \in \mathbb{N}}$.

In Problems 3.1 to 3.4 write the first five terms for the indicated sequence.

3.1. $x_n = 1 + (-1)^n \frac{1}{n}$. **3.2.** $x_n = n(1 - (-1)^n)$.

3.3. $x_n = \frac{3n+5}{2n-3}$. **3.4.** $x_n = (-1)^n \arcsin \frac{\sqrt{3}}{2} + \pi n$.

In Problems 3.5 to 3.10 write the formula for the general term of the given sequence.

3.5. $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$ **3.6.** $0, 2, 0, 2, \ldots$

3.7. $2, \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \ldots$

3.8. $1, 0, -3, 0, 5, 0, -7, 0, \ldots$

3.9. $-3, \frac{5}{3}, -\frac{7}{5}, \frac{9}{7}, -\frac{11}{9}, \ldots$

3.10. $0, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \ldots$

In Problems 3.11 to 3.16 find the largest (smallest) term in the sequence $(x_n)_{n\in\mathbb{N}}$ having the upper (lower) bound.

3.11. $x_n = 6n - n^2 - 5$. **3.12.** $x_n = e^{10n-n^2-24}$.

3.13. $x_n = \frac{\sqrt{n}}{9+n}$. **3.14.** $x_n = 3n^2 - 10n - 14$.

3.15. $x_n = 2n + \frac{512}{n^2}$. **3.16.** $x_n = -\frac{n^2}{2^n}$.

2. Limit of a Sequence. A number a is said to be the *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$, i.e. $\lim\limits_{n\to\infty} x_n = a$, if for any $\varepsilon > 0$ there is an ordinal number $N(\varepsilon)$ such that for $n > N(\varepsilon)$ the inequality $|x_n - a| < \varepsilon$ is fulfilled. In this case the sequence itself is called *convergent*.

Cauchy's criterion: *for a sequence $(x_n)_{n\in\mathbb{N}}$ to have a limit, it is necessary and sufficient that for any $\varepsilon > 0$ there exists an ordinal number $N(\varepsilon)$ such that for $n > N(\varepsilon)$ the inequality $|x_{n+p} - x_n| < \varepsilon$ is fulfilled for any $p \in \mathbb{N}$.*

A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *infinitely small* if $\lim\limits_{n\to\infty} x_n = 0$.

A *sequence* $(x_n)_{n\in\mathbb{N}}$ is said to be *infinitely large* (converging to infinity) (written $\lim\limits_{n\to\infty} x_n = \infty$) if for any $E > 0$ there exists an ordinal number $N(E)$ such that for $n > N(E)$ the inequality $|x_n| > E$ is fulfilled. In this case, if, beginning with some ordinal number, all terms of the sequence are positive (negative), then the following notation is used

$$\lim_{n\to\infty} x_n = +\infty \quad (\lim_{n\to\infty} x_n = -\infty).$$

A number a is called a *limit point* of a sequence $(x_n)_{n\in\mathbb{N}}$, if for any $\varepsilon > 0$ there can be found an infinite number of terms in this sequence satisfying the condition $|x_n - a| < \varepsilon$.

Bolzano-Weierstrass principle: *any bounded sequence has at least one limit point.*

The largest (smallest) of the limit points of a sequence is called the limit *superior* (the limit *inferior*) of this sequence and is denoted by the symbol $\overline{\lim\limits_{n\to\infty}} x_n$ ($\underline{\lim\limits_{n\to\infty}} x_n$).

3.17. Using the logic symbols, write the following sentences and their negations:

(a) The sequence is bounded;

(b) The sequence increases monotonically;

(c) The number a is the limit of the sequence;

(d) The sequence $(x_n)_{n\in\mathbb{N}}$ is infinitely large;

(e) The number a is a limit point of the sequence.

3.18. Find $a = \lim\limits_{n\to\infty} x_n$ and determine the ordinal number $N(\varepsilon)$ such that $|x_n - a| < \varepsilon$ for all $n > N(\varepsilon)$ if

(a) $x_n = 0.\underbrace{33\ldots3}_{n}$, $\varepsilon = 0.001$;

(b) $x_n = \dfrac{\sqrt{n^2+1}}{n}$, $\varepsilon = 0.005$;

(c) $x_n = \dfrac{1}{n}\sin\dfrac{\pi n}{2}$, $\varepsilon = 0.001$;

(d) $x_n = \dfrac{5n^2+1}{7n^2-3}$, $\varepsilon = 0.005$.

In Problems 3.19 to 3.33 compute the indicated limits:

3.19. $\lim\limits_{n\to\infty} \dfrac{n-1}{3n}$. **3.20.** $\lim\limits_{n\to\infty} \dfrac{(n+1)}{2n^2}$.

3.21. $\lim\limits_{n\to\infty} \dfrac{5n+1}{7-9n}$.

3.22. $\lim\limits_{n\to\infty} \dfrac{(n+2)^3-(n-2)^3}{95n^3+39n}$.

3.23. $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^4+3n+1}}{n-1}$.

3.24. $\lim\limits_{n\to\infty} \dfrac{3n^2-7n+1}{2-5n-6n^2}$.

3.25. $\lim\limits_{n\to\infty} \left(\dfrac{2n-1}{5n+7} - \dfrac{1+2n^3}{2+5n^3}\right)$.

3.26. $\lim\limits_{n\to\infty} (\sqrt{n+2}-\sqrt{n})$.

3.27. $\lim\limits_{n\to\infty} n^{3/2}(\sqrt{n^3+1}-\sqrt{n^3-2})$.

3.28. $\lim\limits_{n\to\infty} \dfrac{2^n+3^n}{2^n-3^n}$.

3.29. $\lim\limits_{n\to\infty} \left(\dfrac{1}{n^2} + \dfrac{2}{n^2} + \ldots + \dfrac{n-1}{n^2}\right)$.

3.30. $\lim\limits_{n\to\infty} \dfrac{1^2+2^2+\ldots+n^2}{n^3}$.

3.31. $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^2}\sin(n^2)}{n-1}$.

3.32. $\lim\limits_{n\to\infty}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}\right)$.

3.33. $\lim\limits_{n\to\infty}\left(\frac{1}{5}-\frac{1}{25}+\ldots+(-1)^{n-1}\frac{1}{5^n}\right)$.

3.34. Prove that if the sequence $(x_n)_{n\in\mathbb{N}}$ is infinitely small and $\forall\, n\in\mathbb{N}$ $(x_n\neq 0)$, then the sequence $\left(\frac{1}{x_n}\right)_{n\in\mathbb{N}}$ is infinitely large.

Which of the given sequences are infinitely large (Problems 3.35 to 3.38):

3.35. $x_n=2^{\sqrt{n}}$. **3.36.** $x_n=n^{(-1)^n}$.

3.37. $x_n=n\sin\frac{\pi n}{2}$.

3.38. $x_n=\log(\log n)$, $n\geqslant 2$.

In Problems 3.39 to 3.41 find all the limit points of the given sequence.

3.39. $x_n=\frac{2+(-1)^n}{2-(-1)^n}$. **3.40.** $x_n=\cos\left(\frac{\pi n}{4}\right)$.

3.41. $x_n=\arcsin\frac{(-1)^n}{2}$.

3.42. Prove:

(a) $\varliminf\limits_{n\to\infty}x_n+\varliminf\limits_{n\to\infty}y_n\leqslant\varliminf\limits_{n\to\infty}(x_n+y_n)\leqslant\varliminf\limits_{n\to\infty}x_n+\varlimsup\limits_{n\to\infty}y_n$;

(b) $\varliminf\limits_{n\to\infty}x_n+\varlimsup\limits_{n\to\infty}y_n\leqslant\varlimsup\limits_{n\to\infty}(x_n+y_n)\leqslant\varlimsup\limits_{n\to\infty}x_n+\varlimsup\limits_{n\to\infty}y_n$.

For each of the following sequences $(x_n)_{n\in\mathbb{N}}$ find $\inf\{x_n\}$, $\sup\{x_n\}$, $\varliminf\limits_{n\to\infty}x_n$ and $\varlimsup\limits_{n\to\infty}x_n$:

3.43. $x_n=1+\frac{1}{n}$. **3.44.** $x_n=\frac{n+1}{n}\cos^2\frac{\pi n}{4}$.

3.45. $x_n=(-1)^n(2n+1)$.

3.46. $x_n=\frac{n+2}{n-1}\sin\frac{\pi n}{3}$, $n\geqslant 2$.

3.47. $x_n=\frac{2+(-1)^n}{2}-\frac{1}{n}$.

3.48. Prove that the equality $\varliminf\limits_{n\to\infty}x_n=\varlimsup\limits_{n\to\infty}x_n$ is a necessary and sufficient condition for the sequence $(x_n)_{n\in\mathbb{N}}$ to have a limit.

SEC. 1.4.
LIMIT OF A FUNCTION. CONTINUITY

1. Limit of a Function. Let a function $y = f(x)$ be defined on a set D. A number a is called the *limit of the function* $y = f(x)$ *at a point* x_0 written $\lim\limits_{x \to x_0} f(x) = a$ if for any $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ such that for any $x \in D$ the condition $0 < |x - x_0| < \delta(\varepsilon)$ implies the inequality $|f(x) - a| < \varepsilon$.

Cauchy's criterion. *For a function* $y = f(x)$ *to have a limit at a point* x_0 *it is necessary and sufficient that for any* $\varepsilon > 0$ *there is* $\delta(\varepsilon) > 0$ *such that* $|f(x') - f(x'')| < \varepsilon$, *whenever* $|x' - x_0| < \delta(\varepsilon)$ *and* $|x'' - x_0| < \delta(\varepsilon)$.

A number a is said to be the *limit of a function* $x = f(x)$ *as* x *tends to infinity* (written $\lim\limits_{x \to \infty} f(x) = a$) if for any $\varepsilon > 0$ there exists a number $A(\varepsilon) > 0$ such that $|f(x) - a| < \varepsilon$, whenever $|x| > A(\varepsilon)$.

We shall use the following *remarkable limits*:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \tag{1}$$

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to 0} (1 + x)^{1/x} = e, \tag{2}$$

where $e = 2.71828...$ is the base of natural logarithms.

Let us also introduce the following notion of a *one-sided* (unilateral) limit. The number a is termed the *right-hand (left-hand) limit of the function* $y = f(x)$ at the point x_0 (written $\lim\limits_{x \to x_0 + 0} f(x) = a$ ($\lim\limits_{x \to x_0 - 0} f(x) = a$)) if for any $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that the condition $0 < x - x_0 < \delta(\varepsilon)$ ($-\delta(\varepsilon) < x - x_0 < 0$) implies $|f(x) - a| < \varepsilon$. The notion of a one-sided limit at infinity ($\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$) is introduced in a similar way.

In Problems 4.1 to 4.3, using only the definition of the limit of a function, prove that $\lim\limits_{x \to x_0} f(x) = a$ and fill in the given table:

ε	0.1	0.01	0.001
$\delta(\varepsilon)$			

4.1. $f(x) = x^2$, $x_0 = 2$, $a = 4$.

4.2. $f(x) = 1/x$, $x_0 = 1$, $a = 1$.

4.3. $f(x) = \log x$, $x_0 = 1$, $a = 0$.

In Problems 4.4 to 4.11 rewrite the given statements, using the logic symbols.

4.4. $\lim\limits_{x\to 0} f(x) = \infty$. **4.5.** $\lim\limits_{x\to 1-0} f(x) = -\infty$.

4.6. $\lim\limits_{x\to +\infty} f(x) = 0$. **4.7.** $\lim\limits_{x\to +\infty} f(x) = +\infty$.

4.8. $\lim\limits_{x\to 0+0} f(x) = 0$. **4.9.** $\lim\limits_{x\to \infty} f(x) = 2$.

4.10. $\lim\limits_{x\to -\infty} f(x) = -\infty$. **4.11.** $\lim\limits_{x\to -\infty} f(x) = \infty$.

In Problems 4.12 to 4.25 compute the limits of the indicated rational expressions.

4.12. $\lim\limits_{x\to 0} \dfrac{x^2-2}{3x^2-5x+1}$. **4.13.** $\lim\limits_{x\to 3} \dfrac{x^2+3}{x^2-3}$.

4.14. $\lim\limits_{x\to -3} \dfrac{x}{|x+3|}$. **4.15.** $\lim\limits_{x\to 1} \dfrac{(x-1)\sqrt[3]{28-x}}{x^2-1}$.

4.16. $\lim\limits_{x\to 2} \left(\dfrac{1}{2-x} - \dfrac{3}{8-x^3}\right)$. **4.17.** $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^3-x}$.

4.18. $\lim\limits_{x\to 1} \dfrac{x^m-1}{x^n-1}$; $m, n \in \mathbb{N}$. **4.19.** $\lim\limits_{h\to 0} \dfrac{(x+h)^3-x^3}{h}$.

4.20. $\lim\limits_{x\to 1} \dfrac{|x^3+x-2|}{x^3-x^2-x+1}$. **4.21.** $\lim\limits_{x\to \frac{1}{2}} \dfrac{8x^3-1}{6x^2-5x+1}$.

4.22. $\lim\limits_{x\to a} \dfrac{x^2-(a+1)x+a}{x^3-a^3}$.

4.23. $\lim\limits_{x\to \infty} \left(\dfrac{x^3}{2x^2-1} - \dfrac{x^2}{2x+1}\right)$. **4.24.** $\lim\limits_{x\to \infty} \dfrac{x^4-5x}{x^2-3x+1}$.

4.25. $\lim\limits_{x\to \infty} \dfrac{(x+1)^5+(x+2)^5+\ldots+(x+n)^5}{x^5+n^5}$, $n \in \mathbb{N}$.

4.26. Prove that if $P_n(x) = a_0x^n + \ldots + a_n$, $Q_m(x) = b_0x^m + \ldots + b_m$, then

$$\lim_{x\to\infty} \frac{P_n(x)}{Q_m(x)} = \begin{cases} 0 & \text{for } n < m, \\ a_0/b_0 & \text{for } n = m, \\ \infty & \text{for } n > m \text{ and } a_0 > 0. \end{cases}$$

When computing limits containing irrational expressions, use is frequently made of the following techniques: (a) introduction of a new variable for obtaining a rational expression; (b) transfer of an irrationality from the denominator to the numerator or vice versa.

Example 1. Compute $\lim\limits_{x\to 81}\dfrac{3-\sqrt[4]{x}}{9-\sqrt{x}}$.

Let $t=\sqrt[4]{x}$. Then

$$\lim_{x\to 81}\frac{3-\sqrt[4]{x}}{9-\sqrt{x}}=\lim_{t\to 3}\frac{3-t}{9-t^2}=\lim_{t\to 3}\frac{1}{3+t}=\frac{1}{6}.$$

Example 2. Evaluate $\lim\limits_{x\to\infty}(\sqrt{x^2+7}-\sqrt{x^2-7})$.

$$\lim_{x\to\infty}(\sqrt{x^2+7}-\sqrt{x^2-7})$$
$$=\lim_{x\to\infty}\frac{(\sqrt{x^2+7}-\sqrt{x^2-7})(\sqrt{x^2+7}+\sqrt{x^2-7})}{\sqrt{x^2+7}+\sqrt{x^2-7}}$$
$$=\lim_{x\to\infty}\frac{14}{\sqrt{x^2+7}+\sqrt{x^2-7}}=0.$$

In Problems 4.27 to 4.41 evaluate the given limits:

4.27. $\lim\limits_{x\to\infty}\dfrac{3x+1}{5x+\sqrt[3]{x}}$. **4.28.** $\lim\limits_{x\to 10}\dfrac{\sqrt{x-1}-3}{x-10}$.

4.29. $\lim\limits_{x\to 1}\dfrac{\sqrt{x}+\sqrt{x-1}-1}{\sqrt{x^2-1}}$.

4.30. $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$, $x>0$.

4.31. $\lim\limits_{x\to 1}\dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$. **4.32.** $\lim\limits_{x\to 0}\dfrac{|\sqrt{1+x}-1|}{x^2}$.

4.33. $\lim\limits_{x\to\infty}\dfrac{\sqrt{2x}}{\sqrt{3x+\sqrt{3x+\sqrt{3x}}}}$.

4.34. $\lim\limits_{x\to 0}\dfrac{\sqrt[n]{x+1}-1}{x}$. **4.35.** $\lim\limits_{x\to 1}\dfrac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$; $m,\ n\in\mathbb{N}$.

4.36. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{\sqrt{x^2+9}-3}$. **4.37.** $\lim\limits_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2-x}}{\sqrt[3]{2+x}-\sqrt[3]{2-x}}$.

4.38. $\lim\limits_{x\to a}(\sqrt{x-a}-\sqrt{x})$.

4.39. $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$.

4.40. $\lim\limits_{x\to\infty}(\sqrt{4x^2-7x+4}-2x)$.

4.41. $\lim\limits_{x\to\infty} x^{3/2}(\sqrt{x^3+2}-\sqrt{x^3-2})$.

In Problems 4.42 to 4.53 compute the given limit, using remarkable limit (1):

4.42. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$. **4.43.** $\lim\limits_{x\to\pi}\dfrac{\sin 7x}{\tan 3x}$.

4.44. $\lim\limits_{\alpha\to 0}\dfrac{\sin(\alpha^n)}{(\sin\alpha)^m}$, n, m are integers.

4.45. $\lim\limits_{x\to 0} x\cot\pi x$. **4.46.** $\lim\limits_{x\to 0}\dfrac{3\arcsin x}{4x}$.

4.47. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$. **4.48.** $\lim\limits_{\alpha\to 0}\dfrac{\sin 2\alpha-\tan 2\alpha}{\alpha^3}$.

4.49. $\lim\limits_{x\to 0}\dfrac{\cos\alpha x-\cos\beta x}{x^2}$, $\alpha\neq\beta$.

4.50. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\cot x\right)$. **4.51.** $\lim\limits_{x\to\alpha}\tan\dfrac{\pi x}{2\alpha}\sin\dfrac{x-\alpha}{2}$.

4.52. $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}-2\cos x}{\pi-4x}$. **4.53.** $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\pi}{2}-x\right)\tan x$.

4.54. $\lim\limits_{\alpha\to 0}\dfrac{(1-\cos\alpha)^2}{\tan^2\alpha-\sin^2\alpha}$. **4.55.** $\lim\limits_{x\to\pi}\dfrac{1+\cos 5x}{1-\cos 4x}$.

In Problems 4.56 to 4.58 prove the given relationships:

4.56.** $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\log_a e$.

4.57*. $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$. **4.58*.** $\lim\limits_{x\to 0}\dfrac{(1+x)^a-1}{x}=a$.

When computing limits of the form $\lim\limits_{x\to x_0} u(x)^{v(x)}$, where $\lim\limits_{x\to x_0} u(x)=1$, $\lim\limits_{x\to x_0} v(x)=\infty$, remarkable limit (2) is used.

Example 3. Compute $\lim\limits_{x\to\infty}\left(\dfrac{x}{2+x}\right)^{3x}$.

We have

$$\left(\frac{x}{2+x}\right)^{3x}=\left(1+\frac{-2}{2+x}\right)^{3x}=\left(1+\frac{-2}{2+x}\right)^{\frac{2+x}{-2}\left(\frac{-2}{2+x}\cdot 3x\right)}.$$

Since

$$\lim_{x\to\infty}\left(1+\frac{-2}{2+x}\right)^{\frac{2+x}{-2}}=\lim_{t=\frac{-2}{2+x}\to 0}(1+t)^{\frac{1}{t}}=e$$

and

$$\lim_{x\to\infty}\frac{-2}{2+x}\cdot 3x=-6,$$

we obtain

$$\lim_{x\to\infty}\left(\frac{x}{2+x}\right)^{3x}=e^{-6}$$

(here the continuity of a composition of continuous functions has been used).

Using remarkable limit (2) and the results obtained in Problems 4.53 to 4.57, compute the following limits (Problems 4.59 to 4.68):

4.59. $\lim\limits_{x\to\infty}\left(\frac{x+3}{x-2}\right)^{2x+1}$. **4.60.** $\lim\limits_{x\to\infty}\left(\frac{x^2+5}{x^2-5}\right)^{x^2}$.

4.61. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$. **4.62.** $\lim\limits_{x\to 0}(1+\tan^2\sqrt{x})^{\frac{3}{x}}$.

4.63. $\lim\limits_{x\to\infty}x(\ln(2+x)-\ln x)$.

4.64. $\lim\limits_{x\to 0}\frac{1}{x}\ln\sqrt{\frac{1+x}{1-x}}$.

4.65. $\lim\limits_{x\to\infty}x(a^{\frac{1}{x}}-1)$. **4.66.** $\lim\limits_{x\to 1}\frac{a^x-a}{x-1}$.

4.67. $\lim\limits_{x\to a}\frac{\log_a x-1}{x-a}$. **4.68.** $\lim\limits_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$.

4.69. Prove that $\lim\limits_{x\to x_0}f(x)=a$ if and only if for any sequence of arguments $(x_n)_{n\in\mathbb{N}}$, convergent to x_0, the corresponding sequence $(f(x_n))_{n\in\mathbb{N}}$ of values of the function converges to a.

Using the result obtained in Problem 4.69, prove that for the following functions $\lim\limits_{x\to x_0}f(x)$ is not existent (Problems 4.70 to 4.72):

4.70. $f(x)=\cos x$, $x_0=\infty$. **4.71.** $f(x)=\sin\frac{1}{x}$, $x_0=0$.

4.72. $f(x) = x - [x]$, $x_0 = \infty$.

In Problems 4.73 to 4.80 compute the indicated one-sided limit.

4.73. $\lim\limits_{x \to 3 \pm 0} \dfrac{x-3}{|x-3|}$. **4.74.** $\lim\limits_{x \to 2 \pm 0} \dfrac{2+x}{4-x^2}$.

4.75. $\lim\limits_{x \to 0 \pm 0} (2+x)^{\frac{1}{x}}$. **4.76.** $\lim\limits_{x \to 2 \pm 0} 7^{\frac{1}{2-x}}$.

4.77. $\lim\limits_{x \to \pm \infty} \arctan x$. **4.78.** $\lim\limits_{x \to \pm \infty} \left[\dfrac{1}{x}\right]$.

4.79. $\lim\limits_{x \to \frac{\pi}{4} \pm 0} \dfrac{|\tan(4x-\pi)|}{2x-\frac{\pi}{2}}$. **4.80.** $\lim\limits_{x \to 2\pi \pm 0} \dfrac{x^2}{\cos x - 1}$.

4.81. Prove that the limit of the function $y = f(x)$ at the point x_0 exists if and only if there exist at this point the left-hand and right-hand limits and they coincide.

2. Infinitely Small and Infinitely Large Quantities. (Infinitesimals and Infinites.) A function $\alpha(x)$ is called *infinitesimal* as $x \to x_0$, if $\lim\limits_{x \to x_0} \alpha(x) = 0$.

Infinitesimals $\alpha(x)$ and $\beta(x)$ are said to be *comparable* if there exists at least one of the limits $\lim\limits_{x \to x_0} \dfrac{\beta(x)}{\alpha(x)}$ or $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)}$.

Let $\alpha(x)$ and $\beta(x)$ be comparable infinitesimals as $x \to x_0$ and let, for the sake of definiteness, there exist $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)} = C$. Then:

(a) If $C \neq 0$, then $\alpha(x)$ and $\beta(x)$ are infinitesimals of the same order. In particular, for $C = 1$ the infinitesimals $\alpha(x)$ and $\beta(x)$ are said to be *equivalent*, written $\alpha \sim \beta$.

(b) If $C = 0$, then $\alpha(x)$ is an infinitesimal of *higher order* than $\beta(x)$; in this case we write $\alpha = o(\beta)$. If there exists a real number $r > 0$ such that $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{(\beta(x))^r} \neq 0$, then $\alpha(x)$ is said to be an infinitesimal of *order* r relative to $\beta(x)$.

A function $\alpha(x)$ is called *infinitely large* as $x \to x_0$ if $\lim\limits_{x \to x_0} \alpha(x) = \infty$. The notion of comparable infinites and their classification is introduced much in the same way as it was done for infinitesimals.

4.82. Prove that if $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)} = C \neq 0$, then there can be found a number $\delta > 0$ and constants C_1 and C_2 such that

$$|x - x_0| < \delta \Rightarrow C_1 \beta(x) \leqslant \alpha(x) \leqslant C_2 \beta(x).$$

4.83. Prove that $\alpha \sim \beta$ if and only if $\alpha - \beta = o(\alpha)$ or $\alpha - \beta = o(\beta)$.

In Problems 4.84 to 4.93 determine the order of smallness of $\alpha(x)$ relative to $\beta(x) = x$ as $x \to 0$.

4.84. $\alpha(x) = \dfrac{3\sqrt{x^3}}{1-x}$. **4.85.** $\alpha(x) = \sqrt[3]{x^2} - \sqrt{x^3}$.

4.86. $\alpha(x) = \dfrac{1-\cos x}{x}$. **4.87.** $\alpha(x) = \tan x - \sin x$.

4.88. $\alpha(x) = \sin(\sqrt{x+2} - \sqrt{2})$.

4.89. $\alpha(x) = 3\sin^3 x - x^4$.

4.90. $\alpha(x) = \sqrt[3]{1+\sqrt[3]{x}} - 1$.

4.91. $\alpha(x) = \sqrt{1+2x} - 1 - \sqrt{x}$.

4.92. $\alpha(x) = 3^{\sqrt{x}} - 1$. **4.93.** $\alpha(x) = 2^x - \cos x$.

4.94. Prove that $\alpha(x) - \beta(x)$ is of the second order of smallness relative to x as $x \to 0$ if:

(a) $\alpha(x) = \dfrac{1}{1+x}$, $\beta(x) = 1 - x$;

(b) $\alpha(x) = \sqrt{a^2 + x}$, $\beta(x) = a + \dfrac{1}{2a}x$ $(a \neq 0)$;

(c) $\alpha(x) = (1+x)^n$, $\beta(x) = 1 + nx$ $(n \in \mathbb{N})$.

In Problems 4.95 to 4.98 compute approximately the given expression.

4.95. $1/1.03$. **4.96.** $\sqrt{25.3}$.

4.97. $(1.03)^5$. **4.98.** $(0.97)^4$.

4.99. Prove that if

$$\alpha(x) \sim \alpha_1(x) \text{ and } \beta(x) \sim \beta_1(x) \text{ as } x \to x_0,$$

then

$$\lim_{x \to x_0} \frac{\alpha(x)}{\beta(x)} = \lim_{x \to x_0} \frac{\alpha_1(x)}{\beta_1(x)}.$$

Using the result obtained in Problem 4.99, compute the following limits (Problems 4.100 to 4.106):

4.100. $\lim\limits_{x \to 0} \dfrac{\arcsin \dfrac{x}{\sqrt{1-x^2}}}{\ln(1-x)}$.

Since $\arcsin \dfrac{x}{\sqrt{1-x^2}} \sim \dfrac{x}{\sqrt{1-x^2}}$ and $\ln(1-x) \sim (-x)$ as $x \to 0$, then

$$\lim_{x\to 0} \frac{\arcsin \dfrac{x}{\sqrt{1-x^2}}}{\ln(1-x)} = \lim_{x\to 0} \frac{\dfrac{x}{\sqrt{1-x^2}}}{-x} = -1.$$

4.101. $\lim\limits_{x\to 1} \dfrac{1-x}{\log x}$. **4.102.** $\lim\limits_{x\to 0} \dfrac{\cos x - \cos 2x}{1-\cos x}$.

4.103. $\lim\limits_{x\to \frac{1}{2}} \dfrac{4x^2-1}{\arcsin(1-2x)}$. **4.104.** $\lim\limits_{x\to 0} \dfrac{\arctan x^2}{\arcsin 3x \cdot \sin \frac{x}{2}}$.

4.105. $\lim\limits_{x\to 0} \dfrac{1-\cos 4x}{2\sin^2 x + x\tan 7x}$.

4.106. $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}$.

In Problems 4.107 to 4.112 determine the order of growth of the infinite $A(x)$ relative to $B(x) = x$ as $x \to \infty$:

4.107. $A(x) = x^3 + 150x + 10$.

4.108. $A(x) = \sqrt{x^2+3x+5} + |x|$.

4.109. $A(x) = \sqrt{x+\sqrt{x}}$. **4.110.** $A(x) = \sqrt[3]{x^2-x} + \sqrt{x}$.

4.111. $A(x) = \dfrac{5x^6}{3x^4+x^3+2}$. **4.112.** $A(x) = \dfrac{\sqrt{x^5}/2}{x^{7/3}+1}$.

3. Continuity of a Function at a Point. Classification of Points of Discontinuity. A function $y = f(x)$ with the domain of definition D is said to be *continuous* at a point x_0 provided the following three conditions are fulfilled:

(a) the function $y = f(x)$ is defined at the point x_0, i.e. $x_0 \in D$;

(b) there exists $\lim\limits_{x\to x_0} f(x)$;

(c) $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

If condition (a) is fulfilled, then conditions (b) and (c) are equivalent to the following:

$$\lim_{\Delta x\to 0} \Delta f(x_0, \Delta x) = 0,$$

where

$$\Delta f(x_0, \Delta x) = f(x_0 + \Delta x) - f(x_0)$$

is the *increment of the function* $y = f(x)$ at the point x_0 corresponding to the *increment of the argument* $\Delta x = x - x_0$.

If at least one of the three conditions is violated at the point x_0, then x_0 is called a *point of discontinuity* (or simply, a *discontinuity*) of the function $y = f(x)$. The following three cases are distinguished:

(a) $\lim\limits_{x \to x_0} f(x)$ exists but the function is not defined at the point x_0 or the condition $\lim\limits_{x \to x_0} f(x) = f(x_0)$ is violated. In this case x_0 is called a *removable discontinuity*.

(b) $\lim\limits_{x \to x_0} f(x)$ does not exist. If both one-sided limits $\lim\limits_{x \to x_0 + 0} f(x)$ and $\lim\limits_{x \to x_0 - 0} f(x)$ exist (obviously not equal to each other), then x_0 is termed a *discontinuity of the first kind*.

(c) In all other cases x_0 is called a *discontinuity of the second kind*.

4.113. Using the logic symbols, write in terms of "ε-δ" the following statements:

(a) The function $y = f(x)$ with the domain D is continuous at the point $x_0 \in D$.

(b) The function $y = f(x)$ is not continuous at the point $x_0 \in D$.

In Problems 4.114 to 4.118 prove that the given function is continuous at each point of its natural domain of definition.

4.114. $f(x) = x^n$, $n \in \mathbb{N}$.

Using the binomial formula, we obtain

$$\Delta f(x_0, \Delta x) = (x_0 + \Delta x)^n - x_0^n = C_n^1 x_0^{n-1} \Delta x + C_n^2 x_0^{n-2} (\Delta x)^2 + \ldots + (\Delta x)^n.$$

Hence $\lim\limits_{\Delta x \to 0} \Delta f(x_0, \Delta x) = 0$.

4.115. $f(x) = a$, $a > 0$.

4.116. $f(x) = \log_a x$; $a > 0$, $a \neq 1$.

4.117. $f(x) = \sin x$. **4.118.** $f(x) = \arcsin x$.

In Problems 4.119 to 4.121 determine at what choice of the parameters entering into its definition the given function $f(x)$ will be continuous.

4.119. $f(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1}, & x \neq 1, \\ A, & x = 1. \end{cases}$

4.120. $f(x) = \begin{cases} x - 1, & x \leqslant 1, \\ ax^2 - 2, & x > 1. \end{cases}$

4.121. $f(x)=\begin{cases} ax+1, & x\leqslant \pi/2, \\ \sin x+b, & x>\pi/2. \end{cases}$

In Problems 4.122 to 4.137 find the points of discontinuity of the given function, investigate their nature, and if a discontinuity is removable, redefine the given function so as to make it continuous.

4.122. $f(x)=\dfrac{1}{x^2(x-1)}$. **4.123.** $f(x)=\dfrac{|3x-5|}{3x-5}$.

4.124. $f(x)=\dfrac{(1+x)^n-1}{x}$, $n\in\mathbb{N}$. **4.125.** $f(x)=\dfrac{1}{x}\sin x$.

4.126. $f(x)=1-x\sin\dfrac{1}{x}$. **4.127.** $f(x)=3^{\frac{x}{4-x^2}}$.

4.128. $f(x)=(x+1)\operatorname{arc\,tan}\dfrac{1}{x}$.

4.129. $f(x)=\dfrac{|x+2|}{\operatorname{arc\,tan}(x+2)}$.

4.130. $f(x)=\dfrac{3^{\frac{1}{x-2}}-1}{3^{\frac{1}{x-2}}+1}$. **4.131.** $f(x)=\dfrac{1}{x}\ln\dfrac{1+x}{1-x}$.

4.132. $f(x)=\dfrac{\frac{1}{x}-\frac{1}{x+1}}{\frac{1}{x-1}-\frac{1}{x}}$. **4.133.** $f(x)=\dfrac{1-\cos x}{x^2}$.

4.134. $f(x)=\begin{cases} 2^x, & -1\leqslant x<1, \\ x-1, & 1<x\leqslant 4, \\ 1, & x=1. \end{cases}$

4.135. $f(x)=\dfrac{1}{2^{\frac{1}{1-x}}+1}$.

4.136. $f(x)=\begin{cases} 2\sqrt{x}, & 0\leqslant x\leqslant 1, \\ 4-2x, & 1<x<2.5, \\ 2x-7, & 2.5\leqslant x\leqslant 4. \end{cases}$

4.137. $f(x)=\begin{cases}\cos x, & -\frac{\pi}{2}\leqslant x<\frac{\pi}{4},\\ 1, & x=\frac{\pi}{4},\\ x^2-\frac{\pi^2}{16}, & \frac{\pi}{4}<x\leqslant\pi.\end{cases}$

4.138. Prove that all points of discontinuity of a bounded monotone function are discontinuities of the first kind.

4. Continuity on a Set. Uniform Continuity. A function $y=f(x)$ is said to be *continuous on a set* D if it is continuous at each point $x\in D$. It is called *uniformly continuous* on D if for any $\varepsilon>0$ there exists a number $\delta(\varepsilon)>0$ such that for any x', $x''\in D$ the inequality $|x'-x''|<\delta(\varepsilon)$ implies $|f(x')-f(x'')|<\varepsilon$.

Cantor's theorem. *If a function $y=f(x)$ is continuous on the interval $[a, b]$, then it is uniformly continuous on this interval.*

4.139. Prove that if $y=f(x)$ is a function continuous on $[a, b]$, then

(a) it is bounded on $[a, b]$;

(b) reaches on $[a, b]$ its upper and lower bounds (*Weierstrass' theorem*);

(c) attains on any interval $(a', b')\subset[a, b]$ all intermediate values between $f(a')$ and $f(b')$ (*Cauchy's theorem*).

4.140. Prove that if a function $y=f(x)$ is continuous on $[a, \infty)$ and there exists a finite $\lim\limits_{x\to\infty} f(x)$, then this function is bounded on $[a, \infty)$.

4.141. Show that the function

$$f(x)=\begin{cases}\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0\end{cases}$$

attains on any interval $[0, a]$ all the intermediate values between $f(0)$ and $f(a)$, but is not continuous on $[0, a]$.

4.142. Prove that any polynomial of an odd degree has at least one real root.

4.143. Formulate the following statement in terms of "ε-δ": the function $y=f(x)$ is continuous on the set D, but is not uniformly continuous on this set. As examples, consider the following functions:

(a) $f(x)=\frac{1}{x}$, $D=(0, 1]$;

(b) $f(x)=\log x$, $D=(0, 10]$;

(c) $f(x)=\sin\frac{\pi}{x}$, $D=(0, 1]$.

4.144. Prove that if a function $y = f(x)$ is continuous on $[a, +\infty)$ and there exists a finite $\lim\limits_{x \to +\infty} f(x)$, then this function is uniformly continuous on $[a, +\infty)$.

4.145. Show that the unbounded function $f(x) = x + \sin x$ is uniformly continuous throughout the entire axis $-\infty < x < +\infty$.

In Problems 4.146 to 4.152 investigate the given function for uniform continuity on the indicated set.

4.146. $f(x) = \frac{x}{4-x^2}$, $D = [-1, 1]$.

4.147. $f(x) = \ln x$, $D = (0, 1]$.

4.148. $f(x) = \frac{\sin x}{x}$, $D = (0, \pi]$.

4.149. $f(x) = e^x \cos \frac{1}{x}$, $D = (0, 1]$.

4.150. $f(x) = \arctan x$, $D = \mathbb{R}$.

4.151. $f(x) = \sqrt{x}$, $D = [1, +\infty)$.

4.152. $f(x) = x \sin x$, $D = [0, +\infty)$.

SEC. 1.5.
COMPLEX NUMBERS

1. Algebraic Operations on Complex Numbers. *Complex numbers* are defined as all possible ordered pairs $z = (x, y)$ of real numbers for which the operations of addition and multiplication are defined in the following way:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{1}$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \tag{2}$$

The set of all complex numbers is denoted by the symbol $\mathbb{C}$.

The real numbers x and y are called the *real* and *imaginary parts* of the complex number $z = (x, y)$ and are denoted by the symbols $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively.

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are said to be *equal* if and only if $x_1 = x_2$ and $y_1 = y_2$.

From definitions (1) and (2) it follows that any complex number (x, y) can be written as follows:

$$(x, y) = (x, 0) + (0, 1)(y, 0). \tag{3}$$

If now complex numbers of the form $(x, 0)$ are identified* with real numbers x, and the number $(0, 1)$ is denoted by the symbol i, then equality (3) takes the form

$$z = x + iy,$$

the latter being called the *algebraic form* of the complex number $z = (x, y)$.

5.1. Prove that the operations of addition and multiplication of complex numbers possess the following properties:

(a) $z_1 + z_2 = z_2 + z_1$ (*commutativity of addition*);

(b) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (*associativity of addition*);

(c) $z_1 z_2 = z_2 z_1$ (*commutativity of multiplication*);

(d) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (*associativity of multiplication*);

(e) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ (*distributive law*).

5.2. Prove that:

(a) $\forall\, z_1,\ z_2 \neq 0\ \exists\, z\ (z_2 z = z_1)$

(the number z is called the *quotient* and is obtained by the division of z_1 by z_2; it is denoted as $\frac{z_1}{z_2}$);

b) if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}.$$

In Problems 5.3 to 5.12 perform the indicated operations, presenting the result in the algebraic form.

5.3. $(1 - 2i)(2 + i)^2 + 5i$.

The task is to present the given complex number in the form

$$(1 - 2i)(2 + i)^2 + 5i = x + iy.$$

To this end, we may take advantage directly of formulas (1) and (2), but the same result can be obtained in a simpler way. As is shown by the properties listed in Problem 5.1, when adding and multiplying complex numbers represented in the algebraic form, we may treat them as binomials of the form $a + ib$, bearing in mind that $i^2 =$

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* That is, a one-to-one correspondence $(x, 0) \leftrightarrow x$ is established between the sets $\{(x, 0) \mid x \in \mathbb{R}\}$ and $\mathbb{R}$. It follows from (1) and (2) that this correspondence "retains the operations":

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \leftrightarrow x_1 + x_2,$$

$$(x_1, 0) \cdot (x_2, 0) = (x_1 x_2, 0) \leftrightarrow x_1 x_2.$$

$= (0, 1)(0, 1) = (-1, 0) = -1$. Therefore

$$(2 + i)^2 = 4 + 4i + i^2 = 3 + 4i,$$

$$(1 - 2i)(2 + i)^2 = (1 - 2i)(3 + 4i) = 3 - 2i - 8i^2 = 11 - 2i,$$

whence we finally obtain

$$(1 - 2i)(2 + i)^2 + 5i = 11 - 2i + 5i = 11 + 3i.$$

5.4. $(2 + 3i)(3 - i)$. **5.5.** $(1 + 2i)^2$.

5.6. $(1 - i)^3 - (1 + i)^3$. **5.7.** $(2i - i^2)^2 + (1 - 3i)^3$.

5.8. $\dfrac{2-i}{1+i}$.

The required result can be obtained directly by the formula from Problem 5.2. We should like to notice, however, that $(1 + i)(1 - i) = 2$ is a real number. Therefore, multiplying both the numerator and denominator of the given fraction by $1 - i$, we find

$$\frac{2-i}{1+i} = \frac{(2-i)(1-i)}{(1+i)(1-i)} = \frac{1-3i}{2} = \frac{1}{2} - \frac{3}{2}i.$$

5.9. $\dfrac{1}{1+4i} + \dfrac{1}{4-i}$. **5.10.** $\left(\dfrac{1-i}{1+i}\right)^3$.

5.11. $\dfrac{(1+i)(3+i)}{3-i} - \dfrac{(1-i)(3-i)}{3+i}$. **5.12.** $\left(\dfrac{i^5+2}{i^{19}+1}\right)^2$.

In Problems 5.13 and 5.14 find the real solutions of the given equation:

5.13. $(1 + i)x + (-2 + 5i)y = -4 + 17i$.

5.14. $12((2x + i)(1 + i) + (x + y)(3 - 2i)) = 17 + 6i$.

In Problems 5.15 and 5.16 solve the given systems of linear equations:

5.15.
$$(3 - i)z_1 + (4 + 2i)z_2 = 1 + 3i,$$
$$(4 + 2i)z_1 - (2 + 3i)z_2 = 7.$$

5.16.
$$(2 + i)z_1 + (2 - i)z_2 = 6,$$
$$(3 + 2i)z_1 + (3 - 2i)z_2 = 8.$$

If a rectangular Cartesian system (Oxy) of coordinates is specified in the plane, then any complex number $z = x + iy$ can be associated with some point $M(x, y)$ whose abscissa is x and ordinate is y.

The point $M(x, y)$ is said *to represent* the complex number $z = x + iy$. The plane on which complex numbers are represented is called the *complex plane*, the *x-axis* is termed the *real axis* and the *y-axis* the *imaginary axis*.

The number $r = \sqrt{x^2 + y^2}$ is called the *modulus* of the complex number $z = x + iy$ and is denoted $|z|$. The modulus of the number z

is equal to the distance of the point M, representing this number, from the origin.

Any solution φ of the system of equations

$$\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\varphi = \frac{y}{\sqrt{x^2+y^2}} \tag{4}$$

is called the *argument* of the complex number $z = x + iy \neq 0$. All arguments of the number z are distinguished as integers multiple of 2π and are denoted as Arg z. Each value of the argument coincides with the magnitude φ of some angle through which the x-axis should be turned to be brought into coincidence with the radius vector $\overrightarrow{OM}$ of the point M ($\varphi > 0$ if the rotation is accomplished anticlockwise and $\varphi < 0$ if otherwise). The value of Arg z satisfying the condition $0 \leqslant \text{Arg } z < 2\pi$, is called the *principal value* of the argument and is denoted as arg z.

In some cases the principal value of the argument is understood as the value of the Arg z satisfying the condition $-\pi < \text{Arg } z \leqslant \pi$.

It follows from (4) that for any complex number z the equality

$$z = |z|(\cos\varphi + i\sin\varphi)$$

holds true. This equality is called the *trigonometric form* of the number z.

Example 1. Represent the complex number $z = -2 + 2i\sqrt{3}$ in the trigonometric form.

We have

$$r = |z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4,$$

$$\cos\varphi = -1/2, \quad \sin\varphi = \sqrt{3}/2,$$

therefore the principal value of the argument is equal to $\arg z = 2\pi/3$ and, consequently, $z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$.

In Problems 5.17 to 5.23 represent the given complex number in the trigonometric form and depict it by a point on the complex plane.

5.17. $-i$. **5.18.** $1 - i\sqrt{3}$.

5.19. $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

5.20. $\frac{1-i}{1+i}$. **5.21*.** $-\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$.

5.22. $\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}$.

5.23. $1 + \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$.

If $z = x + iy$, then $x - iy$ is called the *conjugate* complex number of z and is denoted by $\bar{z}$.

Prove the following equalities (Problems 5.24 to 5.29):

5.24. $z + \bar{z} = 2 \operatorname{Re} z$ and $z - \bar{z} = 2i \operatorname{Im} z$.

5.25. $\overline{(\bar{z})} = z$. **5.26.** $|\bar{z}| = |z|$.

5.27. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

5.28. $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$. **5.29.** $z\bar{z} = |z|^2$.

5.30. Compute:

(a) $z_1\bar{z}_2$ and $\left(\frac{\bar{z}_1}{z_2}\right)^2$, if $z_1 = 1 - i\sqrt{3}$, $z_2 = \sqrt{3} + i$,

(b) $z_1\bar{z}_2$ and $\frac{\bar{z}_1^2}{z_2}$, if $z_1 = 3 + 2i$, $z_2 = 2 + 2i$.

5.31. Let $p(z)$ be an arbitrary polynomial with real coefficients. Prove that for any $z \in \mathbb{C}$ the equality $\overline{p(z)} = p(\bar{z})$ holds true.

In Problems 5.32 and 5.33 solve the given equations.

5.32. $|z| - z = 1 + 2i$. **5.33.** $|z| + z = 2 + i$.

5.34. Prove the equalities given below and clarify their geometrical meaning:

(a) $|z_1 z_2| = |z_1| \cdot |z_2|$ and $\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|$;

(b) $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg}(z_1 z_2)$ and

$$\operatorname{Arg} z_1 - \operatorname{Arg} z_2 = \operatorname{Arg}\left(\frac{z_1}{z_2}\right)$$

(equalities (b) are understood in the sense that the numbers standing in the left-hand sides coincide with some values of the arguments $z_1 z_2$ and $\frac{z_1}{z_2}$).

In Problems 5.35 to 5.42 clarify the geometrical meaning of the indicated transformations of the complex plane:

5.35. $z \to z - 2$. **5.36.** $z \to z + (3 - i)$. **5.37.** $z \to iz$.

5.38. $z \to \frac{\sqrt{2}}{2}(1 - i)z$. **5.39.** $z \to -z$. **5.40.** $z \to 2z$.

5.41. $z \to \frac{z}{1 - i}$. **5.42.** $z \to \bar{z}$.

5.43. Prove that

(a) the quantity $|z_1 - z_2|$ is equal to the distance between the points M_1 and M_2 representing the complex numbers z_1 and z_2 on the complex plane;

(b) $|z_1 + z_2| \leqslant |z_1| + |z_2|$ and $|z_1 - z_2| \geqslant \geqslant ||z_1| - |z_2||$ (the *triangle inequalities*). What is the geometrical meaning of these inequalities?

5.44. Prove the identities:

(a) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ (what is its geometrical meaning?);

(b)* $|z_1| + |z_2| = \left|\frac{z_1+z_2}{2} + \sqrt{z_1 z_2}\right| + \left|\frac{z_1+z_2}{2} - \sqrt{z_1 z_2}\right|$.

In Problems 5.45 to 5.55 give a geometrical description of the set of all points of the complex plane satisfying the indicated condition.

5.45. $\operatorname{Re} z \geqslant 0$. **5.46.** $0 \leqslant \operatorname{Im} z < 1$. **5.47.** $|\operatorname{Im} z| \leqslant 2$.

5.48. $|z| < 1$. **5.49.** $|z + i| = 2$. **5.50.** $1 < |z + 2| \leqslant 2$.

5.51. $|z| > 1 - \operatorname{Re} z$. **5.52.** $|z - i| = |z + 2|$.

5.53. $0 < \arg z \leqslant \pi/4$. **5.54.** $|\pi - \arg z| < \pi/4$.

5.55. $z = \bar{z}$. **5.56.** Let $z \neq -1$. Prove that $\operatorname{Re} \frac{z-1}{z+1} = 0 \Leftrightarrow |z| = 1$.

Let φ be an arbitrary real number. The symbol $e^{i\varphi}$ denotes the complex number $\cos\varphi + i\sin\varphi$. With the aid of this designation any complex number $z = |z|(\cos\varphi + i\sin\varphi)$ can be written in the *exponential form*

$$z = |z| e^{i\varphi}.$$

In Problems 5.57 to 5.62 represent the given complex numbers in the exponential form.

5.57. $\frac{7 + 24i}{5}$. **5.58.** $5 - 12i$. **5.59.** $-3 - 4i$.

5.60. $-2 + i$. **5.61.** $\sin\alpha - i\cos\alpha$.

5.62. $\sin\alpha + i(1 - \cos\alpha)$.

5.63. Prove that the symbol $e^{i\varphi}$ possesses the following properties:

(a) $\forall n \in \mathbb{Z}$ $(e^{i2\pi n} = 1)$; (b) $\overline{e^{i\varphi}} = e^{-i\varphi}$;

(c) $e^{i\varphi_1} \cdot e^{i\varphi_2} = e^{i(\varphi_1+\varphi_2)}$ and $\frac{e^{i\varphi_1}}{e^{i\varphi_2}} = e^{i(\varphi_1-\varphi_2)}$.

5.64. Represent the given numbers z_1 and z_2 in the exponential form and carry out the indicated operations:

(a) $z_1 z_2$, $\frac{z_1^2}{z_2}$ if $z_1 = 2\sqrt{3} - 2i$, $z_2 = 3 - 3\sqrt{3}i$;

(b) $z_1^2 \overline{z_2}$, $\frac{\overline{z_2}}{z_1}$ if $z_1 = -\sqrt{2} + i\sqrt{2}$, $z_2 = \sqrt{8} - i\sqrt{8}$.

5.65. Prove *Euler's formulas*:

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \quad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}.$$

5.66. Prove *De Moivre's formula*: if $z = re^{i\varphi}$, then

$$z^n = r^n e^{in\varphi},$$

or, in the trigonometric form,

$$z^n = r^n (\cos n\varphi + i \sin n\varphi).$$

Using De Moivre's formula, compute the following expressions (Problems 5.67 to 5.70):

5.67. $(1+i)^{10}$. **5.68.** $\frac{(1+i)^5}{(1-i)^3}$. **5.69.** $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$.

5.70. $(1+i)^8 (1 - i\sqrt{3})^{-6}$.

5.71. Prove the given equalities:

(a) $(1+i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} + i \sin\frac{n\pi}{4}\right)$;

(b) $(\sqrt{3} - i)^n = 2^n \left(\cos\frac{n\pi}{6} - i \sin\frac{n\pi}{6}\right)$.

5.72. Using Euler's formulas, express the following functions in terms of cosines and sines of multiple arcs:
(a) $\cos^3\varphi$; (b) $\sin^3\varphi$.

Using De Moivre's formula, express the following functions in terms of $\cos\varphi$ and $\sin\varphi$ (Problems 5.73 to 5.76):

5.73. $\cos 3\varphi$. **5.74.** $\sin 3\varphi$.

5.75. $\cos 4\varphi$. **5.76.** $\sin 4\varphi$.

Let $a = re^{i\varphi}$ be a fixed complex number. Then the equation $z^n = a$, $n \in \mathbb{N}$, has exactly n distinct solutions $z_0, z_1, \ldots, z_{n-1}$, which are

given by the formula

$$z_k = \sqrt[n]{r}\, e^{i\left(\frac{\varphi}{n}+\frac{2\pi}{n}k\right)} = \sqrt[n]{r}\left(\cos\frac{\varphi+2\pi k}{n} + i\sin\frac{\varphi+2\pi k}{n}\right),$$
$$k = 0, 1, \ldots, n-1$$

(here $\sqrt[n]{r}$ is a positive real number). The numbers z_k, $k = 0, 1, \ldots, n-1$ are called the nth *roots* of the complex number a and are denoted by the symbol $\sqrt[n]{a}$.

Example 2. Find all cube roots of the number $a = -2 + 2i\sqrt{3}$.

Since $a = 4e^{i\frac{2\pi}{3}} = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$, then

$$(\sqrt[3]{a})_k = \sqrt[3]{4}\, e^{i\left(\frac{2\pi}{9}+\frac{2\pi k}{3}\right)}$$
$$= \sqrt[3]{4}\left(\cos\left(\frac{2\pi}{9}+\frac{2\pi}{3}k\right) + i\sin\left(\frac{2\pi}{9}+\frac{2\pi}{3}k\right)\right),$$

where $k = 0, 1, 2$.

For $k=0$: $(\sqrt[3]{a})_0 = \sqrt[3]{4}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)$.

For $k=1$: $(\sqrt[3]{a})_1 = \sqrt[3]{4}\left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right)$.

For $k=2$: $(\sqrt[3]{a})_2 = \sqrt[3]{4}\left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right)$.

5.77. Find and represent on the complex plane all square, cube and fourth roots of unity.

In Problems 5.78 to 5.85 find all values of the indicated radical.

5.78. $\sqrt{i}$. **5.79.** $\sqrt[4]{-1}$. **5.80.** $\sqrt[9]{-9}$.

5.81. $\sqrt{-1+i\sqrt{3}}$. **5.82.** $\sqrt[4]{2\sqrt{3}+2i}$.

5.83. $\sqrt{-1-i}$. **5.84.** $\sqrt{1+i\sqrt{3}}$.

5.85. $\sqrt{(2-2i)^4}$.

5.86. Prove that the square roots of a complex number can be found by the formula

$$\sqrt{z} = \sqrt{x+iy} = \pm\left(\sqrt{\frac{|z|+x}{2}} + i\,\operatorname{sgn} y\sqrt{\frac{|z|-x}{2}}\right).$$

In many cases the use of the exponential form of complex numbers simplifies considerably the computations involved.

Example 3. Compute the sum

$$S(\varphi) = \sin\varphi + \sin 2\varphi + \ldots + \sin n\varphi, \quad \varphi \neq 2\pi m, \quad m \in \mathbb{Z}.$$

Since $\sin\varphi = \operatorname{Im} e^{i\varphi}$, then, using the formula for the sum of a geometrical progression, we obtain

$$S(\varphi) = \operatorname{Im} e^{i\varphi} + \operatorname{Im} e^{i2\varphi} + \ldots + \operatorname{Im} e^{in\varphi}$$
$$= \operatorname{Im}\left(e^{i\varphi} + e^{i2\varphi} + \ldots + e^{in\varphi}\right)$$
$$= \operatorname{Im}\frac{e^{i\varphi}(1-e^{in\varphi})}{1-e^{i\varphi}} = \operatorname{Im}\frac{e^{i\varphi+i\frac{n}{2}\varphi}\left(e^{-i\frac{n}{2}\varphi} - e^{i\frac{n}{2}\varphi}\right)}{e^{i\frac{\varphi}{2}}\left(e^{-i\frac{\varphi}{2}} - e^{i\frac{\varphi}{2}}\right)}$$
$$= \frac{\sin\frac{n\varphi}{2}}{\sin\frac{\varphi}{2}} \operatorname{Im} e^{i\frac{n+1}{2}\varphi} = \frac{\sin\frac{n\varphi}{2}\sin\frac{n+1}{2}\varphi}{\sin\frac{\varphi}{2}}.$$

In Problems 5.87 to 5.89 compute the indicated sums.

5.87. $\cos\varphi + \cos 2\varphi + \cos 3\varphi + \ldots + \cos n\varphi$.

5.88. $\cos\varphi + \cos 3\varphi + \cos 5\varphi + \ldots + \cos(2n-1)\varphi$.

5.89. $\sin\varphi + \sin 3\varphi + \sin 5\varphi + \ldots + \sin(2n-1)\varphi$.

2. Polynomials and Algebraic Equations. *A polynomial* (or *an entire rational function*) of degree n is defined as a function of the form

$$p_n(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \tag{5}$$

where $z \in \mathbb{C}$, $a_0, a_1, \ldots, a_n$ are coefficients (generally speaking, complex ones) and $a_n \neq 0$, $n \in \mathbb{N}$. The equation

$$a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0, \quad a_n \neq 0, \tag{6}$$

is called an *algebraic equation* of the nth degree. A number z_0, for which $p_n(z_0) = 0$, is called the *root* of polynomial (5) or equation (6).

Gauss' theorem (the fundamental theorem of algebra). *Any polynomial of nonzero degree has at least one root* (generally speaking, a complex one).

The number z_0 is a root of the polynomial $p_n(z)$ if and only if $p_n(z)$ is divisible (without a remainder) by the binomial $z - z_0$, i.e.

$$p_n(z) = (z - z_0)\, q_{n-1}(z),$$

where $q_{n-1}(z)$ is a polynomial of degree $n-1$. If $p_n(z)$ is divisible (without a remainder) by $(z-z_0)^k$, $k \geqslant 1$, but is not divisible by $(z-z_0)^{k+1}$, then z_0 is said to be a *root of the order of multiplicity k*

of the polynomial $p_n(z)$, and

$$p_n(z) = (z - z_0)^k q_{n-k}(z),$$

where $q_{n-k}(z_0) \neq 0$.

Gauss' theorem can be stated in a more precise way: *a polynomial of degree n has exactly n roots if each root is counted the number of times equal to the order of its multiplicity.*

If the coefficients of polynomial (5) are real numbers and $z_0 = x_0 + iy_0$ is its complex root, then the conjugate number $\bar{z}_0 = x_0 - iy_0$ is also a root of this polynomial, both roots having the same order of multiplicity (see Problem 5.31).

Let the polynomial $p_n(z)$ have the roots $z_1, z_2, \ldots, z_m$ $(m \leqslant n)$ whose orders of multiplicity are, respectively, $k_1, k_2, \ldots, k_m$ $(k_1 + k_2 + \ldots + k_m = n)$. Then it can be presented in the form of linear factors, i.e. the following identity holds true

$$p_n(z) = a_n (z - z_1)^{k_1} (z - z_2)^{k_2} \ldots (z - z_m)^{k_m}.$$

And if the coefficients of this polynomial are real numbers, then, joining the parentheses corresponding to conjugate complex roots, this polynomial can be presented as a product of linear and quadratic factors with real coefficients.

Example 4. Find the roots of the polynomial $z^6 + 2z^3 + 1$ and factor it.

Since $z^6 + 2z^3 + 1 = (z^3 + 1)^2$, the roots of this polynomial are cube roots of -1:

$$z_1 = -1;$$

$$z_2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2};$$

$$z_3 = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Each of these roots has the order of multiplicity $k = 2$. Linear factorization of this polynomial has the form

$$z^6 + 2z^3 + 1 = (z+1)^2 \left(z - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)^2 \left(z - \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)^2.$$

Joining the last two parentheses into one cofactor, we obtain a factorization with real coefficients

$$z^6 + 2z^3 + 1 = (z + 1)^2 (z^2 - z + 1)^2.$$

Solve the following quadratic equations (Problems 5.90 to 5.93):

5.90. $z^2 + 2z + 5 = 0$. **5.91.** $4z^2 - 2z + 1 = 0$.

5.92. $z^2 + (5 - 2i)z + 5(1 - i) = 0$.

5.93. $z^2 + (2i - 3)z + 5 - i = 0$.

Solve the following binomial equations (Problems 5.94 to 5.97):

5.94. $z^3 - 1 = 0$. **5.95.** $z^3 + 1 = 0$.

5.96. $(z+1)^4 - 16 = 0$. **5.97.** $(z+1)^4 + 16 = 0$.

Solve the following biquadratic equations (Problems 5.98 to 5.101):

5.98. $z^4 + 18z^2 + 81 = 0$. **5.99.** $z^4 + 4z^2 + 3 = 0$.

5.100. $z^4 + 9z^2 + 20 = 0$.

5.101. $z^4 - (1+i)z^2 + 2(1+i) = 0$.

Solve the following trinomial equations (Problems 5.102 and 5.103):

5.102. $z^6 + 4z^3 + 3 = 0$. **5.103.** $z^8 + 15z^4 - 16 = 0$.

5.104*. Show that all the roots of the equation

$$\left(\frac{1+ix}{1-ix}\right)^n = \frac{(1+ia)}{(1-ia)} \quad (a \in \mathbb{R})$$

are real and distinct.

In Problems 5.105 to 5.110 present the given polynomials in the form of linear and quadratic factors with real coefficients:

5.105. $z^4 - 1$. **5.106.** $z^4 + 1$. **5.107.** $z^4 + z^2 + 1$.

5.108. $z^4 + 4z^3 + 11z^2 + 14z + 10$; known is one root: $z_1 = -1 + i$.

5.109. $z^5 + z^4 + z^3 - z^2 - z - 1$; known is the double root: $z_1 = z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

5.110. $z^4 + 6z^3 + 9z^2 + 100$; known is the root: $z_1 = 1 + 2i$.

3. Limit of a Sequence of Complex Numbers. A number a is called the *limit* of a sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ (written $\lim\limits_{n\to\infty} z_n = a$) if for any $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that for $n > N(\varepsilon)$ the inequality $|z_n - a| < \varepsilon$ is fulfilled.

A sequence $(z_n)_{n\in\mathbb{N}}$ is said to be *convergent to infinity* (written $\lim\limits_{n\to\infty} z_n = \infty$) if for any $E > 0$ there exists a number $N(E)$ such that for $n > N(E)$ the inequality $|z_n| > E$ is fulfilled.

5.111. Let $x_n = \operatorname{Re} z_n$ and $y_n = \operatorname{Im} z_n$. Prove that $\lim\limits_{n\to\infty} z_n = a \neq \infty$ if and only if $\lim\limits_{n\to\infty} x_n = \operatorname{Re} a$ and $\lim\limits_{n\to\infty} y_n = \operatorname{Im} a$.

5.112. Let $\lim\limits_{n\to\infty} z_n = a \neq \infty$ and $\lim\limits_{n\to\infty} w_n = b \neq \infty$. Prove that:

(a) $\lim\limits_{n\to\infty} (z_n + w_n) = a + b$; (b) $\lim\limits_{n\to\infty} z_n w_n = ab$.

5.113. Let $\lim\limits_{n\to\infty} z_n = a \neq \infty$ and $\lim\limits_{n\to\infty} w_n = b \neq 0$. Prove that $\lim\limits_{n\to\infty} \frac{z_n}{w_n} = \frac{a}{b}$.

In Problems 5.114 to 5.126 compute the given limits:

5.114. $\lim\limits_{n\to\infty} \left(2 - i + \frac{1}{n}(1+i)\right)$. **5.115.** $\lim\limits_{n\to\infty} \frac{1-in}{1+in}$.

5.116. $\lim\limits_{n\to\infty} \frac{5n}{3n+2} \cdot \frac{in^2}{n^2+n-4i}$. **5.117.** $\lim\limits_{n\to\infty} \frac{(n+2i)(3+7in)}{(2-i)n^2+1}$.

5.118. $\lim\limits_{n\to\infty} \left(1 + \frac{i^n}{n}\right)$. **5.119.** $\lim\limits_{n\to\infty} \frac{1}{n} e^{in\frac{\pi}{4}}$.

5.120. $\lim\limits_{n\to\infty} (2i)^n$. **5.121.** $\lim\limits_{n\to\infty} \left(2^n + i\left(1 - \frac{1}{n}\right)\right)$.

5.122. $\lim\limits_{n\to\infty} \left(\frac{1}{5i} - \frac{1}{25} + \ldots + \frac{1}{(5i)^n}\right)$.

5.123. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{i^k}{3^k}$.

5.124. $\lim\limits_{n\to\infty} \left(n \sin \frac{1}{n} + i\left(1 + \frac{1}{n}\right)^n\right)$.

5.125. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{1}{(2-3i)^k}$. **5.126.** $\lim\limits_{n\to\infty} \left(1 + \frac{3+2i}{n}\right)^n$.

Prove that the following sequences are bounded, but are divergent (Problems 5.127 to 5.130):

5.127. $z_n = i^n$. **5.128.** $z_n = (-1)^n + i\frac{2-n}{2+n}$.

5.129. $z_n = \left(1 + \frac{1}{n}\right) e^{i\frac{n\pi}{2}}$. **5.130.** $z_n = \frac{1}{2}\left(i^n + (-i)^n\right)$.

Show that the following sequences are unbounded, but are not convergent to infinity (Problems 5.131 and 5.132):

5.131. $z_n = n(1 + i^n)$. **5.132.** $z_n = \left(e^{i\frac{\pi n}{2}} - i\right) \ln n$.

5.133. Let $r_n = |z_n|$ and $\varphi_n = \arg z_n$. Prove that $\lim\limits_{n\to\infty} z_n = a$ $(0 < |a| < \infty)$ if and only if $\lim\limits_{n\to\infty} r_n = |a|$ and $\lim\limits_{n\to\infty} \varphi_n = \arg \varphi$ (for a properly chosen domain of principal values of the arguments).

The result obtained in Problem 5.133 is frequently used for computing limits of complex sequences.

Example 5. Let φ be a real number $(\varphi \neq 0)$. Prove that

$$\lim_{n\to\infty} \left(1 + \frac{i\varphi}{n}\right)^n = \cos\varphi + i \sin\varphi = e^{i\varphi}.$$

Consider two real sequences:

$$r_n = \left|\left(1 + \frac{i\varphi}{n}\right)^n\right| = \left(1 + \frac{\varphi^2}{n^2}\right)^{n/2},$$

$$\varphi_n = \arg\left(1 + \frac{i\varphi}{n}\right)^n = n \arg\left(1 + \frac{i\varphi}{n}\right) = n \operatorname{arc\,tan} \frac{\varphi}{n}.$$

Let us evaluate their limits

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \left(\left(1 + \frac{\varphi^2}{n^2}\right)^{\frac{n^2}{\varphi^2}}\right)^{\frac{\varphi^2}{2n}} = e^{\lim\limits_{n\to\infty} \frac{\varphi^2}{2n}} = 1,$$

$$\lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} n \operatorname{arc\,tan} \frac{\varphi}{n} = \varphi \lim_{n\to\infty} \frac{\operatorname{arc\,tan} \frac{\varphi}{n}}{\frac{\varphi}{n}} = \varphi.$$

Hence we obtain

$$\lim_{n\to\infty} \left(1 + \frac{i\varphi}{n}\right)^n = 1 \cdot (\cos\varphi + i \sin\varphi) = e^{i\varphi}.$$

5.134. Let $z = x + iy$. Prove (see Example 5) that

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^x (\cos y + i \sin y) = e^{x+iy} = e^z.$$

In Problems 5.135 to 5.138 prove that the following sequences are convergent and find their limits.

5.135. $z_n = z^n$, $|z| < 1$. **5.136.** $z_n = nz^n$, $|z| < 1$.

5.137. $z_n = 1 + z + \ldots + z^n$, $|z| < 1$.

5.138. $z_n = \dfrac{z^n}{1 + z^{2n}}$, $|z| > 1$.

5.139. Compute

$$\lim_{n\to\infty}\frac{1+z_1+\ldots+z_1^n}{1+z_2+\ldots+z_2^n},$$

if $|z_1|<1$ and $|z_2|<1$.

5.140. Let $\lim\limits_{n\to\infty} z_n = a \neq \infty$. Prove that

$$\lim_{n\to\infty}\frac{z_1+z_2+\ldots+z_n}{n}=a.$$

Answers

1.1. Approximation with deficit: 0.1; 0.10; 0.101. Approximation with excess: 0.2; 0.11; 0.102. **1.2.** (a) $\frac{11}{9}$; (b) $\frac{901}{300}$; (c) $\frac{2183}{19\,800}$. **1.11.** $\log_{1/2}\frac{1}{3}>\log_{1/3}\frac{1}{2}$. **1.12.** $\left(\frac{1}{5}\right)^{\log\frac{1}{7}}=\left(\frac{1}{7}\right)^{\log\frac{1}{5}}$. **1.13.** $\log_{\log_3 2}\frac{1}{2}>1$. **1.18.** $\left\{\frac{7}{6},\frac{3}{2}\right\}$. **1.19.** $\{-1, 0\}$. **1.20.** $\varnothing$. **1.21.** $\{0, 2\}$. **1.22.** $(-\infty, 2]$. **1.23.** $(-\infty, 1]\cup[3, \infty)$. **1.24.** $(3, 4)$. **1.25.** $[1-\sqrt{17}, -1+\sqrt{5}]$. **1.26.** $\left(-\infty, \frac{5-\sqrt{13}}{2}\right)\cup\left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$. **1.27.** $(-\infty, -1]$. **1.28.** (a) $\{1, 2\}\subset\{1, 2, \{1, 2, 3\}\}$; (b) both notations are correct. **1.29.** $A=\{0, 1, 2\}$. **1.30.** $A=\{1\}$. **1.31.** $A=\{1, 2, 3, 4\}$. **1.32.** $A=\{-2, -1, 0, 1, 2\}$. **1.33.** $A=\{1,2, 3\}$. **1.34.** $A=\{\pi/2, \pi, 3\pi/2, 2\pi\}$. **1.35.** See Fig. 3. **1.36.** See Fig. 4 (the boundary of the hatched domain does not belong to the set). **1.37.** See Fig. 5. **1.38.** See Fig. 6 (the dashed line does not belong to the set). **1.39.** See Fig. 7 (the boundary of the hatched domain does not belong to the set). **1.40.** Point (2, 2). **1.41.** See Fig. 8. **1.42.** See Fig. 9 (the boundary of the hatched domain does not belong to the set). **1.43.** $A\cup B=\{-5, 3, 4\}$; $A\cap B=\{4\}$; $A\setminus B=\{-5\}$; $B\setminus A=\{3\}$. **1.44.** $\{2, 4, 8\}$. **1.45.** $\{8k \mid k\in\mathbb{Z}\}$. **1.46.** $\{1, 2, 4\}$. **1.47.** $\{24k \mid k\in\mathbb{N}\}$. **1.49.** $A\cup B=(-1, 4)$; $A\cap B=[1, 2]$; $A\setminus B=(-1, 1)$; $B\setminus A=(1, 4)$. **1.50.** $(0, 1)$. **1.51.** $[0, 1/4]\cup[1/2, 1]$. **1.52.** $\{0\}\cup(1/2, 1)$. **1.53.** $[0, 1/4)\cup(1/4, 3/4)\cup\{1\}$. **1.60.** $\mathbb{Z}$; $\{-1, 0, 1\}$. **1.61.** $\{n\in\mathbb{N} \mid n\neq 3k, k\in\mathbb{N}\}$; $\varnothing$. **1.62.** $\{x\in\mathbb{R} \mid x=1/n, n\in\mathbb{N}\}$, $\{1\}$. **1.63.** (a) All points of the given circle; $\varnothing$; (b) all points of the annulus formed by the given circle and a concentric circle of radius equal to half the radius of the given

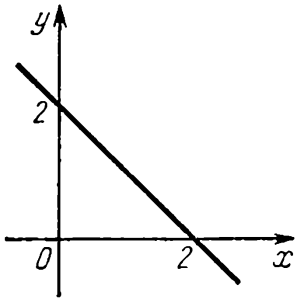

Fig. 3.

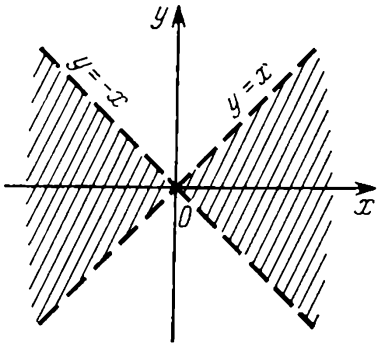

Fig. 4.

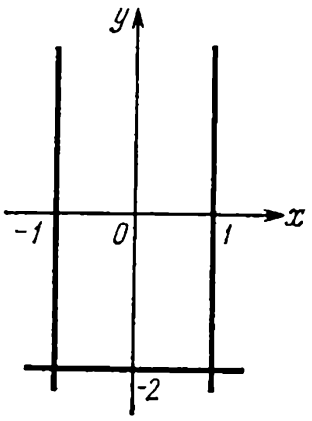

Fig. 5.

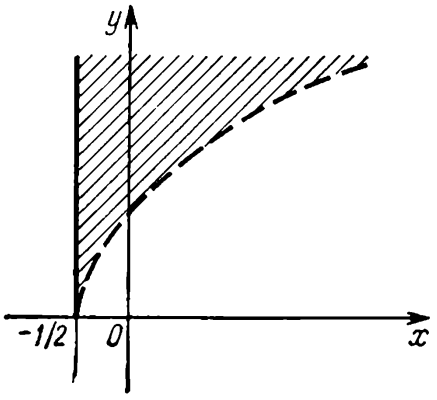

Fig. 6.

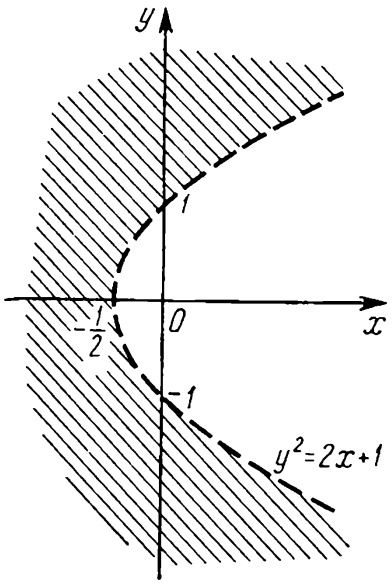

Fig. 7.

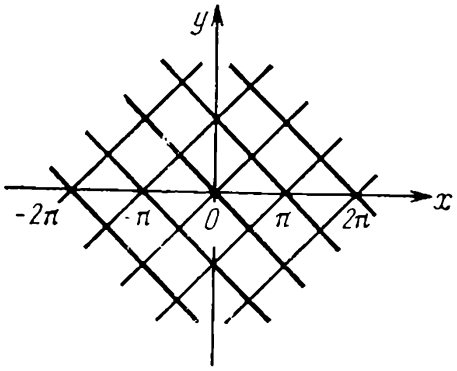

Fig. 8.

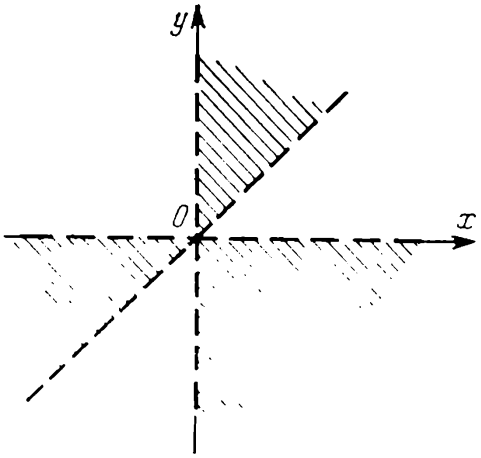

Fig. 9.

circle; $\varnothing$; (c) all points of the circle; centre of the circle. **1.73.** (a) $\min X$ does not exist; $\max X = 1$; (b) $[1, +\infty)$; $(-\infty, 0]$; $\sup X = 1$; $\inf X = 0$. **1.74.** 1/2; not existent; 1/2; 0. **1.75.** 1; —1; 1; —1. **1.76.** Not existent; —5; 0; —5. **1.77.** Not existent; not existent; 0; not existent. **1.78.** Not existent; not existent; 1; 0. **1.79.** $\sup X = \sqrt{2}$. **1.83.** (a) True; (b) false; (c) true; (d) false. **1.84.** False. **1.85.** True. **1.86.** False. **1.87.** True. **1.88.** True. **1.89.** False. **1.90.** (a) True; (b) true. **1.91.** (a) True; (b) true; (c) false. **1.92.** (a) $f(x_0) = 0$; $f(x_0) \neq 0$. (b) $f(x_0) = 0 \wedge \forall x\ (x \neq x_0 \Rightarrow f(x_0) \neq 0)$; $f(x_0) \neq 0 \vee (f(x_0) = 0 \wedge \exists x\ (x \neq x_0 \wedge f(x) = 0))$. (c) $\exists x_0\ (f(x_0) = 0) \wedge (\forall x\ (x \neq x_0 \Rightarrow f(x) \neq 0))$; $\forall x\ (f(x) \neq 0) \vee (\exists x_1, x_2\ (x_1 \neq x_2 \wedge f(x_1) = f(x_2) = 0))$. **1.93.** (a) $\exists M\ \forall x \in X\ (x \leqslant M)$; $\forall M\ \exists x \in X\ (x > M)$. (b) $(m \in X) \wedge (\forall x \in X\ (m \leqslant x))$; $(m \notin X) \vee (\exists x \in X\ (x < m))$. (c) $(\exists m \in X) \wedge (\forall x \in X\ (m \leqslant x))$; $\forall x' \in X \exists x \in X\ (x < x')$. **1.94.** (a) $\exists k \in \mathbb{Z}\ (n = km)$; $\forall k \in \mathbb{Z}\ (n \neq km)$. (b) $(2 \mid n \wedge 3 \mid n) \Rightarrow 6 \mid n$; $(2 \mid n \wedge 3 \mid n) \wedge 6 \nmid n$. (**Note.** Since the original sentence is true, its negation is false.) (c) $\forall n \in \mathbb{N}\ (n \mid p \Rightarrow (n = 1 \vee n = p))$; $\exists n \in \mathbb{N}\ (n \mid p \wedge (n \neq 1 \wedge n \neq p))$.

2.1. $R = \sqrt{\dfrac{1}{\pi H}}$. **2.2.** $V = \dfrac{S^2\sqrt{16\pi^2 - S^2}}{24\pi^2}$.

2.3. $S = \dfrac{3}{4}\tan\alpha$. **2.4.** $v = a(t - t_0)$; $s = \dfrac{a}{2}(t - t_0)^2$; $s = \dfrac{v^2}{2a}$, where $t \geqslant t_0$.

2.5. $$S_{ABNM} = \begin{cases} \dfrac{x^2 h}{a-b}, & 0 \leqslant x \leqslant \dfrac{a-b}{2}, \\ hx + \dfrac{(a-b)h}{4}, & \dfrac{a-b}{2} < x \leqslant \dfrac{a+b}{2}, \\ \dfrac{a+b}{2}h - \dfrac{(a-x)^2 h}{a-b}, & \dfrac{a+b}{2} < x \leqslant a. \end{cases}$$

2.6. $V = \dfrac{1}{4}\pi h(4R^2 - h^2)$, $D = [0, 2R]$. **2.7.** (a) $S = \dfrac{\pi l^2}{2R}\sqrt{4R^2 - l^2}$, $D = [0, 2R]$; (b) $S = 4\pi R^2 \sin\dfrac{\alpha}{2}\cos^2\dfrac{\alpha}{2}$, $D = [0, \pi]$; (c) $S = 4\pi R^2 \cos\beta \sin^2\beta$, $D = [0, \pi/2]$. **2.8.** 0, —6, 4. **2.9.** —1, 0, 1, 2, 4. **2.10.** 0, $a^3 - 1$, $a^3 + 3a^2 + 3a$, $a^3 - 3a^2 + 3a - 2$, $16a^3 - 2$. **2.11.** 1, $\dfrac{1+x}{1-x}$, $-\dfrac{x}{2+x}$, $\dfrac{2}{1+x}$, $\dfrac{x-1}{x+1}$, $\dfrac{1+x}{1-x}$. **2.12.** $D = (-3, \infty)$, $E = (-\infty, +\infty)$. **2.13.** $D = (-\infty, 5/2)$, $E = [0, +\infty)$. **2.14.** $D = \bigcup\limits_{k \in \mathbb{N} \cup \{0\}} [4\pi^2 k^2, \pi^2(2k+1)^2]$, $E = [0, \pi]$. **2.15.** $D = \left[-\dfrac{3}{2},\right.$

$\frac{5}{2}\Big]$, $E=[0,\ \pi]$. **2.16.** $D=\bigcup\limits_{k\in\mathbb{Z}}\left(\frac{2\pi}{3}\left(3k+\frac{1}{2}\right),\ \frac{2\pi}{3}\left(3k+\frac{5}{2}\right)\right)$, $E=[-\infty,\ \ln 3]$. **2.17.** $D=[-1,\ 1]$, $E=[0,\ 1]$. **2.18.** $E=(2,\ 3)$, $E=\left(-\infty,\ \log\frac{1}{4}\right]$. **2.19.** $D=[-1,\ 1]$, $E=[0,\ \pi/4]$. **2.20.** $D=[0,\ 2]$, $E=[1,\ 2\pi]$. **2.21.** $D=(-\infty,\ +\infty)$, $E=[1/e^2,\ +\infty)$. **2.22.** $G=[0,\ 4]$. **2.23.** $G=[1,\ 2]$. **2.24.** $G=(-\infty,\ 0)\cup(1,\ +\infty)$. **2.25.** $G=(0,\ 1/2]$. **2.26.** $G=(1,\ 3]$. **2.27.** $G=[0,\ \sqrt{2}/2)$. **2.28.** $D_0=\{-1\}$, $D_+=(-1,\ +\infty)$, $D_-=(-\infty,\ -1)$. **2.29.** $D_0=\{-1,\ 2\}$, $D_+=(-1,\ 2)$, $D_-=(-\infty,\ -1)\cup(2,\ +\infty)$. **2.30.** $D_0=\left\{x\in\mathbb{R}\mid x=\left|\frac{1}{n},\ n\in\mathbb{Z}\setminus\{0\}\right.\right\}$, $D_+=\bigcup\limits_{n\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{2n},\ \frac{1}{2n+1}\right)$, $D_-=\bigcup\limits_{n\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{2n+1},\ \frac{1}{2(n+1)}\right)$. **2.31.** $D_0=\{1\}$, $D_+=(-\infty,\ 0)\cup(1,\ +\infty)$, $D_-=(0,\ 1)$. **2.37.** $f(x)=x^2-2$. **2.38.** $f(x)=\frac{1+\sqrt{1+x^2}}{x}$. **2.39.** $f(x)=\sin x$. **2.40.** Even. **2.41.** Neither even, nor odd. **2.42.** Neither even, nor odd. **2.43.** Odd. **2.44.** Neither even, nor odd. **2.45.** Odd. **2.47.** $T=2\pi/7$. **2.48.** $T=\pi/2$. **2.49.** Nonperiodic. **2.50.** Nonperiodic. **2.51.** Nonperiodic. **2.52.** $T=6\pi$. **2.53.** If $a=0$, then the inverse does not exist; if $a\neq 0$, then the inverse is $y=\frac{x-b}{a}$ and $D=(-\infty,\ +\infty)$. **2.54.** The inverse is

$$y=\begin{cases}-\sqrt[3]{-x}+1, & \text{if } x\leqslant 0,\\ \sqrt[3]{x}+1, & \text{if } x>0,\end{cases}\qquad D=(-\infty,\ +\infty).$$

2.55. The inverse is not existent. **2.56.** The inverse is $y=\frac{1}{2}e^x$, $D=(-\infty,\ +\infty)$. **2.57.** The inverse is $y=2\log_2 x$, $D=(0,\ +\infty)$. **2.58.** The inverse is $y=\frac{1-x}{1+x}$, $x\neq -1$. **2.60.** (a) $y=-\sqrt{x+1}$, $D=\left[-\frac{3}{4},\ +\infty\right)$; (b) $y=\sqrt{x+1}$, $D=\left[-\frac{3}{4},\ +\infty\right)$. **2.61.** (a) $y=\arcsin x$, $D=[-1,\ 1]$; (b) $y=\pi+\arcsin x$, $D=[-1,\ 1]$. **2.62.** $y=\begin{cases}x, & x\leqslant 0,\\ x/2, & x>0.\end{cases}$ **2.63.** (a) $y=\frac{1}{2}\arccos(2x-1)$, $D=[0,\ 1]$; (b) $y=\pi-\frac{1}{2}\arccos(2x-1)$, $D=$

$=[0, 1]$; (c) $y=\pi+\frac{1}{2}\arccos(2x-1)$, $D=[0, 1]$. **2.65.** $f\circ g=$ $=1-x^2$, $g\circ f=(1-x)^2$. **2.66.** $f\circ g=x$, $x>0$; $g\circ f=x$. **2.67.** $f\circ g=x$, $g\circ f=\begin{cases} x+\pi, & x\in[-\pi, -\pi/2), \\ x, & x\in[-\pi/2, \pi/2], \\ x-\pi, & x\in(\pi/2, \pi]. \end{cases}$ **2.68.** $f\circ g=0$, $g\circ f=g$. **2.69.** (a) x; (b) $x/\sqrt{1+3x^2}$. **2.70.** $f(u)=\sqrt{u}$, $u=x^2$. **2.71.** $f(u)=\sin u$, $u=\cos v$, $v=\sqrt{x}$. **2.72.** $f(u)=2^u$, $u=\sin v$, $v=x^2$. **2.73.** $f(u)=\arcsin u$, $u=e^v$, $v=\sqrt[3]{x}$. **2.74.** $f(u)=\sin u$, $u=2^v$, $v=x^2$. **2.75.** $f(u)=u^{-1/3}$, $u=v^2$, $v=\tan t$, $t=\log_3 x$. **2.77.** $\left\{\left(\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right) \middle| k\in\mathbb{Z}\right\}$. **2.78.** $\{(-1, 2), (1, 2)\}$. **2.79.** $\{(2k\pi, k\pi) \mid k\in\mathbb{Z}\}$. **2.80.** $\{(k\pi, 1) \mid k\in\mathbb{Z}\}$. **2.81.** (a) Straight line passing through the origin and the point (1, 2); (b) straight line parallel to the x-axis passing through the point $(0, -2)$; (c) straight line passing through the point $\left(0, -\frac{1}{3}\right)$ parallel to the bisectors of the second and fourth quadrants. **2.82.** (a) Parabola $y=x^2$ displaced downward along the y-axis by 1; (b) parabola $y=x^2$ stretched along the y-axis to double size and displaced rightward along the x-axis by 1; (c) parabola $y=x^2$ reflected in the x-axis, contracted along the y-axis to half size, displaced leftward along the x-axis by 2, and upward along the y-axis by 3/2. **2.83.** (a) Hyperbola $y=\frac{1}{x}$ displaced downward along the y-axis by 1 and rightward along the x-axis by 1; (b) hyperbola $y=-\frac{1}{x}$ reflected with respect to the x-axis, stretched along the y-axis to a double size, displaced downward along the y-axis by 1/2 and leftward along the x-axis by 1. **2.84.** (a) Sinusoid $y=\sin x$ contracted to half size along the x-axis and displaced leftward along the x-axis by $\pi/6$; (b) sinusoid $y=\sin x$ reflected with respect to the x-axis, elongated along the y-axis to a double size, stretched along the x-axis to a double size, and displaced rightward along the x-axis by $2\pi/3$. **2.85.** (a) Tangent curve $y=\tan x$ stretched threefold along the y-axis, stretched threefold along the x-axis, and displaced leftward along the x-axis by $3\pi/4$; (b) tangent curve $y=\tan x$ reflected with respect to the x-axis, compressed to half size along the y-axis, contracted to half size along the x-axis, and displaced leftward along the x-axis by $3\pi/4$. **2.86.** (a) Graph of the inverse trigonometric function $y=\arcsin x$ stretched fourfold along the y-axis and displaced rightward along the x-axis by 1; (b) graph of the function $y=\arcsin x$ reflected in the x-axis, contracted to 3/2 its size along the y-axis, and displaced leftward along the x-axis by 1/2. **2.87.** (a) Graph of the inverse trigonometric function $y=\arctan x$ reflected with respect to the x-axis, stretched threefold along the y-axis, and displaced leftward along the x-axis

by 5/2; (b) graph of the function $y = \arctan x$ contracted to 5/2 its size along the y-axis and displaced rightward along the x-axis by 6. **2.88.** (a) Graph of the exponential function $y = 2^x$ reflected in the y-axis and displaced rightward along the x-axis by 1; (b) graph of the function $y = 2^x$ reflected in the y-axis, contracted twofold along the x-axis, and displaced rightward along the x-axis by 1. **2.89.** (a) Graph of the logarithmic function $y = \log x$ displaced upward along the y-axis by 1 and rightward along the x-axis by 1/10; (b) graph of the function $y = \log x$ reflected in the x-axis, displaced upward along the y-axis by $\log 2$ and leftward along the x-axis by 4.

2.90*. $y = \begin{cases} -2x, & x \in (-\infty, -2], \\ 4, & x \in (-2, 2], \\ 2x, & x \in (2, +\infty). \end{cases}$

2.91. $y = \begin{cases} (x+1)^2 - 1, & x \in (-\infty, 0], \\ x^2, & x \in (0, +\infty). \end{cases}$

2.92. $y = \begin{cases} (x+3)^2, & x \in (-\infty, 0], \\ (x-3)^2, & x \in (0, +\infty). \end{cases}$

2.93. $y = \begin{cases} 6\left(x+\frac{1}{12}\right)^2 - \frac{25}{24}, & x \in \left(-\infty, -\frac{1}{6}\right] \cup [0, +\infty), \\ -6\left(x+\frac{1}{12}\right)^2 - \frac{23}{24}, & x \in \left(-\frac{1}{6}, 0\right). \end{cases}$

2.94. $y = \begin{cases} -(x+1)^2 + 1, & x \in (-\infty, 1], \\ (x+1)^2 - 1, & x \in (1, +\infty). \end{cases}$

2.95. $y = \begin{cases} 2(x-1), & x \in (-\infty, 1], \\ 0, & x \in (1, +\infty). \end{cases}$

2.96. $y = \begin{cases} 2 - \frac{7}{x+2}, & x \in (-\infty, -2) \cup [3/2, +\infty), \\ -2 + \frac{7}{x+2}, & x \in (-2, 3/2). \end{cases}$

2.97. $y = \begin{cases} 1 - \frac{1}{x+2}, & x \in (-\infty, -2), \\ -1 + \frac{1}{x+2}, & x \in (-2, 0], \\ 1 - \frac{3}{x+2}, & x \in (0, +\infty). \end{cases}$

2.98. For $x \in (-\infty, 0)$—straight line $y = -1$, for $x \in (0, +\infty)$—straight line $y = 1$, for $x = 0$—straight line $y = 0$. **2.99.** For $x \in [n, n+1)$, $n \in \mathbb{Z}$—straight line $y = n$. **2.100.** For $x \in [n, n+1)$, $n \in \mathbb{Z}$—straight line $y = x - n$.

* Here and henceforward all similar problems of this section are supplied with answers which actually provide that form of the original function which yields readily its graph.

2.101. $y=\begin{cases} 2^{-x}-1, & x\in(-\infty,\ 0], \\ 2^{x}-1, & x\in(0,\ +\infty). \end{cases}$

2.102. $y=\begin{cases} 3^{x+1}+2, & x\in(-\infty,\ -1], \\ \left(\frac{1}{3}\right)^{x+1}+2, & x\in(-1,\ +\infty). \end{cases}$

2.103. $y=\begin{cases} \log_{1/2}(3-x), & x\in(-\infty,\ 3), \\ \log_{1/2}(x-3), & x\in(3,\ +\infty). \end{cases}$

2.104. $y=\begin{cases} -\log_2(x+1), & x\in(-1,\ 0], \\ \log_2(x+1), & x\in(0,\ +\infty). \end{cases}$

2.105. $y=\begin{cases} x+\frac{\pi}{4}-2k\pi, & x\in\left[2k\pi-\frac{3\pi}{4},\ 2k\pi+\frac{\pi}{4}\right], \\ x+\frac{\pi}{4}-(2k+1), x\in\left((2k+1)\,\pi-\frac{3\pi}{4}, (2k+1)\,\pi+\frac{\pi}{4}\right). \end{cases}$ $k\in\mathbb{Z}$,

2.106. $y=\begin{cases} 3x-2k\pi, & x\in\left[2k\frac{\pi}{3},\ (2k+1)\frac{\pi}{3}\right], \\ 3x-(2k+1)\,\pi, & x\in\left((2k+1)\frac{\pi}{3},\ 2(k+1)\frac{\pi}{3}\right), \end{cases}$ $k\in\mathbb{Z}$.

2.107. $y=\begin{cases} \sqrt{2}\cos\left(x-\frac{\pi}{4}\right), & x\in[2k\pi,\ (2k+1)\,\pi], \\ \sqrt{2}\cos\left(x+\frac{\pi}{4}\right), & x\in((2k+1)\,\pi,\ 2(k+1)\,\pi), \end{cases}$ $k\in\mathbb{Z}$.

2.108. $y=\begin{cases} -\arctan(x-1), & x\in(-\infty,\ 1], \\ \arctan(x-1), & x\in(1,\ +\infty). \end{cases}$

2.109. $y=\begin{cases} x, & x\in\left(2k\pi-\frac{\pi}{2},\ 2k\pi+\frac{\pi}{2}\right), \\ 0, & x=\frac{\pi}{2}+k\pi, \\ -x, & x\in\left((2k+1)\,\pi-\frac{\pi}{2},\ (2k+1)\,\pi+\frac{\pi}{2}\right), \end{cases}$ $k\in\mathbb{Z}$.

2.110. $y=\begin{cases} \cot\left(x+\frac{\pi}{4}\right), & x\in\left[k\pi-\frac{\pi}{4},\ k\pi+\frac{\pi}{4}\right], \\ -\cot\left(x+\frac{\pi}{4}\right), & x\in\left(k\pi+\frac{\pi}{4},\ (k+1)\,\pi-\frac{\pi}{4}\right), \end{cases}$ $k\in\mathbb{Z}$.

2.111. $y=\frac{1}{2}(1-\cos x)$. **2.112.** Line segment $y=\frac{x}{5}+\frac{2}{5}$, $x\in[-7,\ 3]$. **2.113.** The coordinate axes. **2.114.** A curve symmetrical about both coordinate axes; in the first quadrant: a part of the parabola

$y = -(x-1)^2 + 4$ for $x \in [0, 3]$ and a part of the parabola $y = (x-1)^2 - 4$ for $x \in (0, +\infty)$. **2.115.** A square with the vertices (1, 0), (0, 1), (−1, 0), (0, −1). **2.116.** A square with sides $x = \pm \frac{1}{2}$, $y = \pm \frac{1}{2}$. **2.117.** A curve symmetrical about both coordinate axes and the bisectors of the first and third quadrants; in the domain $G = \{(x, y) \mid x \geqslant 0,\ y \geqslant 0,\ x \geqslant y\}$—a ray $y = x - 1$. **2.118.** A curve symmetrical about both coordinate axes; in the first quadrant for $y \leqslant \frac{1}{2}$—a line segment $x = \frac{\sqrt{3}}{2}$, for $y > \frac{1}{2}$—a line segment $y = 1 - \frac{x}{\sqrt{3}}$.

3.1. $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \ldots$ **3.2.** 2, 0, 6, 0, 10, ... **3.3.** −8, 11, $\frac{14}{3}, \frac{17}{5}, \frac{20}{7}, \ldots$ **3.4.** $\frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}, \ldots$ **3.5.** $x_n = \frac{(-1)^n}{n+1}$. **3.6.** $x_n = 1 + (-1)^n$. **3.7.** $x_n = \frac{2n}{2n-1}$. **3.8.** $x_n = n \cos \frac{\pi(n-1)}{2}$. **3.9.** $x_n = (-1)^n \frac{2n+1}{2n-1}$. **3.10.** $x_n = \sin \frac{(n-1)\pi}{4}$. **3.11.** The largest term $x_3 = 4$. **3.12.** The largest term $x_5 = e$. **3.13.** The largest term $x_9 = 1/6$. **3.14.** The smallest term $x_2 = -22$. **3.15.** The smallest term $x_8 = 24$. **3.16.** The smallest term $x_3 = -9/8$. **3.17.** (a) $\exists A > 0\ \forall n \in \mathbb{N}\ (|x_n| \leqslant A)$; $\forall A > 0\ \exists n \in \mathbb{N}\ (|x_n| > A)$. (b) $\forall n \in \mathbb{N}\ (x_n < x_{n+1})$; $\exists n \in \mathbb{N}\ (x_n \geqslant x_{n+1})$. (c) $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n > N \Rightarrow |x_n - a| < \varepsilon)$; $\exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n \in \mathbb{N}\ (n > N \wedge |x_n - a| \geqslant \varepsilon)$. (d) $\forall E > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n > N \Rightarrow |x_n| > E)$; $\exists E > 0\ \forall N \in \mathbb{N}\ \exists n \in \mathbb{N}\ (n > N \wedge |x_n| \leqslant E)$. (e) $\forall \varepsilon > 0\ \exists n \in \mathbb{N}\ (|x_n - a| < \varepsilon)$; $\exists \varepsilon > 0\ \forall n \in \mathbb{N}\ (|x_n - a| \geqslant \varepsilon)$. **3.18.** (a) $a = 1/3$, $N = 3$; (b) $a = 1$, $N = 10$; (c) $a = 0$, $N = 999$; (d) $a = 5/7$, $N = 3$. **3.19.** $\frac{1}{3}$. **3.20.** 0. **3.21.** −5/9. **3.22.** 0. **3.23.** ∞. **3.24.** −1/2. **3.25.** 0. **3.26.** 0. **3.27.** $\frac{3}{2}$. **3.28.** −1. **3.29.** 1/2. **3.30.** 1/3. **3.31.** 0. **3. 32.** 1. **3.33.** 1/6. **3.35.** Infinitely large. **3.36.** Not infinitely large. **3.37.** Not infinitely large. **3.38.** Infinitely large. **3.39.** 1/3, 3. **3.40.** 0, $\pm\sqrt{2}/2$, 1, $-\sqrt{2}/2$ −1. **3.41.** $\pi/6$, $-\pi/6$. **3.43.** $\inf\{x_n\} = \underline{\lim\limits_{n\to\infty}} x_n = \overline{\lim\limits_{n\to\infty}} x_n = 1$, $\sup\{x_n\} = 2$. **3.44.** $\underline{\lim\limits_{n\to\infty}} x_n = \inf\{x_n\} = 0$, $\overline{\lim\limits_{n\to\infty}} x_n = 1$, $\sup\{x_n\} = 5/4$. **3.45.** The sequence is unbounded; $\overline{\lim\limits_{n\to\infty}} x_n = +\infty$, $\underline{\lim\limits_{n\to\infty}} x_n = -\infty$. **3.46.** $\inf\{x_n\} = -\sqrt{3}$, $\sup\{x_n\} = 2\sqrt{3}$, $\underline{\lim\limits_{n\to\infty}} x_n = -\frac{\sqrt{3}}{2}$, $\overline{\lim\limits_{n\to\infty}} x_n =$

$= \frac{\sqrt{3}}{2}$. **3.47.** $\inf\{x_n\} = -\frac{1}{2}$, $\sup\{x_n\} = \frac{3}{2}$, $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$, $\overline{\lim\limits_{n\to\infty}} x_n = \frac{3}{2}$.

4.4. $\forall E > 0\ \exists \delta > 0\ (0 < |x| < \delta \Rightarrow |f(x)| > E)$. **4.5.** $\forall E > 0\ \exists \delta > 0\ (-\delta < x - 1 < 0 \Rightarrow f(x) < -E)$. **4.6.** $\forall \varepsilon > 0\ \exists A > 0\ (x > A \Rightarrow |f(x)| < \varepsilon)$. **4.7.** $\forall E > 0\ \exists A > 0\ (x > A \Rightarrow f(x) > E)$. **4.8.** $\forall \varepsilon > 0\ \exists \delta > 0\ (0 < x < \delta \Rightarrow |f(x)| < \varepsilon)$. **4.9.** $\forall \varepsilon > 0\ \exists A > 0\ (|x| > A \Rightarrow |f(x) - 2| < \varepsilon)$. **4.10.** $\forall E > 0\ \exists A > 0\ (x < -A \Rightarrow f(x) < -E)$. **4.11.** $\forall E > 0\ \exists A > 0\ (x < -A \Rightarrow |f(x)| > E)$. **4.12.** -2. **4.13.** 2. **4.14.** $-\infty$. **4.15.** 3/2. **4.16.** ∞. **4.17.** 0. **4.18.** m/n. **4.19.** $3x^2$. **4.20.** ∞. **4.21.** 6. **4.22.** $(a-1)/3a^2$. **4.23.** 1/4. **4.24.** ∞. **4.25.** $n+1$. **4.27.** 3/5. **4.28.** 1/6. **4.29.** $\sqrt{2}/2$. **4.30.** $\frac{1}{2\sqrt{x}}$. **4.31.** 3. **4.32.** ∞. **4.33.** $\sqrt{2/3}$. **4.34.** $1/n$. **4.35.** m/n. **4.36.** 3/2. **4.37.** $3\sqrt[6]{2}$. **4.38.** 0. **4.39.** 1/2. **4.40.** $-7/4$. **4.41.** 2. **4.42.** 3. **4.43.** 7/3. **4.44.** 0 for $n > m$, 1 for $n = m$, ∞ for $n < m$. **4.45.** $1/\pi$. **4.46.** 3/4. **4.47.** 2. **4.48.** 4. **4.49.** $\frac{\beta^2 - \alpha^2}{2}$. **4.50.** 0. **4.51.** $-\alpha/\pi$. **4.52.** $-\sqrt{2}/4$. **4.53.** 1. **4.54.** $\frac{1}{2}$. **4.55.** $\frac{25}{16}$. **4.56.** *Hint*: bearing in mind that $\frac{\log_a(1+x)}{x} = \log_a(1+x)^{1/x}$, and taking advantage of the continuity of the function $f(x) = \log_a x$ (see Problem 4.105), we may write: $\lim\limits_{x\to 0} \frac{\log_a(1+x)}{x} = \lim\limits_{x\to 0} \log_a(1+x)^{1/x} = \log_a\left(\lim\limits_{x\to 0}(1+x)^{1/x}\right) = \log_a e$. **4.57.** *Hint*: make the substitution $a^x - 1 = y$. **4.58.** *Hint*: make the substitution $(1+x)^\alpha - 1 = y$. Then $\alpha \ln(1+x) = \ln(1+y)$. Consequently, $\frac{(1+x)^\alpha - 1}{x} = \frac{y}{x} = \frac{y}{\ln(1+y)} \cdot \alpha \cdot \frac{\ln(1+x)}{x}$. **4.59.** e^{10}. **4.60.** e^{10}. **4.61.** $e^{-1/2}$. **4.62.** e^3. **4.63.** 2. **4.64.** 1. **4.65.** $\ln a$. **4.66.** $a \ln a$. **4.67.** $\frac{1}{a}\log_a e$. **4.68.** $a - b$. **4.73.** $+1$, -1. **4.74.** $-\infty$, $+\infty$. **4.75.** $+\infty$, 0. **4.76.** 0, $+\infty$. **4.77.** $\pi/2$, $-\pi/2$. **4.78.** 0, -1. **4.79.** 2, -2. **4.80.** $-\infty$, $-\infty$. **4.84.** 3/2. **4.85.** 2/3. **4.86.** 1. **4.87.** 3. **4.88.** 1. **4.89.** 3. **4.90.** 1/3. **4.91.** 1/2. **4.92.** 1/2. **4.93.** 1/2. **4.95.** 0.97. **4.96.** 5.03. **4.97.** 1.15. **4.98.** 0.88. **4.101.** $-\ln 10$. **4.102.** 3. **4.103.** -2. **4.104.** 2/3. **4.105.** 8/9. **4.106.** $3\sqrt{2}/2$. **4.107.** 3. **4.108.** 1. **4.109.** 1/2. **4.110.** 2/3. **4.111.** 2. **4.112.** 1/6. **4.119.** $A = 3$. **4.120.** $a = 2$. **4.121.** $b = \pi a/2$. **4.122.** $x_1 - 0$, $x_2 = 1$ are points of discontinuity of the second kind. **4.123.** $x = 5/3$ is a point of discontinuity of the first kind. **4.124.** $x = 0$ is a removable

discontinuity; $f(0) = n$. **4.125.** $x = 0$ is a removable discontinuity; $f(0) = 1$. **4.126.** $x = 0$ is a removable discontinuity; $f(0) = 1$. **4.127.** $x_1 = 2$, $x_2 = -2$ are points of discontinuity of the second kind. **4.128.** $x = 0$ is a discontinuity of the first kind. **4.129.** $x = -2$ is a discontinuity of the first kind. **4.130.** $x = 2$ is a discontinuity of the first kind. **4.131.** $x = 0$ is a removable discontinuity; $f(0) = 2$; $x = \pm 1$ are points of discontinuity of the second kind. **4.132.** $x_1 = 0$ is a removable discontinuity, $f(0) = -1$; $x_2 = 1$ is a removable discontinuity, $f(1) = 0$; $x_3 = -1$ is a discontinuity of the second kind. **4.133.** $x = 0$ is a removable discontinuity; $f(0) = 1$ 2. **4.134.** $x = 1$ is a discontinuity of the first kind. **4.135.** $x = 1$ is a discontinuity of the first kind. **4.136.** $x = 2.5$ is a discontinuity of the first kind. **4.137.** $x = \pi/4$ is a discontinuity of the first kind. **4.143.** $(\forall \varepsilon > 0 \; \forall x_0 \in D \; \exists \delta > 0 \; (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)) \wedge \wedge (\exists \varepsilon > 0 \forall \delta > 0 \exists x', x'' \in D \; (|x' - x''| < \delta \wedge |f(x') - f(x'')| > > \varepsilon))$. **4.146.** Uniformly continuous. **4.147.** Not uniformly continuous. **4.148.** Uniformly continuous. **4.149.** Not uniformly continuous. **4.150.** Uniformly continuous. **4.151.** Not uniformly continuous. **4.152.** Not uniformly continuous.

5.4. $9+7i$. **5.5.** $-3+4i$. **5.6.** $-4i$. **5.7.** $-29+22i$. **5.9.** $\frac{5}{17} - \frac{3}{17} i$. **5.10.** i. **5.11.** $\frac{14}{5} i$. **5.12.** $\frac{1}{2} + \frac{3}{2} i$. **5.13.** $x = 2$, $y = 3$. **5.14.** $x = \frac{1}{3}$, $y = \frac{1}{4}$. **5.15.** $z_1 = 1$, $z_2 = i$. **5.16.** $z_1 = 2 + i$, $z_2 = 2 - i$. **5.17.** $\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$. **5.18.** $2\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$. **5.19.** $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. **5.20.** $\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$. **5.21.** $\cos \frac{6\pi}{7} + i \sin \frac{6\pi}{7}$. *Hint*: determine the angle φ satisfying the conditions: $\varphi \in [0, 2\pi)$, $\cos \varphi = -\cos \frac{\pi}{7}$, $\sin \varphi = \sin \frac{\pi}{7}$. **5.22.** $\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$. **5.23.** $2 \cos \frac{\pi}{14} \left(\cos \frac{\pi}{14} + i \sin \frac{\pi}{14}\right)$. **5.30.** (a) $-4i$, $\frac{1}{2} + i \frac{\sqrt{3}}{2}$; (b) $10 - 2i$, $-\frac{7}{4} - \frac{17}{4} i$. **5.32.** $\frac{3}{2} - 2i$. **5.33.** $\frac{3}{4} + i$. **5.35.** A shift by the vector $\boldsymbol{a}(-2, 0)$. **5.36.** A shift by the vector $\boldsymbol{a}(3, -1)$. **5.37.** Anticlockwise rotation through an angle of $\pi/2$. **5.38.** Clockwise rotation through an angle of $\pi/4$. **5.39.** Symmetry about the origin. **5.40.** A homothetic transformation with centre at the origin and the ratio of similitude $k = 2$. **5.41.** Anticlockwise rotation through an angle of $\pi/4$ followed by a homothetic transformation with centre at the origin and ratio $1/\sqrt{2}$. **5.42.** Reflection with respect to the real axis. **5.44.** (a) The sum of the squares of the lengths

of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides. (b) *Hint*: use identity (a). 5.45. Half-plane $x \geqslant 0$. 5.46. Strip $0 \leqslant y < 1$. 5.47. Strip $|y| \leqslant 2$. 5.48. Interior of a circle of radius 1 with centre at the origin. 5.49. Circle $x^2 + (y+1)^2 = 4$. 5.50. Annulus bounded by two concentric circles γ_1: $(x+2)^2 + y^2 = 1$ and γ_2: $(x+2)^2 + y^2 = 4$ (γ_1 does not belong to the annulus). 5.51. $D = \{(x, y) \mid y^2 > 1 - 2x\}$. 5.52. Straight line $2x + y + \frac{3}{2} = 0$. 5.53. Sector bounded by the rays $l_1 = \{(x, y) \mid y = 0, x \geqslant 0\}$ and $l_2 = \{(x, y) \mid y = x, x \geqslant 0\}$ (the ray l_1 does not belong to the sector). 5.54. Sector bounded by the rays $l_1 = \{(x, y) \mid y = x, x < 0\}$ and $l_2 = \{(x, y) \mid y = -x, x \leqslant 0\}$. 5.55. The x-axis. 5.57. $5e^{i \arctan \frac{24}{7}}$. 5.58. $13e^{i \arctan \left(-\frac{12}{5}\right)}$. 5.59. $5e^{i\left(\arctan \frac{4}{3} + \pi\right)}$. 5.60. $\sqrt{5}\, e^{i\left(\pi - \arctan \frac{1}{2}\right)}$. 5.61. $e^{i\left(\alpha + \frac{3\pi}{2}\right)}$. 5.62. $2 \sin \frac{\alpha}{2}\, e^{i \frac{\alpha}{2}}$ for $\sin \frac{\alpha}{2} > 0$, $-2 \sin \frac{\alpha}{2}\, e^{i\left(\pi + \frac{\alpha}{2}\right)}$ for $\sin \frac{\alpha}{2} < 0$. 5.64. (a) $24e^{-\frac{\pi i}{2}}$, $\frac{8}{3}$; (b) $16e^{i \frac{7\pi}{4}}$, $2e^{-i \frac{\pi}{2}}$. 5.67. $32i$. 5.68. 2. 5.69. $512(1 - i\sqrt{3})$. 5.70. $-\frac{1}{4}$. 5.72. (a) $\frac{1}{4}(3 \cos \varphi + \cos 3\varphi)$; (b) $\frac{1}{4}(3 \sin \varphi - \sin 3\varphi)$. 5.73. $4 \cos^3 \varphi - 3 \cos \varphi$. 5.74. $3 \sin \varphi - 4 \sin^3 \varphi$. 5.75. $\cos^4 \varphi - 6 \cos^2 \varphi \sin^2 \varphi + \sin^4 \varphi$. 5.76. $4 \sin \varphi \cos^3 \varphi - 4 \cos \varphi \sin^3 \varphi$. 5.77. Square roots of unity: $z_1 = 1$, $z_2 = -1$; cube roots of unity: $z_1 = 1$, $z_2 = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$, $z_3 = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$; fourth roots of unity: $z_1 = 1$, $z_2 = i$, $z_3 = -1$, $z_4 = -i$. 5.78. $\pm \frac{\sqrt{2}}{2}(1+i)$. 5.79. $\pm \frac{\sqrt{2}}{2}(1+i)$, $\pm \frac{\sqrt{2}}{2}(1-i)$. 5.80. $\cos\left(\frac{\pi}{6} + \frac{2\pi}{9}k\right) + i \sin\left(\frac{\pi}{6} + \frac{2\pi}{9}k\right)$, $k = 0, 1, \ldots, 8$. 5.81. $\pm \frac{\sqrt{2}}{2}(1 + i\sqrt{3})$. 5.82. $\sqrt{2}\left(\cos\left(\frac{\pi}{24} + \frac{\pi}{2}k\right) + i \sin\left(\frac{\pi}{24} + \frac{\pi}{2}k\right)\right)$, $k = 0, 1, 2, 3$. 5.83. $\sqrt[10]{2}\left(\cos\left(\frac{\pi}{4} + \frac{2\pi}{5}k\right) + i \sin\left(\frac{\pi}{4} + \frac{2\pi}{5}k\right)\right)$, $k = 0, 1, 2, 3, 4$. 5.84. $\sqrt[6]{2}\left(\cos\left(\frac{\pi}{18} + \frac{\pi}{3}k\right) + i \sin\left(\frac{\pi}{18} + \frac{\pi}{3}k\right)\right)$, $k = 0, 1, 2, 3, 4, 5$. 5.85. $2\sqrt[5]{2}\left(\cos\left(\frac{(2k-1)\pi}{5} + i \sin \frac{(2k-1)\pi}{5}\right)\right)$, $k = 0, 1,$

2, 3, 4. **5.87.** $\dfrac{\sin\frac{n\varphi}{2}\cos\frac{n+1}{2}\varphi}{\sin\frac{\varphi}{2}}$. **5.88.** $\dfrac{\sin 2n\varphi}{2\sin\varphi}$. **5.89.** $\dfrac{\sin^2 n\varphi}{\sin\varphi}$.

5.90. $-1\pm 2i$. **5.91.** $\dfrac{1}{4}\pm\dfrac{\sqrt{3}}{4}i$. **5.92.** $-2+i$, $-3+i$. **5.93.** $1+i$, $2-3i$. **5.94.** 1, $-\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}$. **5.95.** -1, $\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}$. **5.96.** 1, -3, $-1\pm 2i$. **5.97.** $(-1+\sqrt{2})\pm i\sqrt{2}$, $(-1-\sqrt{2})\pm i\sqrt{2}$. **5.98.** $z_{1,2}=z_{3,4}=\pm 3i$. **5.99.** $\pm i$, $\pm\sqrt{3}i$. **5.100.** $\pm 2i$, $\pm\sqrt{5}i$. **5.101.** $\pm(1+i)$, $\pm\sqrt[4]{2}\left(\cos\dfrac{\pi}{8}-i\sin\dfrac{\pi}{8}\right)$. **5.102.** $\dfrac{1}{2}(1\pm i\sqrt{3})$, -1, $\dfrac{\sqrt[3]{3}}{2}(1\pm i\sqrt{3})$, $-\sqrt[3]{3}$. **5.103.** ± 1, $\pm i$, $\pm\sqrt{2}(1\pm i)$. **5.104.** $x=\tan\dfrac{\alpha+\pi k}{n}$, $k=0,\ 1,\ \ldots,\ n-1$, where $\alpha=\arctan a$.

Hint: set $a=\tan\alpha$, $x=\tan y$ and make use of the trigonometric form of a complex number. **5.105.** $(z-1)(z+1)(z^2+1)$. **5.106.** $(z^2-z\sqrt{2}+1)(z^2+z\sqrt{2}+1)$. **5.107.** $(z^2-z+1)\times(z^2+z+1)$. **5.108.** $(z^2+2z+2)(z^2+2z+5)$. **5.109.** $(z-1)\times(z^2+z+1)^2$. **5.110.** $(z^2-2z+5)(z^2+8z+20)$. **5.114.** $2-i$. **5.115.** -1. **5.116.** $\dfrac{5}{3}i$. **5.117.** $-\dfrac{7}{5}+\dfrac{14}{5}i$. **5.118.** 1. **5.119.** 0. **5.120.** ∞. **5.121.** ∞. **5.122.** $\dfrac{-1-5i}{26}$. **5.123.** $\dfrac{3}{10}(3+i)$. **5.124.** $1+ie$. **5.125.** $\dfrac{11+3i}{10}$. **5.126.** $e^3(\cos 2+i\sin 2)$. **5.135.** 0. **5.136.** 0. **5.137.** $\dfrac{1}{1-z}$. **5.138.** 0. **5.139.** $\dfrac{1-z_2}{1-z_1}$.

Chapter 2

VECTOR ALGEBRA AND ANALYTICAL GEOMETRY

SEC. 2.1.
VECTOR ALGEBRA

1. Linear Operations on Vectors. The *vector* (*geometrical vector*) $\boldsymbol{a}$ is defined as the set of all directed line segments having the same length and direction. Each line segment $\overline{AB}$ from this set is said to represent the vector $\boldsymbol{a}$ (that is, it is obtained by applying the vector $\boldsymbol{a}$ to the point A). The length of the line segment $\overline{AB}$ is the *length* (*modulus*) of the vector $\boldsymbol{a}$ and is denoted as $|\boldsymbol{a}| = |AB|$. A vector of length 0 is called a *null vector* and is denoted by $\mathbf{0}$.

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be *equal* (written $\boldsymbol{a} = \boldsymbol{b}$) if the sets of directed line segments, representing them, coincide.

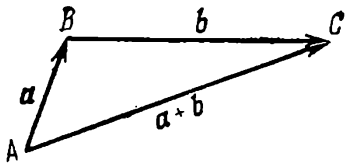

Fig. 10.

In a number of problems it is often convenient not to distinguish between a vector and some directed segment representing this vector. For instance, the expression "to construct a vector" should be understood just in this sense.

Let a line segment $\overline{AB}$ represent the vector $\boldsymbol{a}$. Applying a given vector $\boldsymbol{b}$ to the point B, we obtain a directed line segment $\overline{BC}$. The vector represented by the directed line segment $\overline{AC}$ is called the *sum* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and is denoted $\boldsymbol{a} + \boldsymbol{b}$ (Fig. 10).

The *product of a vector* $\boldsymbol{a}$ *by a real number* λ will again be a vector, denoted $\lambda\boldsymbol{a}$, such that:

(1) $|\lambda\boldsymbol{a}| = |\lambda| \cdot |\boldsymbol{a}|$;

(2) the vectors $\boldsymbol{a}$ and $\lambda\boldsymbol{a}$ are in the same direction for $\lambda > 0$ and are in opposite directions for $\lambda < 0$.

1.1. Prove that the operation of addition of vectors possesses the following properties:

(a) $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$;

(b) $\boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{a}_2 + \boldsymbol{a}_1$ (*commutativity*);

(c) $\boldsymbol{a}_1 + (\boldsymbol{a}_2 + \boldsymbol{a}_3) = (\boldsymbol{a}_1 + \boldsymbol{a}_2) + \boldsymbol{a}_3$ (*associativity*);

(d) $\forall\, \boldsymbol{a}\ \exists!\, \boldsymbol{b}\ (\boldsymbol{a} + \boldsymbol{b} = \mathbf{0})$

(the vector $\boldsymbol{b}$ is said to be *opposite* to the vector $\boldsymbol{a}$ and is denoted by the symbol $-\boldsymbol{a}$);

(e) $\forall\, \boldsymbol{a}_1,\ \boldsymbol{a}_2\ \exists!\, \boldsymbol{a}_3 (\boldsymbol{a}_1 + \boldsymbol{a}_3 = \boldsymbol{a}_2)$
(the vector $\boldsymbol{a}_3$ is called the *difference* of the vectors $\boldsymbol{a}_2$ and $\boldsymbol{a}_1$ and is symbolized $\boldsymbol{a}_2 - \boldsymbol{a}_1$).

1.2. Prove the following equalities:

(a) $-\boldsymbol{a} = (-1)\,\boldsymbol{a}$;

(b) $\boldsymbol{a}_2 - \boldsymbol{a}_1 = \boldsymbol{a}_2 + (-\boldsymbol{a}_1)$;

(c) $\boldsymbol{a} = |\boldsymbol{a}|\,\boldsymbol{a}_0$, where $\boldsymbol{a}_0$ is the *unit vector* of the vector $\boldsymbol{a}$, i.e. a vector of unit length in the same direction with the vector $\boldsymbol{a}$ $(\boldsymbol{a} \neq \boldsymbol{0})$.

1.3. In a parallelepiped $ABCDA'B'C'D'$ the vectors $\boldsymbol{m}$, $\boldsymbol{n}$, $\boldsymbol{p}$ are represented by the edges $\overline{AB}$, $\overline{AD}$, $\overline{AA'}$, respectively. Construct the vectors:

(a) $\boldsymbol{m} + \boldsymbol{n} + \boldsymbol{p}$;

(b) $(1/2)\,\boldsymbol{m} + (1/2)\,\boldsymbol{n} - \boldsymbol{p}$;

(c) $-\boldsymbol{m} - \boldsymbol{n} + (1/2)\,\boldsymbol{p}$.

1.4. Given two vectors: $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Construct the following vectors: $3\boldsymbol{a}_1$, $(1/2)\,\boldsymbol{a}_2$, $\boldsymbol{a}_1 + 2\boldsymbol{a}_2$, $(1/2)\,\boldsymbol{a}_1 - \boldsymbol{a}_2$.

1.5. Prove that:

(a) the operation of multiplication of a vector by a number possesses the following properties:

$$0\boldsymbol{a} = \lambda\boldsymbol{0} = \boldsymbol{0}, \quad (\lambda_1\lambda_2)\,\boldsymbol{a} = \lambda_1\,(\lambda_2\boldsymbol{a});$$

(b) the operations of addition of vectors and multiplication of them by numbers are related by the following two *distributive properties*:

$$\lambda\,(\boldsymbol{a}_1 + \boldsymbol{a}_2) = \lambda\boldsymbol{a}_1 + \lambda\boldsymbol{a}_2 \text{ and } (\lambda_1 + \lambda_2)\,\boldsymbol{a} = \lambda_1\boldsymbol{a} + \lambda_2\boldsymbol{a}.$$

1.6. Prove the following equalities:

(a) $\boldsymbol{a} + (1/2)\,(\boldsymbol{b} - \boldsymbol{a}) = (1/2)\,(\boldsymbol{a} + \boldsymbol{b})$;

(b) $\boldsymbol{a} - (1/2)\,(\boldsymbol{a} + \boldsymbol{b}) = (1/2)\,(\boldsymbol{a} - \boldsymbol{b})$.

What is their geometrical meaning?

1.7. $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are medians of a triangle ABC. Prove the equality $\overline{AD} + \overline{BE} + \overline{CF} = \boldsymbol{0}$.

1.8. $\overline{AK}$ and $\overline{BM}$ are medians of a triangle ABC. Express the vectors $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ in terms of $\boldsymbol{p} = \overline{AK}$ and $\boldsymbol{q} = \overline{BM}$

1.9. In a parallelogram $ABCD$: $\overline{AB} = \boldsymbol{a}$, $\overline{AD} = \boldsymbol{b}$. Express in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ the vectors $\overline{MA}$, $\overline{MB}$, $\overline{MC}$, and $\overline{MD}$, where M is the point of intersection of the diagonals of the parallelogram.

1.10. In a triangle ABC $\overline{AM} = \alpha\overline{AB}$ and $\overline{CN} = \beta\overline{CM}$. Setting $\overline{AB} = \boldsymbol{a}$ and $\overline{AC} = \boldsymbol{b}$, express $\overline{AN}$ and $\overline{BM}$ in terms of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

1.11. In a regular hexagon $ABCDEF$: $\overline{AB} = \boldsymbol{p}$, $\overline{BC} = \boldsymbol{q}$. Express in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ the vectors $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FA}$, $\overline{AC}$, $\overline{AD}$, and $\overline{AE}$.

1.12. M is the point of intersection of the medians of a triangle ABC and O is an arbitrary point in space. Prove the equality $\overline{OM} = (1/3)(\overline{OA} + \overline{OB} + \overline{OC})$.

1.13. Two triangles ABC and $A'B'C'$ are given in space; M and M' are the points of intersection of their medians, respectively. Express the vector $\overline{MM'}$ in terms of $\overline{AA'}$, $\overline{BB'}$ and $\overline{CC'}$.

1.14. E and F are the midpoints of the sides $[AD]$ and $[BC]$ of a quadrangle $ABCD$. Prove that $\overline{EF} = (1/2)(\overline{AB} + \overline{DC})$, and then derive the theorem on the midline of a trapezoid.

1.15. In a trapezoid $ABCD$ the ratio of the length of the base $[AD]$ to the length of the base $[BC]$ is equal to λ. Setting $\overline{AC} = \boldsymbol{a}$ and $\overline{BD} = \boldsymbol{b}$, express the vectors $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is said to be *linearly dependent* if there exist numbers $\lambda_1, \ldots, \lambda_n$ such that at least one of them is nonzero and $\lambda_1\boldsymbol{a}_1 + \ldots + \lambda_n\boldsymbol{a}_n = 0$. Otherwise the system is called *linearly independent*.

1.16. In a triangle ABC $\overline{AM} = \alpha\overline{AB}$ and $\overline{AN} = \beta\overline{AC}$.

(a) For what ratio of α and β are the vectors $\overline{MN}$ and $\overline{BC}$ collinear?

(b) Let α and β be such that the vectors $\overline{MN}$ and $\overline{BC}$ are noncollinear. Setting $\overline{BC} = \boldsymbol{p}$ and $\overline{MN} = \boldsymbol{q}$, express the vectors $\overline{AB}$ and $\overline{AC}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$.

1.17. Prove the following geometrical criteria of a linear dependence:

(a) a system $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is linearly dependent if and only if the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are *collinear*, i.e. their directions coincide or are opposite;

(b) a system $\{a_1, a_2, a_3\}$ is linearly dependent if and only if the vectors a_1, a_2, and a_3 are *coplanar*, i.e. parallel to some plane;

(c) any system of $n \geqslant 4$ vectors is linearly dependent.

1.18. Laid off along the side $[AD]$ of a parallelogram $ABCD$ is a vector $\overline{AK}$ of length $|\overline{AK}| = (1/5)|\overline{AD}|$, and along the diagonal $[AC]$ a vector $\overline{AL}$ of length $|\overline{AL}| = (1/6)|\overline{AC}|$. Prove that the vectors $\overline{KL}$ and $\overline{LB}$ are collinear and find λ such that $\overline{KL} = \lambda \cdot \overline{LB}$.

1.19. Decompose the vector $s = a + b + c$ along the three noncoplanar vectors: $p = a + b - 2c$, $q = a - b$, $r = 2b + 3c$.

1.20. Find the linear dependence among the given four noncoplanar vectors: $p = a + b$, $q = b - c$, $r = a - b + c$, $s = b + (1/2)\,c$.

1.21. Given four vectors: a, b, c, and d. Find their sum if it is known that $a + b + c = \alpha d$, $b + c + d = \beta a$, and the vectors a, b, and c are noncoplanar.

1.22. Prove that for any given vectors a, b, and c the vectors $a + b$, $b + c$, and $c - a$ are coplanar.

1.23. Given three noncoplanar vectors a, b, and c. Prove that the vectors $a + 2b - c$, $3a - b + c$, $-a + 5b - 3c$ are coplanar.

1.24. Given three noncoplanar vectors a, b, and c. Find the values of λ for which the vectors $\lambda a + b + c$, $a + \lambda b + c$, $a + b + \lambda c$ are coplanar.

1.25. Given three noncoplanar vectors a, b, and c. Compute the values of λ and μ for which the vectors $\lambda a + \mu b + c$ and $a + \lambda b + \mu c$ are collinear.

2. Basis and Coordinates of a Vector. An ordered triple of noncoplanar vectors e_1, e_2, e_3 is called the *basis* in the set of all geometrical vectors. Every geometrical vector a can be uniquely represented in the form

$$a = X_1 e_1 + X_2 e_2 + X_3 e_3; \tag{1}$$

the numbers X_1, X_2, X_3 are called the *coordinates* of the vector a in the basis $\mathfrak{B} = (e_1, e_2, e_3)$. Notation (1) is also called the decomposition of the vector a in the basis $\mathfrak{B}$.

Analogously, an ordered pair e_1, e_2 of noncollinear vectors is called the basis $\mathfrak{B} = (e_1, e_2)$ in the set of geometrical vectors coplanar with some plane.

Finally, every nonzero vector e forms the basis $\mathfrak{B} = (e)$ in the set of all geometrical vectors collinear with some direction.

If a vector $\boldsymbol{a}$ is a *linear combination* of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ with the coefficients $\lambda_1, \ldots, \lambda_n$, i.e.

$$\boldsymbol{a} = \sum_{k=1}^{n} \lambda_k \boldsymbol{a}_k,$$

then each coordinate $X_i(a)$ of the vector $\boldsymbol{a}$ is equal to the sum of the products of the coefficients $\lambda_1, \ldots, \lambda_n$ by the like coordinates of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$:

$$X_i(\boldsymbol{a}) = \sum_{k=1}^{n} \lambda_k X_i(\boldsymbol{a}_k), \quad i = 1, 2, 3.$$

The basis $\mathfrak{B} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is said to be rectangular if the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are pairwise perpendicular and have a unit length. In this case the following notation is used

$$\boldsymbol{e}_1 = \boldsymbol{i}, \quad \boldsymbol{e}_2 = \boldsymbol{j}, \quad \boldsymbol{e}_3 = \boldsymbol{k}. \tag{2}$$

The *projection* of a vector $\boldsymbol{a}$ onto a vector $\boldsymbol{e}$ is defined as the number $\mathrm{pr}_{\boldsymbol{e}} \boldsymbol{a} = |\boldsymbol{a}| \cos \varphi$, where $\varphi = \widehat{(\boldsymbol{a}, \boldsymbol{e})}$ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{e}$ $(0 \leqslant \varphi \leqslant \pi)$.

The coordinates X, Y, Z of a vector $\boldsymbol{a}$ in a rectangular basis coincide with the projections of the vector $\boldsymbol{a}$ onto the basis unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, respectively, and the length of the vector $\boldsymbol{a}$ is

$$|\boldsymbol{a}| = \sqrt{X^2 + Y^2 + Z^2}. \tag{3}$$

The numbers

$$\cos \alpha = \cos \widehat{(\boldsymbol{a}, \boldsymbol{i})} = \frac{X}{\sqrt{X^2 + Y^2 + Z^2}},$$

$$\cos \beta = \cos \widehat{(\boldsymbol{a}, \boldsymbol{j})} = \frac{Y}{\sqrt{X^2 + Y^2 + Z^2}},$$

$$\cos \gamma = \cos \widehat{(\boldsymbol{a}, \boldsymbol{k})} = \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}$$

are termed *direction cosines* of the vector $\boldsymbol{a}$.

1.26. Given a tetrahedron $OABC$. In the basis formed by the edges $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ find the coordinates of:

(a) the vector $\overline{DE}$, where D and E are the midpoints of the edges $\overline{OA}$ and $\overline{BC}$, respectively;

(b) the vector $\overline{OF}$, where F is the point of intersection of the medians of the base ABC.

1.27. In a tetrahedron $OABC$ the median $[AL]$ of the face ABC is divided by the point M in a ratio $|\overline{AM}| : |\overline{ML}| = 3 : 7$. Find the coordinates of the vector $\overline{OM}$ in the basis formed by the edges $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$.

1.28. A point O is taken outside the plane of a parallelogram $ABCD$. In the basis formed by the vectors $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ find the coordinates of the vectors:

(a) $\overline{OM}$, where M is the point of intersection of the diagonals of the parallelogram;

(b) $\overline{OK}$, where K is the midpoint of the side $[AD]$.

1.29. In a trapezoid $ABCD$ the ratio of the lengths of the bases $|\overline{AB}|/|\overline{CD}| = \lambda$ is known. Find the coordinates of the vector $\overline{CB}$ in the basis formed by the vectors $\overline{AB}$ and $\overline{AD}$.

In what follows, if it is not stated otherwise, vectors are represented by their coordinates in some rectangular basis. The notation $\boldsymbol{a}(X, Y, Z)$ means that the coordinates of the vector $\boldsymbol{a}$ are equal to X, Y, and Z, i.e. $\boldsymbol{a} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$.

1.30. In a triangle ABC $\overline{AM} = \alpha\overline{AB}$, $\overline{AN} = \beta\overline{AC}$ ($\alpha, \beta \neq 0, 1$ and $\alpha\beta \neq 1$) and O is the point of intersection of (CM) and (BN).

In the basis formed by the vectors $\overline{OM}$ and $\overline{ON}$ find the coordinates of

(a)** the vector $\overline{AO}$;

(b)* the vectors $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

1.31. In a triangle ABC $\overline{AK} = \alpha\overline{AB}$, $\overline{BM} = \beta\overline{BC}$, and $\overline{CF} = \gamma\overline{CA}$. Let P, Q, and R be the points of intersection of the following pairs of straight lines: (BF) and (CK), (CK) and (AM), (AM) and (BF), respectively. Find the coordinates of the vectors $\overline{AP}$, $\overline{BQ}$, and $\overline{CR}$ in the basis formed by the vectors $\overline{AB}$ and $\overline{AC}$.

1.32. Given a regular pentagon $ABCDE$. In the basis formed by the vectors $\overline{AB}$ and $\overline{AE}$ find the coordinates of the following vectors:

(a) $\overline{AC}$ and $\overline{AD}$;

(b) $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$.

1.33. Given a triangle ABC in which $\overline{AM} = \frac{2}{3}\overline{AB}$ and $\overline{AN} = \frac{3}{2}\overline{AC}$. The straight line (MN) intersects (BC) at point K.

(a) Find the coordinates of the vector $\overline{AK}$ in the basis made up from the vectors $\overline{AB}$ and $\overline{AC}$.

(b) Prove that the vectors $\boldsymbol{p} = \overline{AB} + \overline{KN}$, $\boldsymbol{q} = \overline{BC} + \overline{NM}$, and $\boldsymbol{r} = \overline{CA} + \overline{MK}$ are collinear and determine the coefficient γ in the equality $\boldsymbol{p} = \gamma\boldsymbol{q}$.

1.34. In a tetrahedron $ABCD$ $[DM]$ is the median of the face BCD and Q is the centre of gravity of this face. Find the coordinates of the vectors $\overline{DM}$ and $\overline{AQ}$ in the basis $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$.

1.35. Given vectors $\boldsymbol{a}_1$ $(-1, 2, 0)$, $\boldsymbol{a}_2$ $(3, 1, 1)$, $\boldsymbol{a}_3$ $(2, 0, 1)$, and $\boldsymbol{a} = \boldsymbol{a}_1 - 2\boldsymbol{a}_2 + (1/3)\,\boldsymbol{a}_3$. Compute:

(a) $|\boldsymbol{a}_1|$ and the coordinates of the unit vector $\boldsymbol{a}_{1,0}$ of the vector $\boldsymbol{a}_1$;

(b) $\cos(\widehat{\boldsymbol{a}_1, \boldsymbol{j}})$;

(c) the coordinate X of the vector $\boldsymbol{a}$;

(d) $\mathrm{pr}_{\boldsymbol{j}}\,\boldsymbol{a}$.

1.36. Given vectors $\boldsymbol{e}$ $(-1, 1, 1/2)$ and $\boldsymbol{a}$ $(2, -2, -1)$. Get convinced that they are collinear and find the decomposition of the vector $\boldsymbol{a}$ in the basis $\mathfrak{B} = (\boldsymbol{e})$.

1.37. Given in a plane are vectors $\boldsymbol{e}_1$ $(-1, 2)$, $\boldsymbol{e}_2$ $(2, 1)$, and $\boldsymbol{a}$ $(0, -2)$. Make sure that $\mathfrak{B} = (\boldsymbol{e}_1, \boldsymbol{e}_2)$ is the basis in the set of all vectors in the plane. Construct the given vectors and find the decomposition of the vector $\boldsymbol{a}$ with respect to the basis $\mathfrak{B}$.

1.38. Given a triple of noncoplanar vectors $\boldsymbol{e}_1$ $(1, 0, 0)$, $\boldsymbol{e}_2$ $(1, 1, 0)$, and $\boldsymbol{e}_3$ $(1, 1, 1)$. Compute the coordinates of the vector $\boldsymbol{a} = -2\boldsymbol{i} - \boldsymbol{k}$ in the basis $\mathfrak{B} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ and write the corresponding decomposition with respect to this basis.

1.39. Given vectors $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j}$, $\boldsymbol{b} = -3\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$. Find:

(a) the coordinates of the unit vector $\boldsymbol{a}_0$;

(b) the coordinates of the vector $\boldsymbol{a} - (1/2)\,\boldsymbol{b} + \boldsymbol{c}$;

(c) the decomposition of the vector $\boldsymbol{a} + \boldsymbol{b} - 2\boldsymbol{c}$ in the basis $\mathfrak{B} = (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$;

(d) $\mathrm{pr}_{\boldsymbol{j}} (\boldsymbol{a} - \boldsymbol{b})$.

1.40. Find the coordinates of the unit vector $\boldsymbol{a}_0$ if $\boldsymbol{a}$ (6, 7, —6).

1.41. Find $Z(a)$ if $X(a) = 3$, $Y(a) = -9$, and $|a| = 12$.

1.42. Find the length and the direction cosines of the vector $\boldsymbol{p} = 3\boldsymbol{a} - 5\boldsymbol{b} + \boldsymbol{c}$ if $\boldsymbol{a} = 4\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$, and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$.

1.43. Find the vector $\boldsymbol{x}$, collinear with the vector $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$, forming an acute angle with the unit vector $\boldsymbol{j}$ and having the length $|\boldsymbol{x}| = 15$.

1.44. Find the vector $\boldsymbol{x}$ forming equal acute angles with all the three basis unit vectors if $|\boldsymbol{x}| = 2\sqrt{3}$.

1.45. Find the vector $\boldsymbol{x}$ forming an angle of 60° with the unit vector $\boldsymbol{j}$, an angle of 120° with the unit vector $\boldsymbol{k}$ if $|\boldsymbol{x}| = 5\sqrt{2}$.

1.46. For what values of α and β are the vectors $\boldsymbol{a} = -2\boldsymbol{i} + 3\boldsymbol{j} + \alpha\boldsymbol{k}$ and $\boldsymbol{b} = \beta\boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k}$ collinear?

1.47*. Find the vector $\boldsymbol{x}$ directed along the bisector of the angle between the vectors $\boldsymbol{a} = 7\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = -2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ if $|\boldsymbol{x}| = 5\sqrt{6}$.

1.48. Given vectors $\boldsymbol{a}$ (1, 5, 3), $\boldsymbol{b}$ (6, —4, —2), $\boldsymbol{c}$ (0, —5, 7), and $\boldsymbol{d}$ (—20, 27, —35). Select the numbers α, β, and γ so that the vectors $\alpha\boldsymbol{a}$, $\beta\boldsymbol{b}$, $\gamma\boldsymbol{c}$, and $\boldsymbol{d}$ form a closed polygonal line if the initial point of each subsequent vector is brought to coincidence with the terminal point of the preceding one.

1.49. In a tetrahedron $OABC$ the plane angles of the trihedral angle with vertex O are right ones. The point H is the foot of the perpendicular drawn from the vertex O to the plane of the face ABC. Find the coordinates of the vector $\overline{OH}$ in the basis $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ if $|\overline{OA}| = \boldsymbol{a}$, $|\overline{OB}| = \boldsymbol{b}$, and $|\overline{OC}| = \boldsymbol{c}$.

3. Rectangular Cartesian Coordinates of a Point. Simplest Problems of Analytical Geometry. A *rectangular Cartesian coordinate system* (symbolized $\langle O, \mathfrak{B} \rangle$) is said to be introduced in three-dimensional space if we are given:

(1) some point O called the *origin of coordinates* (or, simply, the origin);

(2) some rectangular basis $\mathfrak{B} = (i, j, k)$ in the set of all geometrical vectors.

The axes Ox, Oy, and Oz drawn through the point O in the directions of the basis unit vectors i, j, and k are called the *coordinate axes* of the coordinate system $\langle O, \mathfrak{B}\rangle = Oxyz$.

If M is an arbitrary point in space, then the directed line segment $\overline{OM}$ is called the *radius vector* of the point M. The *coordinates of the point* M in a system $\langle O, \mathfrak{B}\rangle$ are defined as the coordinates of its radius vector $\overline{OM}$ as a geometrical vector in the basis $\mathfrak{B}$, i.e.

$$x(M) = X(\overline{OM}), \quad y(M) = Y(\overline{OM}), \quad z(M) = Z(\overline{OM}).$$

If $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ are two arbitrary points in space, then the coordinates of the vector $\overline{M_1M_2}$ are

$$X = x_2 - x_1, \quad Y = y_2 - y_1, \quad Z = z_2 - z_1. \tag{4}$$

Hence, by virtue of (3), the distance between the two points is expressed by the formula

$$\rho(M_1, M_2) = |\overline{M_1M_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

When solving the problems of analytical geometry, it is advisable to use the methods of vector algebra as frequently as possible.

Example 1. Given in a triangle ABC two vertices A (1, 0, —1), B (2, 2, 1) and the point E (—1, 2, 1) of intersection of its medians. Find the coordinates of the vertex C.

Since the coordinates of the vertex A are given, to compute the coordinates of the vertex C, it is sufficient to find the coordinates of the vector $\overline{AC}$. Let $\overline{BF}$ be the median drawn from the vertex B. Then

$$\overline{AC} = 2\overline{AF} = 2(\overline{BA} + \overline{BF}) = 2\left(\overline{AB} + \frac{3}{2}\overline{BE}\right) \tag{5}$$

(here use is made of the fact that the point E divides the median in the ratio 2 : 1). Further, from the hypothesis, with the aid of formula (4), we compute the coordinates of the vectors $\overline{AB}$ (1, 2, 2) and $\overline{BE}$ (—3, 0, 0), whence, by (5), we obtain $\overline{AC}$ (—7, 4, 4), and finally, using formula (4) once again, we find the coordinates of the point C:

$$x(C) = x(A) + X(\overline{AC}) = -6;$$

$$y(C) = y(A) + Y(\overline{AC}) = 4;$$

$$z(C) = z(A) + Z(\overline{AC}) = 3.$$

Let on a straight line l there be given points M_1, M_2, and M, where $M_1 \neq M_2$. Consider the vectors $\overline{M_1M}$ and $\overline{MM_2}$. Since they are collinear, there can be found a real number λ such that $\overline{M_1M} = \lambda \cdot \overline{MM_2}$. The number λ is said to be the *ratio* in which the point M divides the directed line segment $\overline{M_1M_2}$. This ratio is positive if the point M is inside the line segment $\overline{M_1M_2}$, negative (and $\lambda \neq -1$) if M is found outside $\overline{M_1M_2}$, and is equal to zero if $M = M_1$.

Example 2. Knowing the coordinates of the points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ and the ratio λ in which the point M divides the directed line segment $\overline{M_1M_2}$, find the coordinates of the point M.

Let O be the origin and $\overline{OM_1} = r_1$, $\overline{OM_2} = r_2$, $\overline{OM} = r$. Since

$$\overline{M_1M} = r - r_1, \quad \overline{MM_2} = r_2 - r,$$

hen

$$r - r_1 = \lambda (r_2 - r),$$

whence (since $\lambda \neq -1$)

$$r = \frac{r_1 + \lambda r_2}{1 + \lambda}.$$

The obtained formula yields the solution of the problem in the vector form. Passing to coordinates in this formula, we get

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}, \quad z = \frac{z_1 + \lambda z_2}{1 + \lambda}. \tag{6}$$

1.50. The point $M(1, -5, 5)$ is specified by its coordinates in a rectangular Cartesian coordinate system $\langle O, \mathfrak{B} = (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})\rangle$. Find the coordinates of this point in the system $\langle O', \mathfrak{B}' = (\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')\rangle$ if:

(a) $\overline{OO'} = -2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i}' = \boldsymbol{i}, \boldsymbol{j}' = \boldsymbol{j}, \boldsymbol{k}' = \boldsymbol{k}$;

(b) $O' = O$ and $\boldsymbol{i}' = -\boldsymbol{j}, \boldsymbol{j}' = \boldsymbol{k}, \boldsymbol{k}' = \boldsymbol{i}$;

(c) $\overline{OO'} = \boldsymbol{j}$ and $\boldsymbol{i}' = \frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j}), \boldsymbol{j}' = \frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j}), \boldsymbol{k}' = \boldsymbol{k}$

(prior to solving the problem, make sure that $\mathfrak{B}'$ is a rectangular basis).

1.51. Given three vertices $A(3, -4, 7)$, $B(-5, 3, -2)$, and $C(1, 2, -3)$ of a parallelogram $ABCD$. Find its fourth vertex D opposite to B.

1.52. Given two adjacent vertices of a parallelogram $A(-2, 6)$, $B(2, 8)$ and the point of intersection of its diagonals $M(2, 2)$. Find the two other vertices.

1.53. Determine the coordinates of the vertices of a triangle given the midpoints of its sides: $K(2, -4)$, $M(6, 1)$, and $N(-2, 3)$.

1.54. On the axis of abscissas find the point M whose distance from the point $A(3, -3)$ is equal to 5.

1.55. On the axis of ordinates find the point M equidistant from the points $A(1, -4, 7)$ and $B(5, 6, -5)$.

1.56. Given the vertices of a triangle: $A(3, -1, 5)$,

B (4, 2, —5), and C (—4, 0, 3). Find the length of the median drawn from the vertex A.

1.57. A line segment with the end points A (3, —2) and B (6, 4) is divided into three equal parts. Find the coordinates of the division points.

1.58. Determine the coordinates of the end points of a line segment which is divided into three equal parts by the points C (2, 0, 2) and D (5, —2, 0).

1.59*. Given four points: A (1, 2, 3), B (2, —2, 1), C (3, 0, 3), and D (16, 10, 18). E is the point of intersection of the plane OAB (O is the origin) and a straight line drawn through the point D parallel to the straight line (OC). Find the coordinates of the point E.

1.60*. Given two points: A (2, 5, 2) and B (14, 5, 4); C is the point of intersection of the coordinate plane Oxy and a straight line drawn through the point B parallel to the straight line (OA). Find the coordinates of the point C.

1.61. Given the vertices of a triangle: A (1, —1, —3), B (2, 1, —2), and C (—5, 2, —6). Compute the length of the bisector of its interior angle at the vertex A.

Let us find the decomposition of the vector $\overline{AE}$ with respect to the basis formed by the vectors $\overline{AB}$ and $\overline{AC}$.

Let $e_1 = \overline{AB}/|\overline{AB}|$ and $e_2 = \overline{AC}/|\overline{AC}|$ be the unit vectors of the vectors $\overline{AB}$ and $\overline{AC}$. Then the vector $\overline{AE}$ is in the same direction with the vector $e = e_1 + e_2$ (cf. Problem 1.47), i.e. there exists a number $\lambda > 0$ such that

$$\overline{AE} = \lambda e = \lambda \left(\frac{\overline{AB}}{|\overline{AB}|} + \frac{\overline{AC}}{|\overline{AC}|} \right). \tag{7}$$

On the other hand,

$$\overline{AE} = \overline{AC} + \overline{CE} = \overline{AC} + \mu \overline{CB} = \overline{AC} + \mu (\overline{AB} - \overline{AC})$$
$$= \mu \overline{AB} + (1 - \mu) \overline{AC}, \qquad \mu > 0. \tag{8}$$

Formulas (7) and (8) represent two decompositions of the vector $\overline{AE}$ with respect to the basis formed by the vectors $\overline{AB}$ and $\overline{AC}$. By virtue of the uniqueness of the decomposition of a vector with respect to a basis, we have

$$\frac{\lambda}{|\overline{AB}|} = \mu \text{ and } \frac{\lambda}{|\overline{AC}|} = 1 - \mu. \tag{9}$$

Solving system (9), we find

$$\lambda = \frac{1}{1/|\overline{AC}| + 1/|\overline{AB}|} = \frac{|\overline{AB}| \cdot |\overline{AC}|}{|\overline{AB}| + |\overline{AC}|},$$

hence formula (7) takes the form

$$\overline{AE} = \frac{|\overline{AC}|}{|\overline{AB}| + |\overline{AC}|}\overline{AB} + \frac{|\overline{AB}|}{|\overline{AB}| + |\overline{AC}|}\overline{AC}. \qquad (10)$$

From the conditions of the problem we find:

$\overline{AB}$ (1, 2, 1) and $|\overline{AB}| = \sqrt{6}$, $\overline{AC}$ (−6, 3, −3) and $|\overline{AC}| = 3\sqrt{6}$, and, by virtue of (10), we obtain

$$\overline{AE} = \frac{3}{4}\overline{AB} + \frac{1}{4}\overline{AC},$$

whence

$$\overline{AE}\left(-\frac{3}{4}, \frac{9}{4}, 0\right) \text{ and } |\overline{AE}| = \frac{3}{4}\sqrt{10}.$$

1.62. A triangle is given by the coordinates of its vertices: A (3, −2, 1), B (3, 1, 5), C (4, 0, 3). Compute the distance from the origin to the point of intersection of the medians of this triangle.

1.63. Given the vertices of a triangle: A (1, 0, 2), B (1, 2, 2), and C (5, 4, 6). The point L divides the line segment $\overline{AC}$ in the ratio $\lambda = 1/3$, $[CE]$ is the median drawn from the vertex C. Find the coordinates of the point M of intersection of the straight lines (BL) and (CE).

4. Scalar Product of Vectors. The *scalar product* of two non-zero vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ is defined as the number

$$(\boldsymbol{a}_1, \boldsymbol{a}_2) = |\boldsymbol{a}_1| \, |\boldsymbol{a}_2| \cos \widehat{(\boldsymbol{a}_1, \boldsymbol{a}_2)}.$$

The designation $\boldsymbol{a}_1\boldsymbol{a}_2$ is also used for a scalar product.

Geometrical properties of a scalar product:

(1) $\boldsymbol{a}_1 \perp \boldsymbol{a}_2 \Leftrightarrow \boldsymbol{a}_1\boldsymbol{a}_2 = 0$ (the *condition for perpendicularity of vectors*);

(2) if $\varphi = \widehat{(\boldsymbol{a}_1, \boldsymbol{a}_2)}$, then $0 \leqslant \varphi < \pi/2 \Leftrightarrow \boldsymbol{a}_1\boldsymbol{a}_2 > 0$ and $\pi/2 < \varphi \leqslant \pi \Leftrightarrow \boldsymbol{a}_1\boldsymbol{a}_2 < 0$.

Algebraic properties of a scalar product:

(1) $\boldsymbol{a}_1\boldsymbol{a}_2 = \boldsymbol{a}_2\boldsymbol{a}_1$;

(2) $(\lambda\boldsymbol{a}_1)\,\boldsymbol{a}_2 = \lambda\,(\boldsymbol{a}_1\boldsymbol{a}_2)$;

(3) $\boldsymbol{a}\,(\boldsymbol{b}_1 + \boldsymbol{b}_2) = \boldsymbol{a}\boldsymbol{b}_1 + \boldsymbol{a}\boldsymbol{b}_2$.

If the vectors $\boldsymbol{a}_1$ (X_1, Y_1, Z_1) and $\boldsymbol{a}_2$ (X_2, Y_2, Z_2) are represented

by their coordinates in a rectangular basis, then the scalar product

$$\boldsymbol{a}_1\boldsymbol{a}_2 = X_1X_2 + Y_1Y_2 + Z_1Z_2.$$

This formula, in particular, yields the formula for determining the cosine of the angle between two vectors

$$\cos(\widehat{\boldsymbol{a}_1, \boldsymbol{a}_2}) = \frac{\boldsymbol{a}_1\boldsymbol{a}_2}{|\boldsymbol{a}_1|\,|\boldsymbol{a}_2|} = \frac{X_1X_2+Y_1Y_2+Z_1Z_2}{\sqrt{X_1^2+Y_1^2+Z_1^2}\,\sqrt{X_2^2+Y_2^2+Z_2^2}}.$$

1.64. Prove the algebraic properties of a scalar product.

1.65. $|\boldsymbol{a}_1| = 3$, $|\boldsymbol{a}_2| = 4$, $(\widehat{\boldsymbol{a}_1, \boldsymbol{a}_2}) = 2\pi/3$. Compute: (a) $\boldsymbol{a}_1^2 = \boldsymbol{a}_1\boldsymbol{a}_1$; (b) $(3\boldsymbol{a}_1 - 2\boldsymbol{a}_2)(\boldsymbol{a}_1 + 2\boldsymbol{a}_2)$; (c) $(\boldsymbol{a}_1 + \boldsymbol{a}_2)^2$.

1.66. $|\boldsymbol{a}_1| = 3$, $|\boldsymbol{a}_2| = 5$. Determine for what value of α the vectors $\boldsymbol{a}_1 + \alpha\boldsymbol{a}_2$ and $\boldsymbol{a}_1 - \alpha\boldsymbol{a}_2$ will be mutually perpendicular.

1.67. Compute the length of the diagonals of a parallelogram constructed on the vectors $\boldsymbol{a} = \boldsymbol{p} - 3\boldsymbol{q}$, $\boldsymbol{b} = 5\boldsymbol{p} + 2\boldsymbol{q}$ if it is known that $|\boldsymbol{p}| = 2\sqrt{2}$, $|\boldsymbol{q}| = 3$, and $(\widehat{\boldsymbol{p}, \boldsymbol{q}}) = \pi/4$.

1.68. Determine the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ if it is known that $(\boldsymbol{a} - \boldsymbol{b})^2 + (\boldsymbol{a} + 2\boldsymbol{b})^2 = 20$ and $|\boldsymbol{a}| = 1$, $|\boldsymbol{b}| = 2$.

1.69. In a triangle ABC $\overline{AB} = 3\boldsymbol{a} - 4\boldsymbol{b}$ and $\overline{BC} = \boldsymbol{a} + 5\boldsymbol{b}$. Find the length of its altitude $\overline{CH}$ if it is known that $\boldsymbol{a}$ and $\boldsymbol{b}$ are mutually perpendicular unit vectors.

1.70. Compute $\mathrm{pr}_{\boldsymbol{a}+\boldsymbol{b}}(2\boldsymbol{a} - \boldsymbol{b})$ if $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$ and $(\widehat{\boldsymbol{a}, \boldsymbol{b}}) = 120°$.

1.71. It is known that $\overline{AB} = 2\boldsymbol{a} - 6\boldsymbol{b}$ and $\overline{AC} = 3\boldsymbol{a} + \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are mutually perpendicular unit vectors. Determine the angles of the triangle ABC.

1.72. Find the angle formed by the unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ if it is known that the vectors $\boldsymbol{a} = \boldsymbol{e}_1 + 2\boldsymbol{e}_2$ and $\boldsymbol{b} = 5\boldsymbol{e}_1 - 4\boldsymbol{e}_2$ are mutually perpendicular.

1.73. Find the angle α at the vertex of an isosceles triangle if it is known that the medians drawn from the end points of the base of this triangle are mutually perpendicular.

1.74. Applied to a vertex of a cube are three forces equal to 1, 2, 3 and directed along the diagonals of the faces of the cube passing through this vertex. Find the resultant of the three forces.

1.75. Given a rectangle $ABCD$ and an arbitrary point M

lying outside of it. Prove the equality $\overline{MA} \cdot \overline{MC} = \overline{MB} \cdot \overline{MD}$. Hence show that $|\overline{MA}|^2 + |\overline{MC}|^2 = |\overline{MB}|^2 + |\overline{MD}|^2$.

1.76. $ABCD$ is an isosceles trapezoid, $AB = \boldsymbol{a}$ is the base, $\overline{AD} = \boldsymbol{b}$ is the lateral side, the angle between $\overline{AB}$ and $\overline{AD}$ is equal to $\pi/3$. Express the vectors $\overline{DC}$, $\overline{CB}$, $\overline{AC}$ and $\overline{DB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

1.77. Knowing that $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 1$, $|\boldsymbol{c}| = 4$ and $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$, compute $\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}$.

1.78. Given vectors $\boldsymbol{a}_1$ (4, —2, —4) and $\boldsymbol{a}_2$ (6, —3, 2). Compute:

(a) $\boldsymbol{a}_1\boldsymbol{a}_2$; (b) $(2\boldsymbol{a}_1 - 3\boldsymbol{a}_2)(\boldsymbol{a}_1 + 2\boldsymbol{a}_2)$; (c) $(\boldsymbol{a}_1 - \boldsymbol{a}_2)^2$;
(d) $|2\boldsymbol{a}_1 - \boldsymbol{a}_2|$; (e) $\mathrm{pr}_{a_1} \boldsymbol{a}_2$; (f) $\mathrm{pr}_{a_2} \boldsymbol{a}_1$;
(g) the direction cosines of the vector $\boldsymbol{a}_1$;
(h) $\mathrm{pr}_{a_1+a_2}(\boldsymbol{a}_1 - 2\boldsymbol{a}_2)$; (i) $\cos(\widehat{\boldsymbol{a}_1, \boldsymbol{a}_2})$.

1.79. Given two points A (2, 2) and B (5, —2). On the axis of abscissas find a point M such that $\widehat{AMB} = \pi/2$.

1.80. Find the lengths of the sides and the magnitudes of the angles of a triangle with vertices A (—1, —2, 4), B (—4, —2, 0), and C (3, —2, 1).

1.81. For the given vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ compute $\mathrm{pr}_c(2\boldsymbol{a} - 3\boldsymbol{b})$:

(a) $\boldsymbol{a} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{c} = 4\boldsymbol{i} + 3\boldsymbol{j}$;
(b) $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{c} = 6\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$.

1.82. Prove that the quadrangle with vertices A (—3, 5, 6), B (1, —5, 7), C (8, —3, —1), and D (4, 7, —2) is a square.

1.83. Find the cosine of the angle φ between the diagonals (AC) and (BD) of a parallelogram given three of its vertices: A (2, 1, 3), B (5, 2, —1), and C (—3, 3, —3).

1.84. Compute the work done by the force $\boldsymbol{F} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ on a material point when the latter is displaced from the position A (—1, 2, 0) to the position B (2, 1, 3).

1.85. Given two vectors: $\boldsymbol{a}$ (1, 1) and $\boldsymbol{b}$ (1, —1). Find the cosine of the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfying the system of equations $2\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{a}$, $\boldsymbol{x} + 2\boldsymbol{y} = \boldsymbol{b}$.

1.86. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are of equal length and

form pairwise equal angles. Find the coordinates of the vector $\boldsymbol{c}$ if $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{j} + \boldsymbol{k}$.

If $\boldsymbol{c} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$, then it follows from the hypothesis that the vector $\boldsymbol{c}$ satisfies the system of equations

$$\boldsymbol{ca} = X + Y = \boldsymbol{ab} = 1;$$

$$\boldsymbol{cb} = Y + Z = \boldsymbol{ab} = 1;$$

$$|\boldsymbol{c}|^2 = X^2 + Y^2 + Z^2 = |\boldsymbol{a}|^2 = |\boldsymbol{b}|^2 = 2.$$

Solving this system, we find $X_1 = -1/3$, $Y_1 = 4/3$, $Z_1 = -1/3$ or $X_2 = 1$, $Y_2 = 0$, $Z_2 = 1$.

1.87. Three rays $[OA)$, $[OB)$ and $[OC)$ form pairwise equal angles of magnitude $\pi/3$. Find the angle between the bisectors of the angles $\angle AOB$ and $\angle BOC$.

1.88. Find the coordinates of the vector $\boldsymbol{x}$ which is collinear with the vector $\boldsymbol{a}$ (2, 1, —1) and satisfies the condition $\boldsymbol{ax} = 3$.

1.89. Vector $\boldsymbol{x}$ is perpendicular to the vectors $\boldsymbol{a}_1$(2, 3, —1) and $\boldsymbol{a}_2$ (1, —2, 3) and satisfies the condition $\boldsymbol{x}\,(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = -6$. Find the coordinates of $\boldsymbol{x}$.

If we are given some vector $\boldsymbol{e}$, then the *orthogonal component* of an arbitrary vector $\boldsymbol{a}$ *along the vector* $\boldsymbol{e}$ is defined as such a vector $\boldsymbol{a}_e$ which is collinear with $\boldsymbol{e}$, the difference $\boldsymbol{a} - \boldsymbol{a}_e$ being perpendicular to the vector $\boldsymbol{e}$.

Analogously, the *orthogonal component* of vector $\boldsymbol{a}$ in a *plane* P is defined as a vector $\boldsymbol{a}_P$, coplanar with the plane P, the difference $\boldsymbol{a} - \boldsymbol{a}_P$ being perpendicular to this plane.

1.90. For the vector $\boldsymbol{a}$ (—1, 2, 1) find the orthogonal component along the unit basis vector $\boldsymbol{j}$ and the orthogonal component in the plane of vectors $\boldsymbol{i}$ and $\boldsymbol{k}$.

1.91. Given two vectors $\boldsymbol{e}$ (1, 1, 1) and $\boldsymbol{a}$ (—1, 2, 1). Find:

(a) the orthogonal component of the vector $\boldsymbol{a}$ along the vector $\boldsymbol{e}$;

(b) the orthogonal component of the vector $\boldsymbol{a}$ in the plane P which is perpendicular to the vector $\boldsymbol{e}$.

1.92. Given the vertices of a triangle: A (—1, —2, 4), B (—4, —1, 2), and C (—5, 6, —4); $[BD]$ is its altitude drawn through the vertex B. Find the coordinates of the point D.

1.93*. Given three vectors $\boldsymbol{e}_1$ (1, —2, 0), $\boldsymbol{e}_2$ (1, 1, 1), and $\boldsymbol{a}$ (—2, 0, 1). Find the orthogonal component $\boldsymbol{a}_{e_1, e_2}$ of the vector $\boldsymbol{a}$ in the plane of the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

If the basis $\mathfrak{B} = (e_1, e_2, e_3)$ is rectangular, then the coordinates of an arbitrary vector $a = X_1e_1 + X_2e_2 + X_3e_3$ in this basis can be computed by the formula

$$X_i = ae_i, \quad i = 1, 2, 3. \tag{11}$$

In particular, formula (11) makes it possible to find readily the relation between the coordinates of one and the same vector in different rectangular bases.

Example 3. Let the basis $\mathfrak{B}' = (i', j')$ have been obtained from the basis $\mathfrak{B} = (i, j)$ by rotating the latter about the point O through an angle φ ($\varphi > 0$ if the rotation is performed anticlockwise and $\varphi < 0$ if otherwise) (see Fig. 11). Find the relation between the coordinates of the vector a in the bases $\mathfrak{B}$ and $\mathfrak{B}'$.

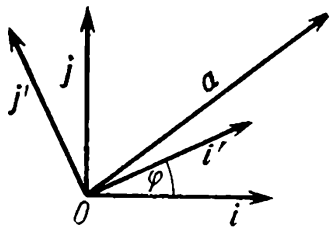

Fig. 11.

Let $a = Xi + Yj$. Then

$$X' = ai' = Xii' + Yji', \tag{12}$$
$$Y' = aj' = Xij' + Yjj'.$$

On the other hand, we have

$$ii' = \cos\varphi, \quad ji' = \cos\left(-\frac{\pi}{2} + \varphi\right) = -\sin\varphi,$$

$$ij' = \cos\left(\frac{\pi}{2} + \varphi\right) = \sin\varphi, \quad jj' = \cos\varphi.$$

Therefore formulas (12) for transforming coordinates take the form

$$X' = X\cos\varphi - Y\sin\varphi, \quad Y' = X\sin\varphi + Y\cos\varphi.$$

1.94. Derive the formulas for transforming the coordinates of points in the plane when passing from the coordinate system $\langle O, \mathfrak{B} = (i, j)\rangle$ to the system $\langle O', \mathfrak{B}' = (i', j')\rangle$, if $\overrightarrow{OO'} = x_0i + y_0j$, and the basis $\mathfrak{B}'$ is obtained from the basis $\mathfrak{B}$ by rotating the latter about the point O through an angle φ.

1.95. Write the formulas for transforming the coordinates of vectors when passing from the basis $\mathfrak{B} = (i, j, k)$ to the basis $\mathfrak{B}' = (i', j', k')$, if

$i' = \cos\varphi \cdot i + \sin\varphi \cdot j$, $j' = -\sin\varphi \cdot i + \cos\varphi \cdot j$, $k' = -k$.

1.96. In a rectangular basis $\mathfrak{B} = (i, j, k)$ vector a is decomposed as $a = -2i + j - k$. Make sure that the triple of vectors

$$i' = i, \quad j' = \frac{1}{\sqrt{2}}j - \frac{1}{\sqrt{2}}k, \quad k' = \frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k$$

also forms a rectangular basis $\mathfrak{B}' = (\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, and find the coordinates of the vector $\boldsymbol{a}$ in this basis.

1.97. Given a noncoplanar triple of vectors $\boldsymbol{e}_1$ (1, —2, 0), $\boldsymbol{e}_2$ (0, 1, 1), and $\boldsymbol{e}_3$ (1, 2, 2) forming an oblique basis (this should be checked!). Vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are decomposed in this basis as follows: $\boldsymbol{a}_1 = X_1^{(1)}\boldsymbol{e}_1 + X_2^{(1)}\boldsymbol{e}_2 + X_3^{(1)}\boldsymbol{e}_3$ and $\boldsymbol{a}_2 = X_1^{(2)}\boldsymbol{e}_1 + X_2^{(2)}\boldsymbol{e}_2 + X_3^{(2)}\boldsymbol{e}_3$. Express the scalar product $\boldsymbol{a}_1\boldsymbol{a}_2$ in terms of the coordinates of the vectors in the given basis.

5. The Vector Product of Vectors. An ordered triple of noncoplanar vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is called *right-handed* if rotations (by the shortest route) from $\boldsymbol{e}_1$ to $\boldsymbol{e}_2$ and from $\boldsymbol{e}_2$ to $\boldsymbol{e}_3$ are performed in an anticlockwise sense for an observer found inside the solid angle formed by these vectors. If the rotation is clockwise, then the triple $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is called *left-handed.*

The *vector product* of a vector $\boldsymbol{a}_1$ by a vector $\boldsymbol{a}_2$ is again a vector symbolized $[\boldsymbol{a}_1, \boldsymbol{a}_2]$ (or $\boldsymbol{a}_1 \times \boldsymbol{a}_2$), and defined by the following three conditions:

(1) the length of the vector $[\boldsymbol{a}_1, \boldsymbol{a}_2]$ is numerically equal to the area of a parallelogram constructed on the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, i.e. $|[\boldsymbol{a}_1, \boldsymbol{a}_2]| = |\boldsymbol{a}_1| \cdot |\boldsymbol{a}_2| \sin (\widehat{\boldsymbol{a}_1, \boldsymbol{a}_2})$;

(2) the vector $[\boldsymbol{a}_1, \boldsymbol{a}_2]$ is perpendicular to the plane of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$;

(3) the ordered triple $\boldsymbol{a}_1, \boldsymbol{a}_2, [\boldsymbol{a}_1, \boldsymbol{a}_2]$ is right-handed.

It follows from the definition of the vector product of two vectors that

$$\boldsymbol{a}_1 \parallel \boldsymbol{a}_2 \Leftrightarrow [\boldsymbol{a}_1, \boldsymbol{a}_2] = \boldsymbol{0}.$$

The algebraic properties of the vector product:

(1) $[\boldsymbol{a}_1, \boldsymbol{a}_2] = -[\boldsymbol{a}_2, \boldsymbol{a}_1]$;

(2) $[\lambda\boldsymbol{a}_1, \boldsymbol{a}_2] = \lambda[\boldsymbol{a}_1, \boldsymbol{a}_2]$;

(3) $[\boldsymbol{a}_1 + \boldsymbol{a}_2, \boldsymbol{b}] = [\boldsymbol{a}_1, \boldsymbol{b}] + [\boldsymbol{a}_2, \boldsymbol{b}]$.

If $\boldsymbol{a}_1$ (X_1, Y_1, Z_1) and $\boldsymbol{a}_2$ (X_2, Y_2, Z_2) are vectors specified by their coordinates in a right-handed rectangular basis, then the decomposition of the vector product in the same basis has the form

$$[\boldsymbol{a}_1, \boldsymbol{a}_2] = (Y_1Z_2 - Z_1Y_2)\,\boldsymbol{i} + (Z_1X_2 - X_1Z_2)\,\boldsymbol{j} + (X_1Y_2 - Y_1X_2)\,\boldsymbol{k},$$

or in symbolic notation (using the notion of a third-order determinant; see Sec. 3.1 in the next chapter)

$$[\boldsymbol{a}_1, \boldsymbol{a}_2] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}. \tag{13}$$

1.98. $|\boldsymbol{a}_1| = 1$, $|\boldsymbol{a}_2| = 2$, and $(\widehat{\boldsymbol{a}_1, \boldsymbol{a}_2}) = 2\pi/3$. Compute:

(a) $|[\boldsymbol{a}_1, \boldsymbol{a}_2]|$; (b) $|[2\boldsymbol{a}_1 + \boldsymbol{a}_2, \boldsymbol{a}_1 + 2\boldsymbol{a}_2]|$;
(c) $|[\boldsymbol{a}_1 + 3\boldsymbol{a}_2, 3\boldsymbol{a}_1 - \boldsymbol{a}_2]|$.

1.99. What condition must be satisfied by the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ for the vectors $\boldsymbol{a}_1 + \boldsymbol{a}_2$ and $\boldsymbol{a}_1 - \boldsymbol{a}_2$ to be collinear?

1.100. Simplify the following expressions:
(a) $[\boldsymbol{i}, \boldsymbol{j} + \boldsymbol{k}] - [\boldsymbol{j}, \boldsymbol{i} + \boldsymbol{k}] + [\boldsymbol{k}, \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}]$,
(b) $[\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}, \boldsymbol{c}] + [\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}, \boldsymbol{b}] + [\boldsymbol{b} - \boldsymbol{c}, \boldsymbol{a}]$,
(c) $[2\boldsymbol{a} + \boldsymbol{b}, \boldsymbol{c} - \boldsymbol{a}] + [\boldsymbol{b} + \boldsymbol{c}, \boldsymbol{a} + \boldsymbol{b}]$,
(d) $2\boldsymbol{i}\,[\boldsymbol{j}, \boldsymbol{k}] + 3\boldsymbol{j}\,[\boldsymbol{i}, \boldsymbol{k}] + 4\boldsymbol{k}\,[\boldsymbol{i}, \boldsymbol{j}]$.

1.101. Prove that $[\boldsymbol{a} - \boldsymbol{b}, \boldsymbol{a} + \boldsymbol{b}] = 2[\boldsymbol{a}, \boldsymbol{b}]$ and explain the geometrical meaning of this identity.

1.102. $|\boldsymbol{a}| = |\boldsymbol{b}| = 5$, $(\widehat{\boldsymbol{a}, \boldsymbol{b}}) = \pi/4$. Compute the area of the triangle constructed on the vectors $\boldsymbol{a} - 2\boldsymbol{b}$ and $3\boldsymbol{a} + 2\boldsymbol{b}$.

1.103. Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are related by the condition $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$. Prove that $[\boldsymbol{a}, \boldsymbol{b}] = [\boldsymbol{b}, \boldsymbol{c}] = [\boldsymbol{c}, \boldsymbol{a}]$. What is the geometrical meaning of this result?

1.104. Prove that for any vectors $\boldsymbol{a}$, $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$ the vectors $[\boldsymbol{a}, \boldsymbol{p}]$, $[\boldsymbol{a}, \boldsymbol{q}]$ and $[\boldsymbol{a}, \boldsymbol{r}]$ are coplanar.

1.105. $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 5$, $(\widehat{\boldsymbol{a}, \boldsymbol{b}}) = \pi/6$. Express in terms of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the unit vector $\boldsymbol{c}_0$ which is perpendicular to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and such that
(a) the triple $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}_0)$ is right-handed;
(b) the triple $(\boldsymbol{b}, \boldsymbol{c}_0, \boldsymbol{a})$ is left-handed.

1.106. Given vectors $\boldsymbol{a}_1$ (3, −1, 2) and $\boldsymbol{a}_2$ (1, 2, −1). Find the coordinates of the vectors:
(a) $[\boldsymbol{a}_1, \boldsymbol{a}_2]$; (b) $[2\boldsymbol{a}_1 + \boldsymbol{a}_2, \boldsymbol{a}_2]$; (c) $[2\boldsymbol{a}_1 - \boldsymbol{a}_2, 2\boldsymbol{a}_1 + \boldsymbol{a}_2]$.

1.107. Compute the area of the triangle whose vertices are A (1, 1, 1), B (2, 3, 4), and C (4, 3, 2).

1.108. In a triangle with vertices A (1, −1, 2), B(5, −6, 2) and C (1, 3, −1) find the altitude $h = |\overline{BD}|$.

1.109. Determine for what values of α and β the vector $\alpha\boldsymbol{i} + 3\boldsymbol{j} + \beta\boldsymbol{k}$ will be collinear with the vector $[\boldsymbol{a}, \boldsymbol{b}]$ if $\boldsymbol{a}$ (3, −1, 1) and $\boldsymbol{b}$ (1, 2, 0).

1.110. For the given vectors $\boldsymbol{a}$ (2, 0, 3), $\boldsymbol{b}$ (−3, 5, 4), and $\boldsymbol{c}$ (3, 4, −1) compute the projection of the vector $[\boldsymbol{a}, \boldsymbol{b}]$ on the vector $(\boldsymbol{a}, \boldsymbol{b})\,\boldsymbol{c}$.

1.111. For the given vectors $\boldsymbol{a}$ (2, 1, —1), $\boldsymbol{b}$ (1, 2, 1), $\boldsymbol{c}$ (2, —1, 3), and $\boldsymbol{d}$ (3, —1, 2) compute the projection of the vector $\boldsymbol{a} + \boldsymbol{c}$ on the vector $[\boldsymbol{b} - \boldsymbol{d}, \boldsymbol{c}]$.

1.112. Find the vector $[\boldsymbol{a}, \boldsymbol{a} + \boldsymbol{b}] + [\boldsymbol{a}, [\boldsymbol{a}, \boldsymbol{b}]]$ if $\boldsymbol{a}$ (2, 1, —3) and $\boldsymbol{b}$ (1, —1, 1).

1.113. Find the vector $[\overline{AB} + \overline{AC}, [\overline{BC}, \overline{AB}]]$ if A (2, 2, 3), B (1, 0, 4), and C (2, 3, 5).

1.114. Three nonzero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are related as follows: $\boldsymbol{a} = [\boldsymbol{b}, \boldsymbol{c}]$, $\boldsymbol{b} = [\boldsymbol{c}, \boldsymbol{a}]$, $\boldsymbol{c} = [\boldsymbol{a}, \boldsymbol{b}]$. Find the lengths of these vectors and the angles between them.

1.115. A force $\boldsymbol{F} = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ is applied at the point A (4, —2, 3). Determine the moment of this force about the point O (3, 2, —1).

1.116. Three forces $\boldsymbol{F}_1$ (2, —1, —3), $\boldsymbol{F}_2$ (3, 2, —1), and $\boldsymbol{F}_3$ (—4, 1, 3) are applied at the point A (—1, 4, 2). Determine the magnitude and the direction cosines of the moment of the resultant of these forces about the point O (2, 3, —1).

1.117. Compute the area of the parallelogram whose diagonals are the vectors $2\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $4\boldsymbol{e}_1 - 5\boldsymbol{e}_2$, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors and $(\widehat{\boldsymbol{e}_1, \boldsymbol{e}_2}) = \pi/4$.

1.118. Find the coordinates of vector $\boldsymbol{x}$ if it is known that it is perpendicular to the vectors $\boldsymbol{a}_1$ (4, —2, —3) and $\boldsymbol{a}_2$ (0, 1, 3), forms an obtuse angle with the unit vector $\boldsymbol{j}$ and $|\boldsymbol{x}| = 26$.

1.119. Find the coordinates of vector $\boldsymbol{x}$ if it is perpendicular to the vectors $\boldsymbol{a}_1$ (2, —3, 1) and $\boldsymbol{a}_2$ (1, —2, 3) and satisfies the condition $\boldsymbol{x}(\boldsymbol{i} + 2\boldsymbol{j} - 7\boldsymbol{k}) = 10$.

1.120. Under what conditions does the equation $\boldsymbol{a}_2 = [\boldsymbol{a}_1, \boldsymbol{x}]$ have a solution with respect to $\boldsymbol{x}$? How many solutions are existent?

1.121. Find the component of the vector $\boldsymbol{a}$ (—1, 2, 0) perpendicular to the plane of the vectors $\boldsymbol{e}_1$ (1, 0, 1) and $\boldsymbol{e}_2$ (1, 1, 1).

1.122. How will expression (13) change if the coordinates of the vectors are specified in a left-handed rectangular basis? Will this formula be correct in the case of an oblique basis?

1.123*. The vector $[\boldsymbol{a}, [\boldsymbol{b}, \boldsymbol{c}]]$ is called the *vector double product* of the given vectors. Prove that the following equal-

ity holds:

$$[a, [b, c]] = b(a, c) - c(a, b).$$

6. Scalar Triple Product. The *scalar triple product* of an ordered triple of vectors a_1, a_2, a_3 is defined as the number $[a_1, a_2]\, a_3$.

Geometrical properties of a scalar triple product:

(1) if V is the volume of the parallelepiped constructed on the vectors a_1, a_2, and a_3, then

$$[a_1, a_2]\, a_3 = \begin{cases} V, \text{ if the triple } (a_1, a_2, a_3) \text{ is right-handed,} \\ -V, \text{ if the triple } (a_1, a_2, a_3) \text{ is left-handed;} \end{cases}$$

(2) for three vectors a_1, a_2, a_3 to be coplanar, it is necessary and sufficient that the condition $[a_1, a_2]\, a_3 = 0$ is fulfilled.

The main algebraic property of a scalar triple product consists in that a circular permutation of the vectors does not change its magnitude, i.e.

$$[a_1, a_2]\, a_3 = [a_2, a_3]\, a_1 = [a_3, a_1]\, a_2.$$

This property allows to introduce the notation $[a_1, a_2]\, a_3 = a_1 a_2 a_3$ (the result is independent of the way in which the square brackets are arranged in the right-hand member). This is how a scalar triple product is written in terms of the coordinates of the vectors specified in a right-handed rectangular basis:

$$a_1 a_2 a_3 = \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}.$$

1.124. Three mutually perpendicular vectors a_1, a_2, a_3 form a right-handed triple; $|a_1| = 4$, $|a_2| = 2$, $|a_3| = 3$. Compute $a_1 a_2 a_3$.

1.125. The vectors a, b, c form a left-handed triple, $|a| = 1$, $|b| = 2$, $|c| = 3$, $(\widehat{a, b}) = 30°$, and c is perpendicular to a and b. Find abc.

1.126. Given three vectors: a_1 (1, —1, 3), a_2 (—2, 2, 1), and a_3 (3, —2, 5). Compute $a_1 a_2 a_3$. What is the orientation of each of the indicated triples:

(a) (a_1, a_2, a_3); (b) (a_2, a_1, a_3); (c) (a_1, a_3, a_2)?

1.127. Do the vectors a_1, a_2, and a_3 form a basis in the set of all vectors if:

(a) a_1 (2, 3, —1), a_2 (1, —1, 3), a_3 (1, 9, —11);

(b) a_1 (3, —2, 1), a_2 (2, 1, 2), a_3 (3, —1, —2).

1.128. Prove that $|a_1 a_2 a_3| \leqslant |a_1|\,|a_2|\,|a_3|$; in what case does an equality take place?

1.129. Prove that for any $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ the vectors $\boldsymbol{a} - \boldsymbol{b}$, $\boldsymbol{b} - \boldsymbol{c}$ and $\boldsymbol{c} - \boldsymbol{a}$ are coplanar. What is the geometrical meaning of this fact?

1.130. Prove the identity

$$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})(\boldsymbol{a} - 2\boldsymbol{b} + 2\boldsymbol{c})(4\boldsymbol{a} + \boldsymbol{b} + 5\boldsymbol{c}) = \boldsymbol{0}.$$

1.131. Prove that if $\alpha [\boldsymbol{a}, \boldsymbol{b}] + \beta [\boldsymbol{b}, \boldsymbol{c}] + \gamma [\boldsymbol{c}, \boldsymbol{a}] = \boldsymbol{0}$, and at least one of the numbers α, β, and γ is nonzero, then the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are coplanar.

1.132. Compute the volume of the tetrahedron $OABC$ if $\overline{OA} = 3\boldsymbol{i} + 4\boldsymbol{j}$; $\overline{OB} = -3\boldsymbol{j} + \boldsymbol{k}$, $\overline{OC} = 2\boldsymbol{j} + 5\boldsymbol{k}$.

1.133. Compute the volume of a tetrahedron with vertices at the points A (2, —3, 5), B (0, 2, 1), C (—2, —2, 3), and D (3, 2, 4).

1.134. In a tetrahedron with vertices at the points A (1, 1, 1), B (2, 0, 2), C (2, 2, 2), and D (3, 4, —3) compute the altitude $h = |\overline{DE}|$.

1.135. Check to see whether the given vectors are coplanar:

(a) $\boldsymbol{a} = -2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{c} = 14\boldsymbol{i} - 13\boldsymbol{j} + 7\boldsymbol{k}$;

(b) $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$.

1.136. For what λ are the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ coplanar:

(a) $\boldsymbol{a}$ (λ, 3, 1), $\boldsymbol{b}$ (5, —1, 2), $\boldsymbol{c}$ (—1, 5, 4);

(b) $\boldsymbol{a}$ (1, 2λ, 1), $\boldsymbol{b}$ (1, λ, 0), $\boldsymbol{c}$ (0, λ, 1)?

1.137. Prove that the four points A (1, 2, —1), B (0, 1, 5), C (—1, 2, 1), and D (2, 1, 3) lie in one plane.

1.138. Find the coordinates of the fourth vertex of a tetrahedron $ABCD$ if it is known that it lies on the y-axis and the volume of the tetrahedron is equal to v:

(a) A (—1, 10, 0), B (0, 5, 2), C (6, 32, 2), $v = 29$;

(b) A (0, 1, 1), B (4, 3, —3), C (2, —1, 1), $v = 2$.

1.139. Show that the volume of the parallelepiped constructed on the diagonals of the faces of the given parallelepiped is equal to the doubled volume of the given parallelepiped.

1.140. Prove the following identities:

(a) $(\boldsymbol{a} + \boldsymbol{c})\,\boldsymbol{b}\,(\boldsymbol{a} + \boldsymbol{b}) = -\boldsymbol{abc}$;

(b) $(\boldsymbol{a} - \boldsymbol{b})(\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c})(\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}) = 3\boldsymbol{abc}$;

(c) $(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{b} + \boldsymbol{c})(\boldsymbol{c} + \boldsymbol{a}) = 2\boldsymbol{abc}$;

(d) $\forall\, \alpha, \beta\ (\boldsymbol{ab}(\boldsymbol{c} + \alpha\boldsymbol{a} + \beta\boldsymbol{b}) = \boldsymbol{abc})$.

SEC. 2.2.
LINEAR GEOMETRICAL OBJECTS

1. Straight Line in the Plane. A straight line in a plane with a rectangular Cartesian coordinate system Oxy can be given by an equation of one of the following forms:

(1) $Ax + By + C = 0$ is the *general equation* of the straight line;

(2) $A(x - x_0) + B(y - y_0) = 0$ is the equation of a straight line through a point $M_0(x_0, y_0)$ and perpendicular to a *normal vector* $\boldsymbol{n}(A, B)$;

(3) $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m}$ is the equation of a straight line through a point $M_0(x_0, y_0)$ and parallel to a *direction vector* $\boldsymbol{q}(l, m)$ (the *canonical equation* of a straight line);

(4) $\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \end{cases} t \in (-\infty, \infty)$ is the *parametric equations* of a straight line; in vector form, they are replaced by one equation

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{q}t,$$

where $\boldsymbol{r}_0(x_0, y_0)$ is the radius vector of the point $M_0(x_0, y_0)$, $\boldsymbol{q}(l, m)$ is the direction vector of the straight line;

(5) $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is the *intercept form* of the equation of a straight line, where a and b are the lengths of the directed segments intercepted by the straight line on the x- and y-axes, respectively;

(6) $x\cos\alpha + y\cos\beta - p = 0$ is the *normal equation* of a straight line, where $\cos\alpha$ and $\cos\beta$ are the direction cosines of the normal vector $\boldsymbol{n}$ directed from the origin towards the straight line, and $p > 0$ is the distance from the origin to the straight line.

General equation (1) is reduced to normal form (6) by multiplying by the normalization factor

$$\mu = -\frac{\operatorname{sgn} C}{\sqrt{A^2 + B^2}}.$$

If a straight line L is given by an equation of form (6), and $M(x, y)$ is some point in the plane, then the expression

$$\delta(M, L) = x\cos\alpha + y\cos\beta - p$$

is called the *deviation of the point M from the straight line L*. The symbol $\delta(M, L)$ indicates the mutual positions of the point M, the straight line L and the origin, namely: if the point M and the origin lie on different sides of the line L, then $\delta(M, L) > 0$, and if M and the origin are on one side of L, then $\delta(M, L) < 0$. The distance $\rho(M, L)$ from the point M to the straight line L is determined by the equality $\rho(M, L) = |\delta(M, L)|$.

Example 1. Write the equation of a straight line L' which is parallel to two given lines L_1: $x + 2y - 1 = 0$, L_2: $x + 2y + 2 = 0$ and passes in the middle between them.

First method. Since the vector $\boldsymbol{n}$ (1, 2), normal to the given lines L_1 and L_2, is at the same time normal to the line L', it suffices to find some point M' equidistant from L_1 and L_2 (i.e. lying between them at an equal distance from either of them). From the equations for L_1 and L_2 we find any two points $M_1 \in L_1$ and $M_2 \in L_2$, for instance, M_1 (1, 0) and M_2 (—2, 0). Then the point M' (—1/2, 0), bisecting the line segment $\overline{M_1M_2}$, lies between L_1 and L_2 and is equidistant from them. Therefore the equation of the straight line L' has the form

$$L'\colon\ x+2y+\frac{1}{2}=0.$$

Second method. An arbitrary point M belongs to L' if and only if $\rho\,(M,\ L_1) = \rho\,(M,\ L_2)$, i.e.

$$|\,\delta\,(M,\ L_1)\,| = |\,\delta\,(M,\ L_2)\,|. \tag{1}$$

To remove the moduli in this relation, let us determine the position of the origin relative to the given lines L_1 and L_2. The normal equations of these lines are

$$L_1\colon\ \frac{1}{\sqrt{5}}x+\frac{2}{\sqrt{5}}y-\frac{1}{\sqrt{5}}=0$$

and

$$L_2\colon\ -\frac{1}{\sqrt{5}}x-\frac{2}{\sqrt{5}}y-\frac{2}{\sqrt{5}}=0.$$

Since the normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ from the point O towards L_1 and L_2, respectively, are in opposite directions, the point O is situated in the strip bounded by L_1 and L_2.

Therefore relation (1) takes the form $\delta\,(M,\ L_1) - \delta\,(M,\ L_2)$, or

$$\frac{1}{\sqrt{5}}x+\frac{2}{\sqrt{5}}y-\frac{1}{\sqrt{5}} \quad -\frac{1}{\sqrt{5}}x-\frac{2}{\sqrt{5}}y-\frac{2}{\sqrt{5}},$$

i.e. $L'\colon\ x+2y+\frac{1}{2}=0.$

In Problems 2.1 to 2.3:

(1) write the equation of the given line, reduce it to the general form, and construct the line;

(2) reduce the general equation to the normal form and indicate the distance from the origin to the straight line.

2.1. A straight line L is given by a point $M_0\,(x_0,\ y_0) \in L$ and a normal vector $\boldsymbol{n}\,(A,\ B)$:

(a) M_0 (—1, 2), $\boldsymbol{n}$ (2, 2); (b) M_0 (2, 1), $\boldsymbol{n}$ (2, 0);

(c) M_0 (1, 1), $\boldsymbol{n}$ (2, —1).

2.2. A straight line L is given by a point $M_0\,(x_0,\ y_0) \in L$ and a direction vector $\boldsymbol{q}\,(l,\ m)$:

(a) M_0 (—1, 2), $\boldsymbol{q}$ (3, —1); (b) M_0 (1, 1), $\boldsymbol{q}$ (0, —1);
(c) M_0 (—1, 1), $\boldsymbol{q}$ (2, 0).

2.3. A straight line L is given by two points M_1 (x_1, y_1) and M_2 (x_2, y_2):
(a) M_1 (1, 2), M_2 (—1, 0); (b) M_1 (1, 1), M_2 (1, —2);
(c) M_1 (2, 2), M_2 (0, 2).

2.4. Given a straight line L and a point M.
(1) Compute the distance $\rho\,(M, L)$ from the point M to the straight line L;
(2) write the equation of the straight line L' passing through the point M and perpendicular to the given line L;
(3) write the equation of the straight line L'' passing through the point M and parallel to the given line L.
(a) L: $-2x + y - 1 = 0$, M (—1, 2);
(b) L: $2y + 1 = 0$, M (1, 0);
(c) L: $x + y + 1 = 0$, M (0, 1).

Let there be given two straight lines L_1 and L_2. Two cases of their mutual positions are possible:
(1) L_1 and L_2 are parallel lines; in particular, they coincide;
(2) L_1 and L_2 intersect.

In Problems 2.5 to 2.9 investigate the mutual positions of the given lines L_1 and L_2. In Case (1) find the distance $\rho\,(L_1, L_2)$ between the straight lines, and in Case (2) find the cosine of the angle $(\widehat{L_1, L_2})$ and the point M_0 of intersection of the given lines.

2.5. L_1: $-2x + y - 1 = 0$, L_2: $2y + 1 = 0$,

2.6. L_1: $\frac{x-1}{-2} = \frac{y}{1}$, L_2: $\frac{x+2}{1} = \frac{y}{0}$,

2.7. L_1: $x + y - 1 = 0$, L_2: $2x - 2y + 1 = 0$,

2.8. L_1: $x + y - 1 = 0$, L_2: $\frac{x}{2} = \frac{y+1}{-2}$,

2.9. L_1: $x + 2y + 1 = 0$, L_2: $2x - 4y - 2 = 0$.

2.10. A triangle ABC is given by the coordinates of its vertices.
(1) Write the equation of the side (AB);
(2) write the equation of the altitude (CD) and compute its length $h = |\,CD\,|$;
(3) find the angle φ between the altitude (CD) and median (BM);

(4) write the equations of the bisectors L_1 and L_2 of the interior and exterior angles at the vertex A.

(a) A (1, 2), B (2, —2), C (6, 1);

(b) A (2, —2), B (6, 1), C (—2, 0).

2.11. Show that the point M (—1, 2) belongs to the straight line L: $x = 2t$, $y = -1 - 6t$. Find the value of the parameter t corresponding to this point.

2.12. Compute the distance from the point M (1, 1) to the straight line L: $x = -1 + 2t$, $y = 2 + t$.

If a straight line is given by the general equation $Ax + By + C = 0$ and $B \neq 0$ (i.e. the given line is not parallel to the y-axis), then this straight line can be described by an *equation in the slope-intercept form* $y = kx + b$.

Example 2. Write the equation of the straight line L' passing through the point M (2, 1) at an angle $\pi/4$ to the straight line L: $2x + 3y + 4 = 0$.

The angle between two lines L and L' is defined as the smaller (acute) angle of the two adjacent angles formed by these lines. Therefore (see Problem 2.13)

$$\tan (\widehat{L_1, L_2}) = \left| \frac{k + \frac{2}{3}}{1 + \left(-\frac{2}{3}\right) k} \right| = 1,$$

where k is the slope of the straight line L'. From this equation we find: $k_1 = 1/5$, $k_2 = -5$. Consequently, the problem has two solutions. Using the coordinates of the point M, we can write for each solution an equation in the slope-intercept form:

$$y = \frac{1}{5} x + \frac{3}{5}, \quad y = -5x + 11,$$

or in the general form

$$x - 5y + 3 = 0, \quad 5x + y - 11 = 0.$$

2.13. Write the equation of the straight line passing through the point M_0 (2, 4) at a distance of $\rho = 1$ from the point A (0, 3).

2.14. Write the equation of the straight line passing through the point M_0 (1, 2) twice as distant from the point A (—2, —5) as from the point B (1, 8).

2.15. Write the equation of the straight line perpendicular to the straight line $2x + 6y - 3 = 0$ and passing at a distance of $\sqrt{10}$ from the point A (5, 4).

2.16. Prove that if two lines L_1 and L_2 are given by equations in the slope-intercept form, then

$$\tan (L_1, L_2) = \left| \frac{k_2 - k_1}{1 + k_1 k_2} \right|.$$

2.17. A light ray emanates from the point M (5, 4) at an angle $\varphi = \operatorname{arc\,tan} 2$ to the x-axis and reflects from it. Write the equations for the incident and reflected rays.

2.18. Write the equation of the straight line intercepting on the axis of abscissas a line segment 2 cm long and forming an angle of 45° with the straight line $2x - y + 3 = 0$.

2.19. In the equation of the straight line $4x + \lambda y - 20 = 0$ choose λ so that the angle between this line and the straight line $2x - 3y + 6 = 0$ be equal to 45°.

2.20. Given in an isosceles triangle ABC: the vertex C (4, 3), the equation $2x - y - 5 = 0$ of the base (AC), and the equation $x - y = 0$ of the lateral side (AB). Write the equation of the side (BC).

2.21. Write the equation of the straight line which passes at a distance $\rho = \sqrt{34}$ from the point A (—1, 2) and forms with the x-axis an angle equal to twice the angle formed by the straight line $2x - 6y + 5 = 0$ with the x-axis.

2.22. Write the equation of the straight line which passes through the point M (8, 6) and cuts off the quadrant a triangle whose area is equal to 12.

2.23. Write the equation of the straight line parallel to the two given lines L_1 and L_2 and passing in the middle between them if

(a) L_1: $3x - 2y - 1 = 0$, L_2: $\dfrac{x-1}{2} = \dfrac{y+5}{3}$;

(b) L_1: $3x - 15y - 1 = 0$, L_2: $\dfrac{x + \frac{1}{2}}{5} = \dfrac{y + \frac{1}{2}}{1}$.

2.24. Write the equation of the straight line passing through the point M (2, 1) at an angle $\pi/4$ to the straight line L: $x = 1 + t$, $y = -2 - (2/3)\,t$.

2.25. Given two opposite vertices of a square: A (1, 3) and C (—1, 1). Find the coordinates of the remaining two vertices and write the equations of its sides.

2.26. Given the equation of one of the sides of a square $x + 3y - 3 = 0$ and the point of intersection of its diagonals N $(-2, 0)$. Write the equations of its remaining sides.

2.27. The point A $(5, -4)$ is a vertex of the square whose diagonal lies on the straight line $x - 7y - 8 = 0$. Write the equations of the sides and the other diagonal of this square.

2.28. Write the equations of the sides of a triangle ABC given its vertex A $(1, 3)$ and the equations of two medians: $x - 2y + 1 = 0$ and $y - 1 = 0$.

2.29*. Prove that the straight line $2x + y + 3 = 0$ intersects the line segment $[M_1M_2]$, where M_1 $(-5, 1)$ and M_2 $(3, 7)$.

2.30. Write the equation of the straight line passing through the point M_0 $(-2, 3)$ equidistant from the points M_1 $(5, -1)$ and M_2 $(3, 7)$.

2.31. Find out whether the point M_0 $(1, -2)$ and the origin lie in the same angle, in adjacent angles or in the vertical angles formed by the intersecting straight lines L_1 and L_2 if

(a) L_1: $2x - y - 5 = 0$, L_2: $3x + y + 10 = 0$;

(b) L_1: $x - 2y - 1 = 0$, L_2: $3x - y - 2 = 0$.

2.32. Which of the angles (acute or obtuse) formed by the straight lines $3x - 5y - 4 = 0$ and $x + 2y + 3 = 0$ contains the point M $(2, -5)$?

2.33. Write the equations of the sides of a triangle, knowing one of its vertices B $(2, 6)$ and the equations of its altitude $x - 7y + 15 = 0$ and the bisector $7x + y + 5 = 0$ drawn from one vertex.

2.34. Write the equations of the sides of a triangle, knowing one of its vertices B $(2, -7)$ and the equations of the altitude $3x + y + 11 = 0$ and the median $x + 2y + 7 = 0$ drawn from different vertices.

2.35. Write the equations of the sides of a triangle, knowing one of its vertices A $(3, -1)$ and the equations of the bisector $x - 4y + 10 = 0$ and median $6x + 10y - 59 = 0$ drawn from different vertices.

2.36. Given the equations $5x + 4y = 0$ and $3x - y = 0$ of the medians of a triangle and the coordinates $(-5, 2)$ of one of its vertices. Write the equations of its sides.

2.37. Given the equations $y + 4 = 0$ and $7x + 4y + 5 = 0$ of the bisectors of two interior angles of a triangle and the equation $4x + 3y = 0$ of the side joining the vertices from which the given bisectors emanate. Write the equations of the remaining two sides of the triangle.

2.38. (a) Prove that the point H of intersection of the altitudes of a triangle lies on one and the same straight line with the point M of intersection of its medians and with the centre N of the circumscribed circle.

(b) Verify the statement of (a) for the triangle with vertices at the points A (5, 8), B (—2, 9), and C (—4, 5). Determine in what ratio λ the directed segment $\overline{MN}$ is divided by the point H.

2.39. In the triangle A (—3, —1), B (1, —5), C (9, 3), $\overline{AM} = 3\overline{MB}$ and $\overline{AN} = 3\overline{NC}$. Show that the point of intersection of the straight lines (BN) and (CM) lies on the median drawn from the vertex A.

2. A Plane and a Straight Line in Space. In a rectangular Cartesian coordinate system $Oxyz$ a plane P can be given by an equation of one of the following forms:

(1) $Ax + By + Cz + D = 0$ is the *general equation* of a plane;

(2) $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ is the equation of a plane passing through a point $M_0(x_0, y_0, z_0)$ and perpendicular to a *normal vector* $\boldsymbol{n}(A, B, C)$;

(3) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is the *intercept form* of the equation of a plane where a, b, c are the lengths of the directed line segments intercepted by the plane on the x-, y-, and z-axes, respectively;

(4) $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$ is the *normal equation* of a plane, where $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of the normal vector $\boldsymbol{n}$ drawn from the origin towards the plane, and $p > 0$ is the distance from the origin to the plane.

The general equation (1) is reduced to the normal form (4) by multiplying the former by the normalization factor

$$\mu = -\frac{\operatorname{sgn} D}{\sqrt{A^2 + B^2 + C^2}}.$$

If a plane P is given by a normal equation of form (4), and $M(x, y, z)$ is some point in space, then the expression

$$\delta(M, P) = x \cos \alpha + y \cos \beta + z \cos \gamma - p$$

is called the deviation of the point M from the plane P. The symbol $\delta(M, P)$ indicates the mutual positions of the point M, the plane P, and the origin, namely: if the point M and the origin lie on different

sides of the plane P, then $\delta(M, P) > 0$, and if M and the origin are found on one side of the plane P, then $\delta(M, P) < 0$.

The distance $\rho(M, P)$ from the point M to the plane P is determined by the equality $\rho(M, P) = |\delta(M, P)|$.

A straight line L in space can be given by:

(1) the *general equations*

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases}$$

where the coefficients A_1, B_1, C_1 are not proportional to the coefficients A_2, B_2, C_2 which is equivalent to its specification as the line of intersection of two planes.

(2) the *parametric equations*

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \\ z = z_0 + nt, \end{cases}$$

or in vector form

$$r(t) = r_0 + qt,$$

where $r_0(x_0, y_0, z_0)$ is the radius vector of some point belonging to the straight line and $q(l, m, n)$ is the direction vector of the straight line;

(3) the *canonical equations*

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n},$$

which is equivalent to describing a straight line as the line of intersection of three planes projecting this straight line on the coordinate planes.

Example 3. Write the equation of the plane P passing through the two points $M_1(1, 1, 1)$ and $M_2(0, 2, 1)$ parallel to the vector $a(2, 0, 1)$.

The problem has a unique solution, since the vectors $\overline{M_1M_2}(-1, 1, 0)$ and $a(2, 0, 1)$ are noncollinear. The vector

$$n = [\overline{M_1M_2}, a] = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = i + j - 2k$$

can be taken as a normal vector to the plane. The equation of the plane has the form $(x - 1) + (y - 1) - 2(z - 1) = 0$ or $x + y - 2z = 0$. Since the last equation has no constant term, the plane passes through the origin.

Other method. Point $M(x, y, z)$ belongs to the desired plane P if and only if the vectors $\overline{M_1M}$, $\overline{M_1M_2}$ and a are coplanar. Hence

$$\overline{M_1M} \cdot \overline{M_1M_2} \cdot a = \begin{vmatrix} x-1 & y-1 & z-1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 0,$$

i.e. $x + y - 2z = 0$.

Example 4. A straight line L is given by the general equations

$$\begin{cases} x + y - z = 0, \\ 2x - y + 2 = 0. \end{cases}$$

Write the canonical equation of this line and the equation of its projection on the xz-plane.

The point M (0, 2, 2) satisfies the general equations of the straight line (check this!) and, consequently, lies on this line. As the direction vector of the line, we can take the vector $\boldsymbol{q} = [\boldsymbol{n}_1, \boldsymbol{n}_2]$, where $\boldsymbol{n}_1$ (1, 1, -1) and $\boldsymbol{n}_2$ (2, -1, 0) are the normal vectors of the planes the line of intersection of which is the given straight line. Thus,

$$\boldsymbol{q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k},$$

and the canonical equations of the given line are:

$$\frac{x}{-1} = \frac{y-2}{-2} = \frac{z-2}{-3}.$$

The obtained proportion is equivalent to the system of three equations

$$\begin{cases} -2x + y - 2 = 0, \\ -3x + z - 2 = 0, \\ -3y + 2z + 2 = 0, \end{cases}$$

describing the three planes projecting the given line on the xy-, xz-, and yz-planes, respectively (the *equations of a straight line in the projection form*). In particular, the equation $-3x + z - 2 = 0$ is the equation of the projection of the given straight line on the xz-plane.

Example 5. Given skew lines

$$L_1: \frac{x}{-2} = \frac{y-1}{0} = \frac{z+2}{1} \quad \text{and} \quad L_2: \frac{x+1}{1} = \frac{y+1}{2} = \frac{z-2}{-1}.$$

Find the distance $\rho\,(L_1, L_2)$ between the given straight lines and write the equation of the common perpendicular L to these lines.

Let us find the equation of the plane P passing through the line L_1 parallel to the line L_2 (Fig. 12). The point M_1 (0, 1, -2) lies on L_1 and, consequently, belongs to the desired plane P. We take as a normal vector to this plane the vector

$$\boldsymbol{n} = [\boldsymbol{q}_1, \boldsymbol{q}_2] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -2\boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k}.$$

The equation of the plane P:

$$-2x - (y - 1) - 4(z + 2) = 0,$$

or, in the general form, $2x + y + 4z + 7 = 0$.

The distance $\rho(L_1, L_2)$ is equal to the distance from any point on L_2, say $M_2(-1, -1, 2)$, to the plane P. The normal equation of the plane P has the form

$$-\frac{2}{\sqrt{21}}x-\frac{1}{\sqrt{21}}y-\frac{4}{\sqrt{21}}z-\frac{7}{\sqrt{21}}=0,$$

whence

$$\rho(L_1, L_2)=|\delta(M_2, P)|=\left|\frac{2}{\sqrt{21}}+\frac{1}{\sqrt{21}}-\frac{8}{\sqrt{21}}-\frac{7}{\sqrt{21}}\right|=\frac{12}{\sqrt{21}}.$$

To write the equation of the common perpendicular L, let us find the equations of the planes P_1 and P_2 passing through the given lines

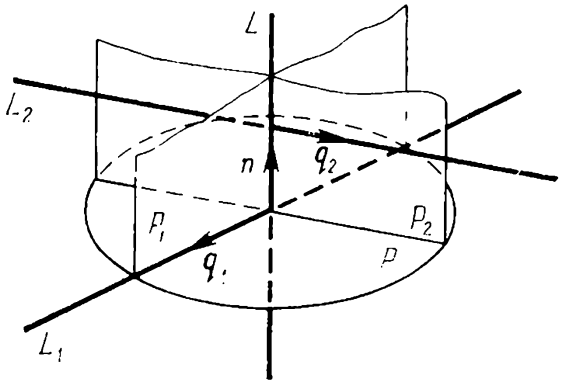

Fig. 12.

L_1 and L_2, respectively, and perpendicular to the plane P. We have $M_1(0, 1, -2) \in P_1$ and $\boldsymbol{n}_1 = [\boldsymbol{q}_1, \boldsymbol{n}] = \boldsymbol{i} - 10\boldsymbol{j} + 2\boldsymbol{k} \perp P_1$, whence P_1: $x - 10y + 2z + 14 = 0$. Analogously, $M_2(-1, -1, 2) \in P_2$ and $\boldsymbol{n}_2 = [\boldsymbol{q}_2, \boldsymbol{n}] = -9\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k} \perp P_2$, whence P_2: $3x - 2y - z + 3 = 0$.

Since $L = P_1 \cap P_2$, we obtain

$$\begin{cases} x - 10y + 2z + 14 = 0, \\ 3x - 2y - z + 3 = 0 \end{cases}$$

which are the general equations of the line L.

2.40. Given a plane P and a point M. Write the equation of the plane P' passing through the point M parallel to the plane P, and compute the distance $\rho(P, P')$ if

(a) P: $-2x + y - z + 1 = 0$, $M(1, 1, 1)$;

(b) P: $x - y - 1 = 0$, $M(1, 1, 2)$.

2.41. Write the equation of the plane P' passing through the given points M_1 and M_2 perpendicular to the given plane P if

(a) P: $-x + y - 1 = 0$, $M_1(1, 2, 0)$, $M_2(2, 1, 1)$;

(b) P: $2x - y + z + 1 = 0$, $M_1(0, 1, 1)$, $M_2(2, 0, 1)$.

2.42. Write the equation of the plane passing through the point M parallel to the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ if

(a) $M(1, 1, 1)$, $\boldsymbol{a}_1(0, 1, 2)$, $\boldsymbol{a}_2(-1, 0, 1)$;

(b) $M(0, 1, 2)$, $\boldsymbol{a}_1(2, 0, 1)$, $\boldsymbol{a}_2(1, 1, 0)$.

2.43. Write the equation of the plane passing through the points M_1 and M_2 parallel to the vector $\boldsymbol{a}$ if

(a) $M_1(1, 2, 0)$, $M_2(2, 1, 1)$, $\boldsymbol{a}(3, 0, 1)$;

(b) $M_1(1, 1, 1)$, $M_2(2, 3, -1)$, $\boldsymbol{a}(0, -1, 2)$.

2.44. Write the equation of the plane passing through the three given points M_1, M_2, and M_3 if

(a) $M_1(1, 2, 0)$, $M_2(2, 1, 1)$, $M_3(3, 0, 1)$;

(b) $M_1(1, 1, 1)$, $M_2(0, -1, 2)$, $M_3(2, 3, -1)$.

Let there be given two planes P_1 and P_2. Two cases of their mutual positions are possible:

(1) P_1 is parallel to P_2, in particular, the planes coincide;

(2) P_1 and P_2 intersect along a straight line.

In Problems 2.45 to 2.48 investigate the mutual positions of the given planes. In Case (1) find the distance $\rho(P_1, P_2)$ between the planes, and in Case (2) find the cosine of the angle between them.

2.45. P_1: $-x + 2y - z + 1 = 0$, P_2: $y + 3z - 1 = 0$.

2.46. P_1: $2x - y + z - 1 = 0$, P_2: $-4x + 2y - 2z - 1 = 0$.

2.47. P_1: $x - y + 1 = 0$, P_2: $y - z + 1 = 0$.

2.48. P_1: $2x - y - z + 1 = 0$, P_2: $-4x + 2y + 2z - 2 = 0$.

2.49. Compute the volume of the pyramid bounded by the plane P: $2x - 3y + 6z - 12 = 0$ and the coordinate planes.

2.50. Through the point $M_0(1, 7, -5)$ pass a plane which would intercept equal positive segments on the coordinate axes.

2.51. Three faces of a tetrahedron situated in the second octant ($x < 0$, $y > 0$, $z > 0$) coincide with the coordinate

planes. Write the equation of the fourth face if the lengths of the edges bounding this face are: $|\overline{AB}| = 6$, $|\overline{BC}| = \sqrt{29}$, $|\overline{AC}| = 5$ and find the length of the altitude $[OH]$ of the tetrahedron.

2.52. Write the equations of the planes bisecting the dihedral angle formed by the planes P_1 and P_2 if:

(a) P_1: $x - 3y + 2z - 5 = 0$, P_2: $3x - 2y - z + 3 = 0$;

(b) P_1: $2x - y + 5z - 3 = 0$, P_2: $2x - 10y + 4z - 2 = 0$.

2.53. Write the equation of the plane equidistant from the two given planes P_1 and P_2 if:

(a) P_1: $4x - y - 2z - 3 = 0$, P_2: $4x - y - 2z - 5 = 0$;

(b) P_1: $5x - 3y + z + 3 = 0$, P_2: $10x - 6y + 2z + 7 = 0$.

2.54. Find out whether the points $M_1(2, -1, 1)$ and $M_2(1, 2, -3)$ lie in one angle, in adjacent or vertical angles formed by the planes P_1 and P_2 if:

(a) P_1: $3x - y + 2z - 3 = 0$, P_2: $x - 2y - z + 4 = 0$;

(b) P_1: $2x - y + 5z - 1 = 0$, P_2: $3x - 2y + 6z - 1 = 0$.

2.55. Given the coordinates of the vertices of a tetrahedron: $A(2, 0, 0)$, $B(5, 3, 0)$, $C(0, 1, 1)$, and $D(-2, -4, 1)$. Write the equations of its faces.

2.56. Write the equation of the plane passing through the point $A(1, 1, -1)$ perpendicular to the planes $2x - y + 5z + 3 = 0$ and $x + 3y - z - 7 = 0$.

2.57. A straight line L is given by its general equations. Write its equations in the canonical and projection forms (see Example 4) if:

(a)
$$L: \begin{cases} 2x - y + 2z - 3 = 0, \\ x + 2y - z - 1 = 0; \end{cases}$$

(b)
$$L: \begin{cases} x + 2y - 3z - 5 = 0, \\ 2x - y + z + 2 = 0. \end{cases}$$

2.58. Write the canonical equations of the straight line passing through the point $M_0(2, 0, -3)$ and parallel to:

(a) the vector $\boldsymbol{q}(2, -3, 5)$;

(b) the straight line $\frac{x-1}{5} = \frac{y+2}{2} = \frac{z+1}{-1}$;

(c) the x-axis;

(d) the z-axis;

(e) the straight line $\begin{cases} 3x-y+2z-7=0, \\ x+3y-2z-3=0; \end{cases}$

(f) the straight line $x = -2 + t, y = 2t, z = 1 - (1/2)\,t$.

2.59. Write the equations of a straight line passing through the two given points M_1 and M_2 if:

(a) $M_1\,(1, -2, 1)$, $M_2\,(3, 1, -1)$;

(b) $M_1\,(3, -1, 0)$, $M_2\,(1, 0, -3)$.

2.60. Given a straight line L: $\frac{x-1}{2}=\frac{y}{1}=\frac{z+1}{0}$ and a point $M\,(0, 1, 2) \notin L$ (verify this!),

(a) write the equation of the plane passing through the straight line L and point M;

(b) write the equation of the plane passing through the point M perpendicular to the line L;

(c) write the equation of the perpendicular dropped from the point M on the line L;

(d) compute the distance $\rho\,(M, L)$;

(e) find the projection of the point M on the line L.

2.61. Given a plane P: $x + y - z + 1 = 0$ and a straight line L: $\frac{x-1}{0}=\frac{y}{2}=\frac{z+1}{1}$, and $L \notin P$ (check!),

(a) compute $\sin\widehat{(P, L)}$ and the coordinates of the point of intersection of the line and the plane;

(b) write the equation of the plane passing through the line L perpendicular to the plane P;

(c) write the equations of the projection of the line L on the plane P.

2.62. Let there be given two straight lines:

$$L_1: \frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

and

$$L_2: \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}.$$

Prove that the lines L_1 and L_2 lie in one plane if and only if the following condition is fulfilled:

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

2.63. Using the result obtained in Problem 2.62, make sure that the straight lines L_1 and L_2 belong to one plane, and write the equation of this plane:

(a) L_1: $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z-5}{4}$, L_2: $\frac{x-7}{3} = \frac{y-2}{2} \quad \frac{z-1}{-2}$;

(b) L_1: $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$, L_2: $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z+3}{-2}$.

2.64. Find the distance between two parallel straight lines

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z}{2} \quad \text{and} \quad \frac{x-7}{3} = \frac{y-1}{4} = \frac{z-3}{2}.$$

2.65. Find the distance from the point A (2, 3, —1) to the given straight line L:

(a)

$$\begin{cases} 2x-2y+z+3=0, \\ 3x-2y+2z+17=0; \end{cases}$$

(b)

$$\begin{cases} x=3t+5, \\ y=2t, \\ z=-2t-25. \end{cases}$$

2.66. Prove that the straight lines

$$L_1: \begin{cases} 2x+2y-z-10=0, \\ x-y-z-22=0 \end{cases} \text{ and } L_2: \frac{x+7}{3} = \frac{y-5}{-1} = \frac{z-9}{4}$$

are parallel and find the distance $\rho\,(L_1, L_2)$.

2.67. Write the equations of the straight line passing through the points of intersection of the plane $x - 3y + 2z + 1 = 0$ with the straight lines

$$\frac{x-5}{5} = \frac{y+1}{-2} = \frac{z-3}{-1} \quad \text{and} \quad \frac{x-3}{4} = \frac{y+4}{-6} = \frac{z-5}{2}.$$

2.68. For what value of λ will the plane $5x - 3y + z + 1 = 0$ be parallel to the straight line $\begin{cases} x - 4z - 1 = 0, \\ y - 3z + 2 = 0? \end{cases}$

2.69. Find the equations of the projection of the straight line $\frac{x-4}{3} = \frac{y+1}{-2} = \frac{z}{4}$ on the plane $x - 3y - z + 8 = 0$.

2.70. Determine the angle between the straight line

$$\begin{cases} x + y + z - 2 = 0, \\ 2x + y - z - 1 = 0 \end{cases}$$

and the plane passing through the points A (2, 3, —1), B (1, 1, 0), and C (0, —2, 1).

2.71. Through the point M_0 (7, 1, 0) and parallel to the plane $2x + 3y - z - 15 = 0$ draw a straight line to intersect the line

$$\frac{x}{1} = \frac{y-1}{4} = \frac{z-3}{2}.$$

2.72. Write the canonical equations of the straight line which passes through the point M_0 (3, —2, —4) parallel to the plane $3x - 2y - 3z - 7 = 0$ and intersects the straight line $\frac{x-2}{3} = \frac{y+4}{-2} = \frac{z-1}{2}$.

2.73. Prove that the distance between the skew lines L_1: $\boldsymbol{r}(t) = \boldsymbol{r}_1 + \boldsymbol{q}_1 t$ and L_2: $\boldsymbol{r}(t) = \boldsymbol{r}_2 + \boldsymbol{q}_2 t$ can be computed by the formula

$$\rho(L_1, L_2) = \frac{|(\boldsymbol{r}_2 - \boldsymbol{r}_1)\, \boldsymbol{q}_1 \boldsymbol{q}_2|}{|[\boldsymbol{q}_1, \boldsymbol{q}_2]|}.$$

In Problems 2.74 to 2.76 for the given straight lines L_1 and L_2 do the following:

(a) prove that they do not lie in one plane, i.e. that they are skew lines;

(b) write the equation of the plane passing through the straight line L_2 parallel to L_1;

(c) compute the distance between the straight lines;

(d) write the equation of the common perpendicular to the given straight lines L_1 and L_2. The initial data:

2.74. L_1: $\frac{x+7}{3} = \frac{y+4}{4} = \frac{z+3}{-2}$, L_2: $\frac{x-21}{6} = \frac{y+5}{-4} = \frac{z-2}{-1}$;

2.75. L_1: $\frac{x-6}{3}=\frac{y-3}{-2}=\frac{z+3}{4}$, L_2: $\frac{x+1}{3}=\frac{y+7}{-3}=\frac{z-4}{8}$;

2.76. L_1: $\frac{x}{1}=\frac{y-9}{4}=\frac{z+2}{-3}$, L_2: $\frac{x-2}{2}=\frac{y}{-2}=\frac{z+7}{9}$.

2.77. A cube $ABCDA'B'C'D'$ is given by its vertices: A (0, 0, 0), B (1, 0, 0), C (1, 1, 0), D (0, 1, 0), A' (0, 0, 1), B' (1, 0, 1), C' (1, 1, 1), D' (0, 1, 1). Do the following:

(a) write the equations of the straight lines $(A'C)$ and (BC');

(b) compute the distance between the straight lines $(A'C)$ and (BC');

(c) write the equation of the common perpendicular to the straight lines $(A'C)$ and (BC');

(d) write the equation of the plane passing through the points P, Q, and H, where P is the centre of the face $ABB'A'$, Q divides $\overline{BC'}$ in a ratio of $\frac{1}{3}$, and H is situated on the edge (BB') so that the length of the vector $\overline{PH}+\overline{HQ}$ is minimal;

(e) determine the angle between the plane obtained in (d) and the diagonal of the cube (BD').

SEC. 2.3.
PLANE CURVES

1. The Equation of a Plane Curve in a Rectangular Cartesian Coordinate System. A curve Γ is said to have in a coordinate system Oxy the equation

$$F(x, y) = 0, \tag{1}$$

if the following condition is fulfilled: a point $M(x, y)$ belongs to the curve Γ if and only if its coordinates x and y satisfy relationship (1). If, in particular, $F(x, y) = f(x) - y$, then equation (1) can be written in the form

$$y = f(x), \tag{2}$$

and in this case the curve Γ coincides with the graph of the function $f(x)$.

This section is dedicated to the relation between the geometrical properties of a curve and its equation in some of the simplest cases.

Example 1. Write the equation of the curve the sum of the squares of the distances from each point of which to the points A $(-a, 0)$, B $(0, a)$, and C $(a, 0)$ is equal to $3a^2$.

Let Γ be the curve satisfying the conditions of the problem; $M(x, y) \in \Gamma$ if and only if

$$\rho^2(M, A) + \rho^2(M, B) + \rho^2(M, C) = 3a^2,$$

or

$$(x + a)^2 + y^2 + x^2 + (y - a)^2 + (x - a)^2 + y^2 = 3a^2.$$

After simple transformations we obtain

$$x^2 + y^2 - \frac{2}{3}ay = 0,$$

or, separating a perfect square,

$$x^2 + \left(y - \frac{1}{3}a\right)^2 = \frac{a^2}{9}.$$

This is just the desired equation of the curve which is a circle of radius $a/3$ centred at the point $M_0(0, a/3)$.

In Problems 3.1 to 3.14 determine what curves are defined by the given equations, and construct these curves.

3.1. $x + |y| = 0$. **3.2.** $|x| + y - x = 0$.

3.3. $x^2 - xy = 0$. **3.4.** $xy + y^2 = 0$. **3.5.** $x^2 - y^2 = 0$.

3.6. $xy = 0$. **3.7.** $y^2 - 9 = 0$. **3.8.** $x^2 - x - 6 = 0$.

3.9. $x^2y - 7xy + 10y = 0$. **3.10.** $x^2 + y^2 = 4$.

3.11. $x^2 + (y + 3)^2 = 1$. **3.12.** $x^2 + 2y^2 = 0$.

3.13. $2x^2 + y^2 + 2 = 0$. **3.14.** $x^2 + |y^2 - 1| = 0$.

3.15. Write the equation of the curve each point of which is equidistant from the points $M_1(3, 2)$ and $M_2(2, 3)$.

3.16. Write the equation of the curve the difference of the squares of the distances from each point of which to the points $M_1(-a, 0)$ and $M_2(a, 0)$ is equal to c.

3.17. Write the equation of the curve the distance of each point of which to the x-axis is twice the distance to the y-axis.

3.18. Write the equation of the curve the sum of the squares of the distances of each point of which to the points $M_1(-3, 0)$ and $M_2(3, 0)$ is equal to 50.

3.19. Write the equation of the curve the distance from each point of which to the point $M_1(-1, 1)$ is half the distance to the point $M_2(-4, 4)$.

3.20. Write the equation of the curve the sum of the distances from each point of which to the points $F_1(-2, 0)$ and $F_2(2, 0)$ is equal to $2\sqrt{5}$.

3.21. Write the equation of the curve the modulus of the difference between the distances of each point of which to the points F_1 (—2, —2) and F_2 (2, 2) is equal to 4.

3.22. Write the equation of the curve each point of which is equidistant from the point F (2, 2) and from the x-axis.

3.23. Prove that each of the following equations defines a circle, find its centre C and radius R:

(a) $x^2 + y^2 - 4x + 6y - 3 = 0$;

(b) $x^2 + y^2 - 8x = 0$; (c) $x^2 + y^2 + 4y = 0$.

3.24. Write the equation of the circle in each of the following cases (designation: C is the centre of a circle, R its radius, M, M_1, M_2, M_3 points on the circles):

(a) C (2, —3), $R = 7$;

(b) M (2, 6), C (—1, 2);

(c) M_1 (3, 2), M_2 (—1, 6)—the end points of the circle;

(d) C (1, —1), the straight line $5x - 12y + 9 = 0$ is tangent to the circle;

(e) M (1, 2), the circle touches the coordinate axes;

(f) M_1 (3, 1), M_2 (—1, 3), $C \in L$: $3x - y - 2 = 0$;

(g)* M_1 (—1, 3), M_2 (0, 2), M_3 (1, —1).

3.25. Write the equation of the diameter of the circle $x^2 + y^2 + 4x - 6y - 17 = 0$ perpendicular to the straight line $5x + 2y - 13 = 0$.

3.26. Compute the shortest distance from the point M_0 to the circle Γ if:

(a) M_0 (6, —8), Γ: $x^2 + y^2 = 9$;

(b) M_0 (—7, 2), Γ: $x^2 + y^2 - 10x - 14y - 151 = 0$.

3.27. Determine the position of a straight line relative to a circle (intersects, touches, or passes outside of the circle) if the line and the circle are given by the indicated equations:

(a) $2x - y - 3 = 0$, $x^2 + y^2 - 3x + 2y - 3 = 0$;

(b) $x - 2y - 1 = 0$, $x^2 + y^2 - 8x + 2y + 12 = 0$;

(c) $x - y + 10 = 0$, $x^2 + y^2 - 1 = 0$.

2. Algebraic Curves of the Second Order (Degree). *An algebraic curve of the second order* (or *of the second degree*) is a plane curve Γ whose

equation in a Cartesian coordinate system has the form

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0, \tag{3}$$

where the coefficients A, B, and C are not all zero (otherwise, Γ is a straight line, i.e. an algebraic curve of the first order).

In the general case it may turn out that equation (3) defines a so-called *degenerate* curve (an empty set, a point, a straight line, a pair of straight lines).

But if a curve Γ is non-degenerate, then for it there can be found a rectangular Cartesian coordinate system in which the equation of this curve has one of the following three forms (the *canonical equation*):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \geqslant b > 0, \tag{4}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0, \tag{5}$$

$$y^2 = 2px, \quad p > 0, \tag{6}$$

the curve Γ being called the *ellipse*, *hyperbola*, or *parabola*, respectively. The coordinate system in which its equation has the form (4), (5) or (6) is termed a *canonical coordinate system* for the given curve.

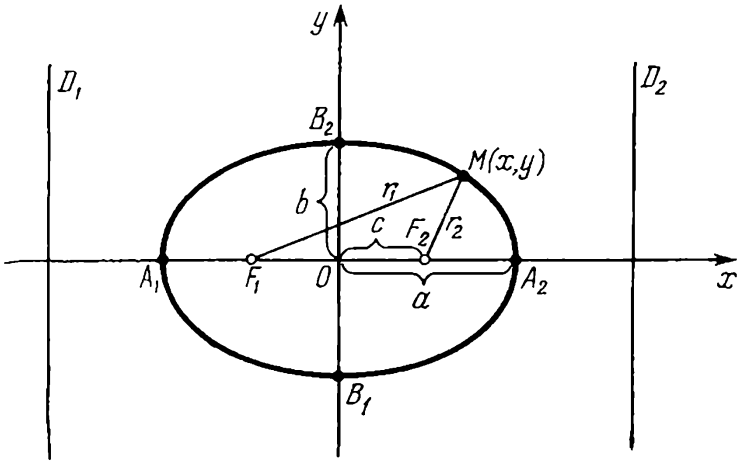

Fig. 13.

The reduction of the general equation of a second-order curve to the canonical form is considered in detail in Chapter 4 (Section 4.3, subsection 4). In the present subsection we are going to study the basic geometrical properties of non-degenerate curves of the second order on the basis of their canonical equations.

The *ellipse* defined by the canonical equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a \geqslant b > 0$, has the form represented in Fig. 13.

The parameters a and b are called the *semi-axes* of the ellipse (the semimajor and semiminor axes, respectively), the points A_1 $(-a, 0)$, A_2 $(a, 0)$, B_1 $(0, -b)$, and B_2 $(0, b)$ its *vertices*, the axes of symmetry

Ox, Oy its *principal axes*, and the centre of symmetry O is the *centre* of the ellipse.

The points $F_1(-c, 0)$ and $F_2(c, 0)$, where $c = \sqrt{a^2 - b^2} \geqslant 0$, are called the *foci* of the ellipse, the vectors $\overrightarrow{F_1M}$ and $\overrightarrow{F_2M}$ *focal radius vectors*, and the numbers $r_1 = |\overrightarrow{F_1M}|$ and $r_2 = |\overrightarrow{F_2M}|$ the *focal radii* of the point M belonging to the ellipse. In a particular case $a = b$ the foci F_1 and F_2 coincide with the centre, and the canonical equation has the form $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$, or $x^2 + y^2 = a^2$, i.e. describes a circle of radius a with centre at the origin.

The number $e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}$ $(0 \leqslant e < 1)$ is called the *eccentricity* of the ellipse and is the measure of its "flattening" (for $e = 0$ the ellipse is a circle).

The straight lines D_1: $x = -a/e$ and D_2: $x = a/e$, perpendicular to the minor axis and passing at a distance a/e from the centre, are called the *directrices* of the ellipse.

3.28. Construct the ellipse $9x^2 + 25y^2 = 225$. Find:
(a) its semi-axes;
(b) the coordinates of the foci;
(c) its eccentricity;
(d) the equations of the directrices.

3.29. Write the canonical equation of the ellipse if:
(a) $a = 3$, $b = 2$; (b) $a = 5$, $c = 4$; (c) $c = 3$, $e = 3/5$; (d) $b = 5$, $e = 12/13$; (e) $c = 2$ and the distance between the directrices is equal to 5; (f) $e = 1/2$ and the distance between the directrices is equal to 32.

3.30. Write the equation of an ellipse with the semi-axes a and b and centre at the point $C(x_0, y_0)$ if it is known that its axes are parallel to the coordinate axes.

3.31. Prove that each of the following equations defines an ellipse, find its centre C, semi-axes, eccentricity and the equations of its directrices:
(a) $5x^2 + 9y^2 - 30x + 18y + 9 = 0$;
(b) $16x^2 + 25y^2 + 32x - 100y - 284 = 0$;
(c) $4x^2 + 3y^2 - 8x + 12y - 32 = 0$.

3.32. Prove the following statements:
(a) If $M(x, y)$ is an arbitrary point of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a \geqslant b$, then the focal radii of this point are

$$r_1(M) = a + ex, \quad r_2(M) = a - ex$$

(see Fig. 13). Hence, in particular, it follows that for any point M of the ellipse the following equality is fulfilled:

$$r_1(M) + r_2(M) = \text{const} = 2a.$$

(b) Let there be given two points $F_1(-c, 0)$ and $F_2(c, 0)$, $c \geqslant 0$. Then the set of points M, satisfying the condition $|\overline{F_1M}| + |\overline{F_2M}| = \text{const} = 2a$, is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2 - c^2$.

3.33. Prove the following statements:

(a) If $M(x, y)$ is an arbitrary point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, $r_1(M)$ and $r_2(M)$ are the focal radii of this point, and $\rho(M, D_1)$ and $\rho(M, D_2)$ its distances to the directrices, then the following equality is fulfilled:

$$\frac{r_1(M)}{\rho(M, D_1)} = \frac{r_2(M)}{\rho(M, D_2)} = \text{const} = e.$$

(b) Let there be given a point $F(c, 0)$ and a straight line D: $x - d = 0$, $d > c > 0$. Then the set of points M, satisfying the condition $\frac{|\overline{FM}|}{\rho(M, D)} = \text{const} = e < 1$, is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = de$ and $b^2 = a^2 - c^2$.

3.34. An ellipse whose axes coincide with the coordinate axes passes through the points $M_1(2, \sqrt{3})$ and $M_2(0, 2)$. Write its equation, find the focal radii of the point M_1 and the distance of this point to the directrices.

3.35. On the ellipse $9x^2 + 25y^2 = 225$ find the point the distance from which to the focus F_2 is four times the distance to the focus F_1.

3.36. Write the equation of the curve in which a point M performs its motion if the sum of the distances from this point to the points $F_1(-1, -1)$ and $F_2(1, 1)$ remains constant and is equal to $2\sqrt{3}$.

3.37. Write the equation of the curve along which a point M moves if the distance from this point to the point $F(3, 0)$ remains half the distance to the straight line $x + y - 1 = 0$.

3.38. Determine the position of a straight line relative to an ellipse (intersects, touches or passes outside of it)

if the straight line and the ellipse are given by the respective equations:

(a) $2x-y-3=0$, $\frac{x^2}{16}+\frac{y^2}{9}=1$,

(b) $2x+y-10=0$, $\frac{x^2}{9}+\frac{y^2}{4}=1$,

(c) $3x+2y-20=0$, $\frac{x^2}{40}+\frac{y^2}{10}=1$.

3.39. Write the equation of the tangent line to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at its point $M_0(x_0, y_0)$.

Let first $y_0 \neq 0$, i.e. the point M_0 coincides with neither of its vertices $A_1(-a, 0)$ and $A_2(a, 0)$. In this case the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ defines implicitly the function $y=y(x)$, $-a<x<a$, whose graph passes through the point $M_0(x_0, y_0)$ and coincides with the appropriate half of the ellipse (the upper for $y_0>0$ and with the lower for $y_0<0$). Differentiating the identity $\frac{x^2}{a^2}+\frac{y^2(x)}{b^2}=1$, with respect to x, we obtain that the derivative $y'(x_0)$ is equal to

$$y'(x_0)=-\frac{b^2x_0}{a^2y_0}.$$

Hence the equation of the tangent line to the ellipse at the point $M_0(x_0, y_0)$ has the form

$$y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0),$$

or, taking into account the equality $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$,

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1. \tag{7}$$

And if $y_0=0$ (and, consequently, $x_0=\pm a$), then the equations of the tangent lines to the ellipse have the form $x=\pm a$, i.e. in this case formula (7) remains true as well.

3.40. Write the equations of the tangent lines to the ellipse $\frac{x^2}{10}+\frac{y^2}{5/2}=1$, parallel to the straight line $3x+2y+7=0$.

3.41. Write the equations of the tangent lines to the ellipse $x^2+4y^2=20$ perpendicular to the straight line $2x-2y-13=0$.

3.42. Prove that the tangent lines to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, drawn through the end points of one and the same diameter, are parallel.

3.43. Write the equations of the tangent lines drawn from the point $A\left(\frac{10}{3}, \frac{5}{3}\right)$ to the ellipse $\frac{x^2}{20}+\frac{y^2}{5}=1$.

3.44. On the ellipse $\frac{x^2}{18}+\frac{y^2}{8}=1$ find the point M_0 nearest to the straight line $2x-3y+25=0$ and compute the distance from the point M_0 to this line.

3.45. Prove that a tangent line to an ellipse at its arbitrary point M forms equal angles with the focal radius vectors $\overrightarrow{F_1M}$ and $\overrightarrow{F_2M}$ of this point.

3.46*. From the left-hand focus of the ellipse $\frac{x^2}{45}+\frac{y^2}{20}=1$ a light ray is directed at an obtuse angle of α to the x-axis ($\tan\alpha=-2$). Write the equation of the straight line on which the ray reflected from the ellipse lies.

The *hyperbola* with the canonical equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a, b>0$, has the form shown in Fig. 14.

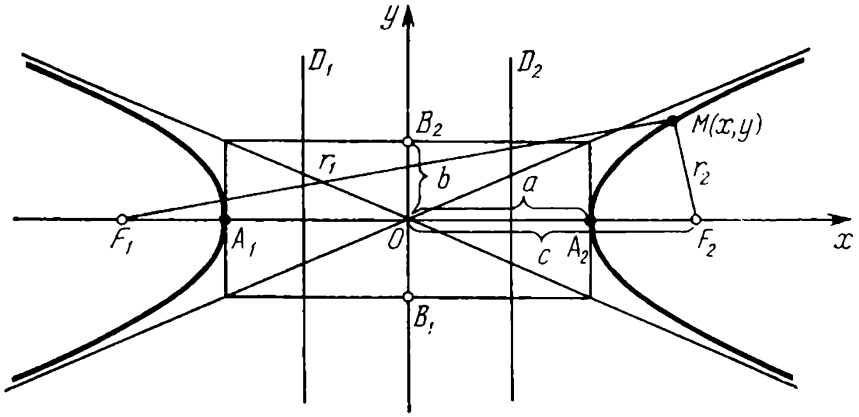

Fig. 14.

The parameters a and b are called the *semi-axes* of the hyperbola, the points $A_1(-a, 0)$ and $A_2(a, 0)$ its *vertices*, the axes of symmetry Ox and Oy are its axes: *transverse* (*real*) and *conjugate*, respectively, and the centre of symmetry O is the *centre* of hyperbola.

The straight lines $y=\pm\frac{b}{a}x$ are the asymptotes to the hyperbola.

The points F_1 $(-c, 0)$ and F_2 $(c, 0)$ where $c = \sqrt{a^2 + b^2} > 0$, are called its *foci*, the vectors $\overrightarrow{F_1M}$ and $\overrightarrow{F_2M}$ are known as *focal radius vectors*, and the numbers $r_1 = |\overrightarrow{F_1M}|$ and $r_2 = |\overrightarrow{F_2M}|$ are *focal radii* of the point M belonging to the hyperbola.

The number $e = \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}}$ $(1 < e < \infty)$ is called the *eccentricity* of the hyperbola and is the measure of its "flattening". In a particular case $a = b$ the hyperbola is called *rectangular* (or *equilateral*); its eccentricity is $e = \sqrt{2}$, and the angle between the asymptotes equals $\pi/2$.

The straight lines D_1: $x = -a/e$ and D_2: $x = a/e$, perpendicular to the real axis and passing at a distance of a/e from the centre are called the *directrices* of the hyperbola.

3.47. Construct the hyperbola $16x^2 - 9y^2 = 144$. Find:
(a) its semi-axes;
(b) the coordinates of the foci;
(c) its eccentricity;
(d) the equations of the asymptotes;
(e) the equations of the directrices.

3.48. Construct the hyperbola $16x^2 - 9y^2 = -144$ called *conjugate* to the hyperbola considered in Problem 3.47. What is the canonical coordinate system for this hyperbola? Find:
(a) its semi-axes;
(b) the coordinates of the foci;
(c) its eccentricity;
(d) the equations of the asymptotes;
(e) the equations of the directrices.

3.49. Write the canonical equation of the hyperbola if:
(a) $a = 2$, $b = 3$; (b) $b = 4$, $c = 5$; (c) $c = 3$, $e = 3/2$; (d) $a = 8$, $e = 5/4$; (e) $c = 10$ and the equations of the asymptotes are $y = \pm 4/3x$; (f) $e = 3/2$ and the distance between the directrices is equal to $8/3$.

3.50. Write the equation of the hyperbola with the semi-axes a and b and centre at the point C (x_0, y_0) if it is known that its real and conjugate axes are parallel to the axes Ox and Oy, respectively.

3.51. Show that each of the following equations defines a hyperbola. Find its centre, semi-axes, eccentricity, and the equations of its asymptotes and directrices:
(a) $16x^2 - 9y^2 - 64x - 54y - 161 = 0$;
(b) $9x^2 - 16y^2 + 90x + 32y - 367 = 0$;
(c) $16x^2 - 9y^2 - 64x - 18y + 199 = 0$.

3.52. Prove the following statements:

(a) If $M(x, y)$ is an arbitrary point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then the focal radii of this point are

$$r_1(M)=a+ex, \quad r_2(M)=-a+ex,$$

if the point M lies on the right-hand branch of the hyperbola, and

$$r_1(M)=-a-ex, \quad r_2(M)=a-ex,$$

if this point lies on its left-hand branch. Hence, in particular, it follows that for any point M of the hyperbola the following equality is fulfilled:

$$|r_1(M)-r_2(M)|=\text{const}=2a.$$

(b) Let there be given two points $F_1(-c, 0)$ and $F_2(c, 0)$, $c>0$. Then the set of points M satisfying the condition $||\overline{F_1M}|-|\overline{F_2M}||=\text{const}=2a$, $a>0$ is a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2=c^2-a^2$.

3.53. Prove the following statements:

(a) If $M(x, y)$ is an arbitrary point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $r_1(M)$ and $r_2(M)$ are the focal radii of this point, and $\rho(M, D_1)$, $\rho(M, D_2)$ are the distances from this point to the directrices, then the following equality is fulfilled:

$$\frac{r_1(M)}{\rho(M, D_1)}=\frac{r_2(M)}{\rho(M, D_2)}=\text{const}=e.$$

(b) Let there be given a point $F(c, 0)$ and a straight line D: $x-d=0$, $c>d>0$. Then the set of points M satisfying the condition $\frac{|\overline{FM}|}{\rho(M, D)}=\text{const}=e>1$, is a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a=de$ and $b^2=c^2-a^2$.

3.54. On making sure that the point $M(-5, 9/4)$ lies on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$, find the focal radii of this point and its distances to the directrices.

3.55. Find the points of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ at a distance of 7 cm from the focus F_1.

3.56. Write the equation of the hyperbola whose foci are the points F_1 $(-3, -4)$ and F_2 $(3, 4)$, and the distance between its directrices is equal to 3.6.

3.57. Write the equation of the hyperbola whose eccentricity $e = \sqrt{5}$, focus F $(2, -3)$ and the equation of the corresponding directrix is $3x - y + 3 = 0$.

3.58. Show that the curve given by the equation $xy = 1$ or $y = 1/x$ is an equilateral hyperbola. Write its canonical equation, find the eccentricity, foci and the equations of its directrices.

3.59*. Write the equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at its point M_0 (x_0, y_0).

3.60. Write the equations of the tangent lines to the hyperbola $\frac{x^2}{16} - \frac{y^2}{64} = 1$, parallel to the straight line $10x - 3y + 9 = 0$.

3.61. Write the equations of the tangent lines to the hyperbola $\frac{x^2}{20} - \frac{y^2}{5} = 1$, perpendicular to the straight line $4x + 3y - 7 = 0$.

3.62. Prove that the tangent lines to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, drawn through the end points of one and the same diameter are parallel.

3.63. Write the equations of the tangent lines drawn from the point A $(-1, -7)$ to the hyperbola $x^2 - y^2 = 16$.

3.64. On the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$, find the point M_0 nearest to the straight line $3x + 2y + 1 = 0$ and compute the distance from the point M_0 to this line.

3.65. Prove that a tangent line to a hyperbola at its arbitrary point M forms equal angles with the focal radius vectors $\overline{F_1M}$ and $\overline{F_2M}$ of this point.

3.66*. From the right-hand focus of the hyperbola $\frac{x^2}{5} - \frac{y^2}{4} = 1$ a light ray is directed to the x-axis at an angle α $(\pi < \alpha < (3/2)\,\pi)$ $(\tan\alpha = 2)$. Write the equation of the straight line on which the ray reflected from the hyperbola lies.

The *parabola* with the canonical equation $y^2 = 2px$, $p > 0$ has the form represented in Fig. 15.

The number p is called the *parameter* of the parabola, the point O its *vertex*, and the axis Ox the *axis* of the parabola.

The point F $(p/2, 0)$ is termed the *focus* of the parabola, the vector $\overline{FM}$ the *focal radius vector*, and the number $r = |\overline{FM}|$ is the *focal radius* of the point M of the parabola.

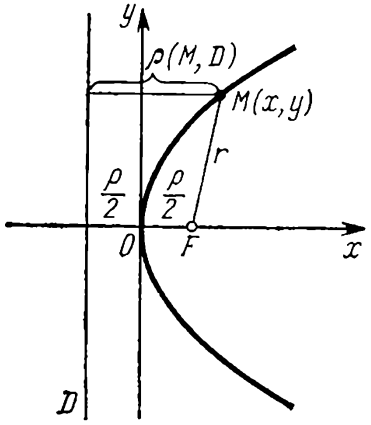

Fig. 15.

The straight line D: $x = -p/2$ perpendicular to the axis and passing at a distance $p/2$ from the vertex of the parabola is called its *directrix*.

3.67. Construct the following parabolas and find their parameters:

(a) $y^2 = 6x$;

(b) $x^2 = 5y$;

(c) $y^2 = -4x$;

(d) $x^2 = -y$.

3.68. Write the equation of the parabola with vertex at the origin if it is known that:

(a) the parabola is situated in the left-hand half-plane symmetrically about the x-axis and $p = 1/2$;

(b) the parabola is situated symmetrically about the y-axis and passes through the point M $(4, -8)$;

(c) its focus is found at the point F $(0, -3)$.

3.69. Write the equation of the parabola whose vertex is found at the point A (x_0, y_0), the parameter is equal to p, the axis is parallel to the x-axis and the parabola is situated about the straight line $x = x_0$:

(a) in the right-hand half-plane;

(b) in the left-hand half-plane.

3.70. Prove that each of the following equations defines a parabola; find the coordinates of its vertex A and the magnitude of the parameter p:

(a) $y^2 = 4x - 8$; (b) $x^2 = 2 - y$;

(c) $y = 4x^2 - 8x + 7$; (d) $y = -\frac{1}{6}x^2 + 2x - 7$;

(e) $x = -\frac{1}{4}y^2 + y$; (f) $x = 2y^2 - 12y + 14$.

3.71. Prove the following statements:

(a) If M (x, y) is an arbitrary point of the parabola $y^2 =$

$= 2px$, $r(M)$ its focal radius, and $\rho(M, D)$ is the distance from the point M to the directrix (see Fig. 15), then the following equality is fulfilled:

$$\frac{r(M)}{\rho(M, D)} = \text{const} = 1.$$

(b) Let there be given a point $F(p/2, 0)$ and a straight line D: $x = -p/2$. Then the set of points M satisfying the condition $\frac{|\overline{FM}|}{\rho(M, D)} = \text{const} = 1$ is a parabola $y^2 = 2px$.

3.72. Compute the focal radius of the point M of the parabola $y^2 = 12x$ if $y(M) = 6$.

3.73. Write the equation of the parabola if the following is known:

(a) the focus $F(4, 3)$ and the directrix D: $y + 1 = 0$;

(b) the focus $F(2, -1)$ and directrix D: $x - y - 1 = 0$.

3.74. Write the equation of the tangent line to the parabola $y^2 = 2px$ at its point $M_0(x_0, y_0)$.

3.75. Write the equation of the tangent line to the parabola $y^2 = 8x$ parallel to the straight line $2x + 2y - 3 = 0$.

3.76. Write the equation of the tangent line to the parabola $x^2 = 16y$ perpendicular to the straight line $2x + 4y + 7 = 0$.

3.77. Write the equations of the tangent lines to the parabola $y^2 = 36x$ drawn from the point $A(2, 9)$.

3.78. On the parabola $y^2 = 64x$ find the point M_0 nearest to the straight line $4x + 3y - 14 = 0$ and compute the distance from the point M_0 to this line.

3.79. Prove that a tangent line to a parabola at its arbitrary point M forms equal angles with the focal radius vector of the point M and the ray emanating from the point M in the direction of the axis of the parabola.

3.80. From the focus of the parabola $y^2 = 12x$ a light ray is directed to the x-axis at an acute angle α $\left(\tan\alpha = \frac{3}{4}\right)$. Write the equation of the straight line on which the ray reflected from the parabola lies.

3. The Equation of a Plane Curve in Polar Coordinates. A polar coordinate system $\langle O, u\rangle$ is said to be introduced in the plane if the following is specified:

(1) an arbitrary point O called the *pole*;

(2) a ray u emanating from the point O termed the *polar axis*.

The polar coordinates of a point $M \neq 0$ are defined as two numbers: the *radius vector* $r(M) = |\overrightarrow{OM}| > 0$ and the *polar* (or *vectorial*) *angle* $\varphi(M)$, i.e. an angle through which the axis u should be rotated to bring its direction to coincidence with the direction of the vector $\overrightarrow{OM}$ (as usual, $\varphi(M) > 0$ if the rotation is performed anticlockwise, and $\varphi(M) < 0$ if it is done clockwise). The notation $M(r, \varphi)$ means that the point M has the polar coordinates r and φ.

The polar angle $\varphi(M)$ has an infinitude of possible values (differing from one another by the quantity of the form $2\pi n$, $n \in \mathbb{Z}$). The value of the polar angle satisfying the condition $0 \leqslant \varphi < 2\pi$ is called *principal*. In some cases the principal value of the polar angle is known as the value of φ satisfying the condition $-\pi < \varphi \leqslant \pi$.

Let the pole of the polar system $\langle O, u \rangle$ coincide with the origin of a right-handed rectangular Cartesian coordinate system Oxy and let the polar axis coincide with the positive direction of the semi-axis of abscissas. Then the relations between the rectangular Cartesian and polar coordinates of an arbitrary point $M \neq 0$ are given by the transformation formulas

$$\begin{aligned} x &= r\cos\varphi, \quad y = r\sin\varphi; \\ r &= \sqrt{x^2+y^2}, \quad \tan\varphi = \frac{y}{x}. \end{aligned} \tag{7}$$

The equation of a curve in polar coordinates has the form $F(r, \varphi) = 0$ or $r = f(\varphi)$. It can be obtained either directly, proceeding from the geometrical properties of the curve, or by passing to polar coordinates in the equation of this curve given in rectangular Cartesian coordinates.

Example 2. Construct the curve given by the equation $r = 6\cos\varphi$.

First of all let us note the following: if a point $M(r, \varphi)$ belongs to the given curve, then for this point $\cos\varphi = \frac{1}{6}r \geqslant 0$, and, consequently, the entire curve is situated in the sector $-\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}$.

In order to construct the curve, let us pass to Cartesian coordinates in its equation. Multiplying both sides of the equation $r = 6\cos\varphi$ by r, we obtain $r^2 = 6r\cos\varphi$, whence, by virtue of formulas (7), we get $x^2 + y^2 = 6x$, or $(x-3)^2 + y^2 = 9$. Hence, the given curve is a circle of radius 3 centred at the point M_0 whose coordinates are $x_0 = 3$, $y_0 = 0$ or $r_0 = 3$, $\varphi_0 = 0$.

Example 3. Write the equation of a straight line in polar coordinates.

If a straight line L passes through the pole and its slope relative to the polar axis is equal to k, then the equation of this line has the form $\tan\varphi = k$.

Suppose now that a straight line L does not pass through the pole. We write the normal equation of this line in a rectangular Cartesian coordinate system:

$$x\cos\alpha + y\cos\beta - p = 0$$

and pass to polar coordinates in this equation. Taking into account that $\cos\beta = \sin\alpha$, we finally obtain:

$$r\cos\varphi\cos\alpha + r\sin\varphi\sin\alpha - p = 0,$$

$$r\cos(\varphi - \alpha) = p,$$

$$r = \frac{p}{\cos(\varphi - \alpha)}. \tag{8}$$

Equation (8) is just the desired equation of a straight line in polar coordinates. It can also be obtained directly from the following obvious fact: $M \in L \Leftrightarrow \mathrm{pr}_{\boldsymbol{n}}\boldsymbol{r} = r\cos(\varphi - \alpha) = \mathrm{const} = p$ (Fig. 16).

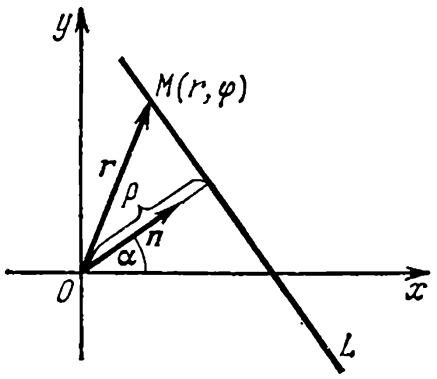

Fig. 16.

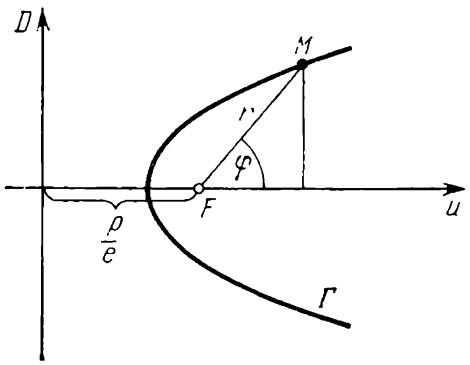

Fig. 17.

Example 4. Let Γ be an ellipse, a branch of a hyperbola or parabola, F the focus of this curve, and D the corresponding directrix. Write the equation of the curve Γ in the polar system of coordinates whose pole coincides with the focus and the polar axis is in the direction of the axis of the curve (Fig. 17).

The common property of the ellipse, hyperbola, and parabola consists in the following (see Problems 3.33, 3.53, and 3.71):

$$M \in \Gamma \Leftrightarrow \frac{\rho(M, F)}{\rho(M, D)} = \mathrm{const} = e, \tag{9}$$

where e is the eccentricity of the curve ($e < 1$ for the ellipse, $e > 1$ for hyperbola and $e = 1$ for parabola).

Let us denote the distance from the focus to directrix by p/e (p is the parameter of the curve called the semi-focal diameter). Then it follows from Fig. 17 that $\rho(M, F) = r$ and $\rho(M, D) = \frac{p}{e} + r\cos\varphi$. Substituting these expressions into (9), we obtain

$$\frac{r}{\frac{p}{e} + r\cos\varphi} = e,$$

whence

$$r = \frac{p}{1 - e \cos \varphi}. \tag{10}$$

Equation (10) is just the sought-for equation in polar coordinates common for the ellipse, hyperbola, and parabola.

In Problems 3.81 to 3.86 write the equations of the given curves in polar coordinates:

3.81. $y = x$. **3.82.** $y = 1$. **3.83.** $x + y - 1 = 0$.

3.84. $x^2 + y^2 = a^2$. **3.85.** $x^2 - y^2 = a^2$.

3.86. $x^2 + y^2 = ax$.

In Problems 3.87 to 3.98 write the equations of the given curves in rectangular Cartesian coordinates and construct these curves:

3.87. $r = 5$. **3.88.** $\tan \varphi = -1$. **3.89.** $r \cos \varphi = 2$.

3.90. $r \sin \varphi = 1$. **3.91.** $r = \dfrac{1/\sqrt{2}}{\cos\left(\varphi + \frac{\pi}{4}\right)}$.

3.92. $r = \dfrac{\sqrt{2}}{\sin\left(\varphi + \frac{\pi}{4}\right)}$. **3.93.** $r = 2a \cos \varphi$.

3.94. $r = 2a \sin \varphi$. **3.95.** $\sin \varphi = 1/\sqrt{5}$.

3.96. $\sin r = 1/2$. **3.97.** $r^2 \sin 2\varphi = 2a^2$.

3.98. $r^2 = a^2 \cos 2\varphi$.

3.99. Write in polar coordinates the equations of:

(a) the straight line perpendicular to the polar axis and cutting off on it a segment equal to 3 units;

(b) the ray emanating from the pole at an angle $\pi/3$ to the polar axis;

(c) the straight line passing through the pole at an angle $\pi/4$ to the polar axis.

3.100. Write in polar coordinates the equation of the given circle if:

(a) its radius $R = 5$, the circle passes through the pole, and its centre lies on the polar axis;

(b) its radius $R = 3$, and the circle touches the polar axis at the pole.

3.101. Determine the polar coordinates of the centre and the radius of each of the following circles:

(a) $r = 4\cos\varphi$; (b) $r = 3\sin\varphi$; (c) $r = -5\sin\varphi$;

(d) $r = 6\cos\left(\frac{\pi}{3} - \varphi\right)$; (e) $r = 8\sin\left(\varphi - \frac{\pi}{3}\right)$;

(f) $r = 8\sin\left(\frac{\pi}{3} - \varphi\right)$.

3.102. Write in polar coordinates the equation of a circle of radius R with centre at a point $C(r_0, \varphi_0)$.

3.103. For the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ write a polar equation assuming that the polar axis is in the same direction with the axis of abscissas and the pole is situated at:

(a) the left-hand focus;

(b) the right-hand focus.

3.104. Write a polar equation for the right-hand branch of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ knowing that the polar axis is in the same direction with the axis of abscissas, and the pole is found:

(a) in the left-hand focus;

(b) in the right-hand focus.

3.105. Write a polar equation for the parabola $y^2 = 6x$ assuming that the polar axis is in the direction of the axis of abscissas, and the pole is situated at the focus of the parabola.

3.106. Write the canonical equations of the following second-order curves:

(a) $r = \frac{9}{5 - 4\cos\varphi}$; (b) $r = \frac{9}{4 - 5\cos\varphi}$; (c) $r = \frac{3}{1 - \cos\varphi}$.

3.107. Derive the polar equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if the polar axis is in the direction of the x-axis, and the pole is situated at the centre of the ellipse.

3.108. Derive the polar equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if the polar axis is in the same direction with the x-axis and the pole is found at the centre of the hyperbola.

3.109. Derive the polar equation of the parabola $y^2 = 2px$ if the polar axis is in the same direction with the x-axis and the pole is found at the vertex of the parabola.

4. Parametric Equations of a Curve. Let there be given two functions $\varphi(t)$ and $\psi(t)$, continuous on some interval I of the number axis (I may be an open interval (a, b), a closed interval $[a, b]$, and also one of the half-open intervals $(a, b]$ or $[a, b)$, the cases when $a = -\infty$ and (or) $b = +\infty$ being not excluded). The equations

$$x = \varphi(t), \quad y = \psi(t), \quad t \in I, \tag{10}$$

are called the *parametric equations* of a curve Γ in rectangular Cartesian coordinates if the following condition is fulfilled: for any value of the *parameter* $t \in I$ the point $M(\varphi(t), \psi(t))$ belongs to the curve Γ and, vice versa, for any point $M(x, y)$ belonging to the curve Γ there exists a value of the parameter $t \in I$, such that $x = \varphi(t)$ and $y = \psi(t)$. By eliminating the parameter t from (10), the equation of the curve can be represented in the form $F(x, y) = 0$.

The parametric equations of a curve in polar coordinates are defined in a similar way.

Example 5. Show that the parametric equations

$$x = a \cos t, \quad y = a \sin t, \quad t \in [0, 2\pi),$$

define a circle $x^2 + y^2 = a^2$.

If a point $M(x, y)$ is such that $x = a \cos t$ and $y = a \sin t$ for some value of $t \in [0, 2\pi)$, then

$$x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2,$$

i.e. the point $M(x, y)$ belongs to the circle $x^2 + y^2 = a^2$.

The converse is also true: if a point $M(x, y)$ belongs to the circle $x^2 + y^2 = a^2$, then, setting $t = (\widehat{\overrightarrow{OM}, \vec{i}})$, $t \in [0, 2\pi)$, we get $x = a \cos t$ and $y = a \sin t$.

Example 6. A curve Γ is given by the polar equation $r = 2R \sin \varphi$. Write the parametric equations of this curve in polar and rectangular Cartesian coordinates, taking the polar angle φ as the parameter.

It is readily seen that the given curve is a circle of radius R with centre at the point $C(0, R)$. The parametric equations of this curve in polar coordinates are

$$r = 2R \sin t, \quad \varphi = t, \quad t \in [0, \pi).$$

The parametric equations in rectangular Cartesian coordinates are obtained if in the conversion formulas $x = r \cos \varphi$, $y = r \sin \varphi$ instead of r and φ their expressions in the form of the functions of the parameter t are substituted. Finally we obtain:

$$x = r(t) \cos \varphi(t) = R \sin 2t,$$

$$y = r(t) \sin \varphi(t) = R(1 - \cos 2t), \quad t \in [0, \pi).$$

3.110. A ray $\Gamma = \{(x, y) \mid x - y + 1 = 0, \ y \geqslant 0\}$ emanates from the point $M_0(-1, 0)$ (check this statement!). Write the parametric equations of this ray, taking as the parameter:

(a) the abscissa x;

(b) the ordinate y;

(c) the distance $\rho(M, M_0)$ from the point $M \in \Gamma$ to the vertex M_0 of the ray;

(d) the polar angle if the pole coincides with the origin, and the polar axis is in the same direction with the x-axis.

3.111. Write the parametric equations of a line segment whose end points are M_1 (1, 1) and M_2 (2, 3), taking as the parameter:

(a) the distance $\rho(M, M_1)$;

(b) the distance $\rho(M, M_2)$.

3.112. Write the parametric equations of the circle of radius R, with centre at the point $M_0(x_0, y_0)$, taking as the parameter t the angle between the x-axis and the vector $\overrightarrow{M_0M}$ measured anticlockwise.

3.113. Write the parametric equations of the circle $x^2 + y^2 = 2Rx$, taking the polar angle as the parameter if the polar axis is in the direction of the x-axis and the pole is found:

(a) at the origin;

(b) at the centre of the circle.

In Problems 3.114 to 3.122 find the equations of the given curves in the form $F(x, y) = 0$ by eliminating the parameter t and construct these curves.

3.114. $x = -1 + 2t$, $y = 2 - t$, $t \in (-\infty, +\infty)$.

3.115. $x = t^2 - 2t + 1$, $y = t - 1$, $t \in (-\infty, +\infty)$.

3.116. $x = -1 + 2\cos t$, $y = 3 + 2\sin t$, $t \in [0, 2\pi)$.

3.117. $x = a\cos t$, $y = b\sin t$, $t \in [0, 2\pi)$.

3.118. $x = 1 + 2\sec t$, $y = -1 + \tan t$, $t \in (-\pi/2, \pi/2)$.

3.119. $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$, $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$, $t \in (0, +\infty)$.

3.120. $x = 2R\cos^2 t$, $y = R\sin 2t$, $t \in [-\pi/2, \pi/2)$.

3.121. $x = R\sin 2t$, $y = 2R\sin^2 t$, $t \in [0, \pi)$.

3.122. $x = 2p\cot^2 t$, $y = 2p\cot t$, $t \in (0, \pi/2]$.

3.123. Write the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, taking as the parameter t the angle between the x-axis and radius vector $\overrightarrow{OM}$, as measured anticlockwise.

3.124. Write the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ taking as the parameter t the angle between the x-axis and radius vector $\overline{OM}$ as measured anticlockwise.

3.125. Write the parametric equations of the parabola $y^2 = 2px$, taking as the parameter:

(a) the ordinate y;

(b) the angle between the x-axis and the vector $\overline{OM}$ as measured anticlockwise;

(c) the angle between the x-axis and the focal radius vector $\overline{FM}$ as measured anticlockwise.

5. **Some Remarkable Curves.** The present subsection is of reference nature; it provides the reader with the equations and basic geometrical

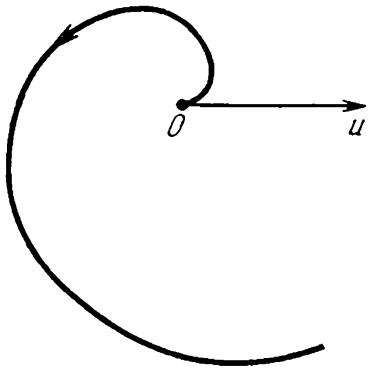

Fig. 18.

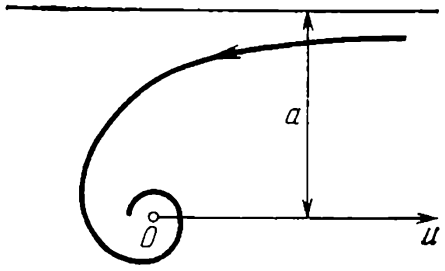

Fig. 19.

properties of a number of algebraic and transcendental curves encountered in applied mathematics as well as in practical engineering design and calculations. The derivation of the equations of these curves may serve as exercises in studying the advanced topics of the course of analytical geometry. A sufficiently profound study of the shape of the curves under consideration can be performed with the aid of the methods used in differential calculus.

1. *Spirals*: *Archimedes' spiral* $r = a\varphi$ (Fig. 18), *hyperbolic spiral* $r = \frac{a}{\varphi}$ (Fig. 19), *logarithmic spiral* $r = a^{\varphi}$ (Fig. 20); the arrow indicates the direction of traverse corresponding to an increase in φ.

2. *Lemniscate of Bernoulli* $(x^2 + y^2)^2 = 2a^2 (x^2 - y^2)$ (Fig. 21), or $r^2 = 2a^2 \cos 2\varphi$ (the pole is placed at the origin). The characteristic property: $| F_1M | \cdot | F_2M | = \text{const} = a^2$, where $F_1 (-a, 0)$, $F_2 (a, 0)$.

3. *Cissoid* $y^2 (2R - x) = x^3$ (Fig. 22), or $r = 2R \tan \varphi \sin \varphi$ (the pole is found at the point O). The characteristic property: $| OM | = | BC |$ for any ray emanating from the point O.

4. *Conchoid* $x^2y^2 + (x + a)^2 (x^2 - b^2) = 0$ (Fig. 23), or $r = \frac{a}{\cos \varphi} \pm b$ (with the pole at the point A). The characteristic prop-

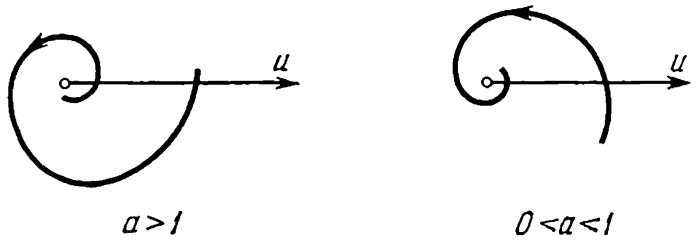

Fig. 20.

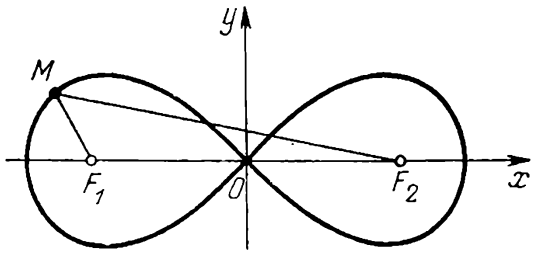

Fig. 21.

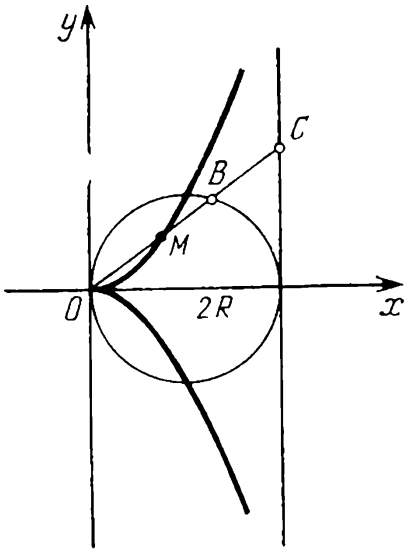

Fig. 22.

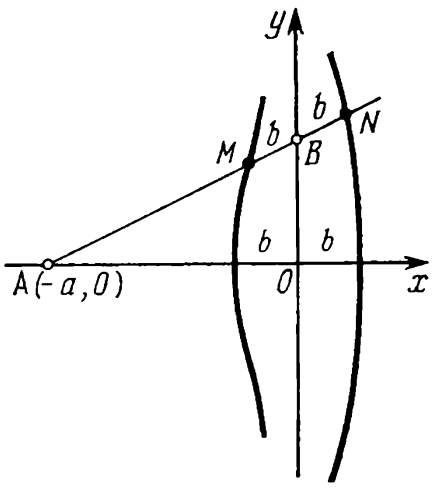

Fig. 23.

erty: $|BM| = |BN| = \text{const} = b$ for any ray emanating from the point A $(-a, 0)$.

5. *Strophoid* $x^2 ((x + a)^2 + y^2) = a^2y^2$ (Fig. 24), or $r = \frac{a}{\cos \varphi} \pm$ $\pm a \tan \varphi$ (the pole is found at the point A). The characteristic prop-

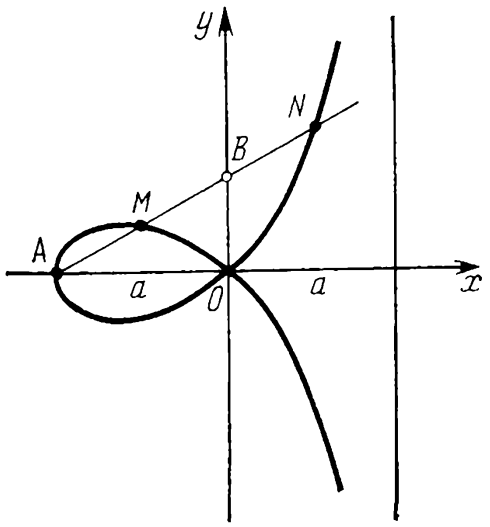

Fig. 24.

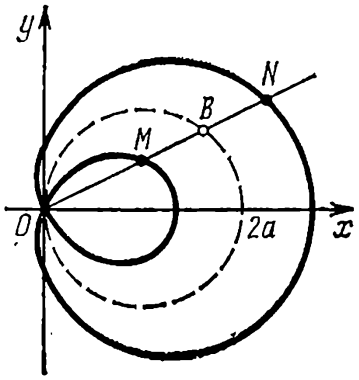

Fig. 25.

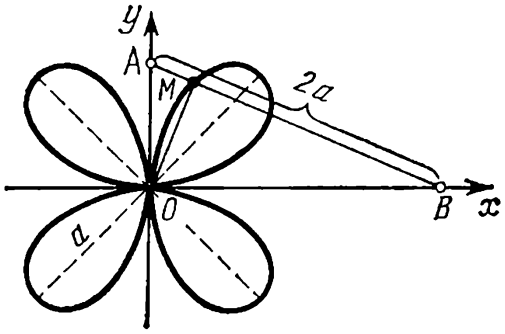

Fig. 26.

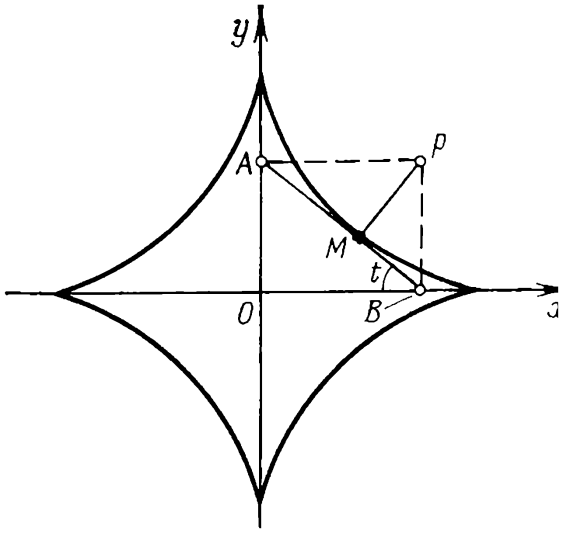

Fig. 27.

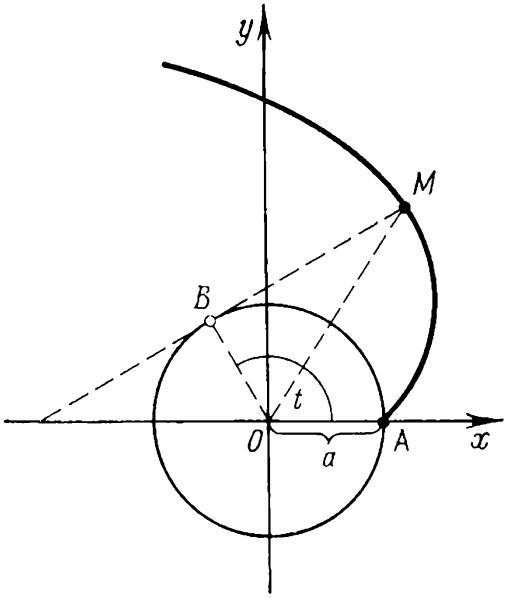

Fig. 28.

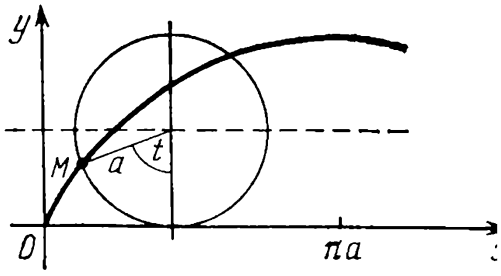

Fig. 29.

erty: $|BM| - |BN| = |OB|$ for any ray emanating from the point $A(-a, 0)$.

6. *Pascal's limacon* $(x^2+y^2-2ax)^2 = b^2(x^2+y^2)$ (Fig. 25), or $r = 2a\cos\varphi \pm b$ (with the pole at the point O). The characteristic property: $|BM| = |BN| = \text{const} = b$ for any ray emanating from the point O.

7. *Four-leafed rose* $(x^2+y^2)^3 = 4a^2x^2y^2$ (Fig. 26), or $r = -a|\sin 2\varphi|$ (the pole is at the point O). The characteristic property: any point M of this curve is the foot of a perpendicular dropped from the origin on the line segment $[AB]$ of constant length $2a$ which performs its motion so that its ends are found all the time on the coordinate axes.

8. *Astroid* $x = a\cos^3 t$, $y = a\sin^3 t$, $t \in [0, 2\pi)$, or $x^{2/3}+y^{2/3} = a^{2/3}$ (Fig. 27). The characteristic property: any point M of this curve is the foot of the perpendicular $[PM]$ to the line segment $[AB]$ of constant length a which moves so that its ends are found on the coordinate axes all the time.

9. *Evolute of a circle* $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$, $t \in [0, \infty)$ (Fig. 28). The characteristic property: any point M of this curve is the end of a thread which, all the time remaining tight, is unwound from the circle $x^2+y^2 = a^2$ (at the starting instant the end of the thread is found at the point $A(a, 0)$).

10. *Cycloid* $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in (-\infty, +\infty)$ (Fig. 29). The characteristic property: the curve coincides with the trajectory of a point M on a circle of radius a which rolls, without sliding, along the x-axis (at the starting instant the point M is found at the origin).

11. *Epicycloid* $x = (a+b)\cos t - a\cos\dfrac{a+b}{a}t$, $y = (a+b)\times$ $\times\sin t - a\sin\dfrac{a+b}{a}t$, $t \in [0, \infty)$ (Fig. 30). The characteristic property: the curve coincides with the trajectory of a point M on a circle of radius a as the circle rolls (without sliding) on the outside of a fixed circle $x^2+y^2 = b^2$ (at the starting instant the point M is found at the position $A(b, 0)$). In a particular case $a = b$ the corresponding curve is called the *cardioid*.

12. *Hypocycloid* $x = (b-a)\cos t + a\cos\dfrac{b-a}{a}t$, $y = (b -$ $-a)\sin t - a\sin\dfrac{b-a}{a}t$, $t \in [0, \infty)$ (Fig. 31). The characteristic property: the curve coincides with the trajectory of a point M on a circle of radius a as the circle rolls (without sliding) on the inside of a fixed circle $x^2+y^2 = b^2$ (at the starting moment the point M is situated at the position $A(b, 0)$). In a particular case $a = b/4$ this curve coincides with the astroid.

13. *Semicubical parabola* $y^2 = ax^3$ (Fig. 32).

14. *Loop-shaped parabola* $ay^2 = x(x-a)^2$ (Fig. 33).

15. *Witch of Agnesi* $y = \dfrac{8a^3}{x^2+4a^2}$ (Fig. 34).

16. *Folium of Descartes* $x^3+y^3-3axy = 0$ (Fig. 35).

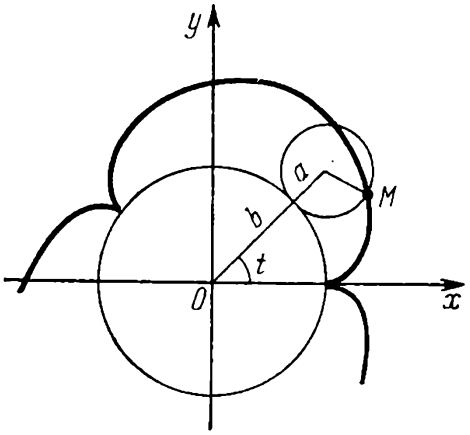

Fig. 30.

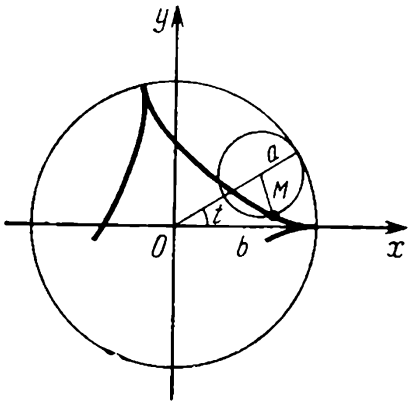

Fig. 31.

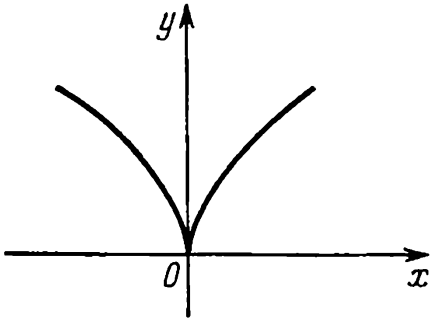

Fig. 32.

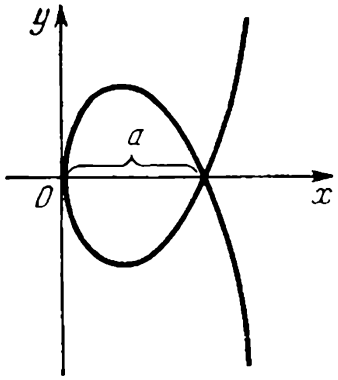

Fig. 33.

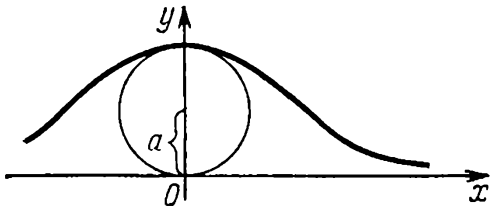

Fig. 34.

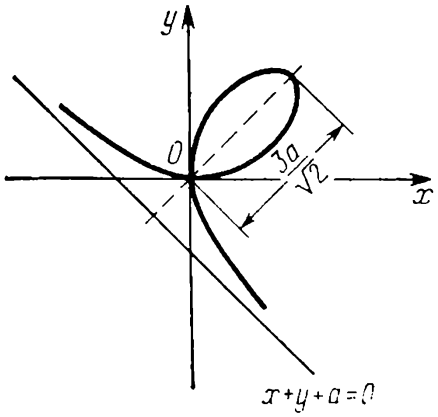

Fig. 35.

SEC. 2.4.
SURFACES AND CURVES IN SPACE

1. Equations of a Surface and a Curve in Rectangular Cartesian Coordinates. A surface S is said to have in a xyz-system of coordinates the equation

$$F(x, y, z) = 0 \tag{1}$$

if the following condition is fulfilled: a point $M(x, y, z)$ belongs to the surface S if and only if its coordinates x, y, and z satisfy relation (1). If, in particular, $F(x, y, z) = f(x, y) - z$, then equation (1) can be written in the form

$$z = f(x, y), \tag{2}$$

and in this case the surface S coincides with the graph of the function of two variables $f(x, y)$.

A curve Γ in space is defined in the general case as the line of intersection of some surfaces S_1 and S_2 (defined nonuniquely), i.e. by giving a system of two equations

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0. \tag{3}$$

Example 1. Write the equation of the surface each point of which is situated twice closer to the point A (2, 0, 0) than to the point B (—4, 0, 0).

If S is the surface specified by the conditions of the problem, then $M(x, y, z) \in S$ if and only if $\rho(M, B) = 2\rho(M, A)$, or

$$\sqrt{(x+4)^2 + y^2 + z^2} = 2\sqrt{(x-2)^2 + y^2 + z^2}.$$

Hence we obtain

$$(x+4)^2 + y^2 + z^2 = 4((x-2)^2 + y^2 + z^2),$$

$$3x^2 - 24x + 3y^2 + 3z^2 = 0$$

or, singling out a perfect square formed by the terms containing x, we get

$$(x-4)^2 + y^2 + z^2 = 16. \tag{4}$$

Equation (4) is just the desired equation of the surface. It is readily seen from this equation that the given surface S is a sphere of radius 4 with centre at the point M_0 (4, 0, 0).

Example 2. Investigate the shape of the curve Γ defined by the equations

$$\begin{cases} (x-1)^2 + y^2 + z^2 = 36, \\ y + z = 0. \end{cases} \tag{5}$$

Determine the form of its projection on the xy-plane.

The curve Γ is given as the line of intersection of the sphere $(x-1)^2 + y^2 + z^2 = 36$ and the plane $y + z = 0$ and, consequently, it is a circle. Since the centre of the sphere C (1, 0, 0) lies in the

cutting plane $y + z = 0$, the centre of the circle coincides with the point C, and its radius is equal to the radius of the sphere, i.e. $R = 4$.

Let us now determine the form of the projection of the circle l' on the xy-plane. Eliminating z from system (5), we obtain $(x - 1)^2 + 2y^2 = 36$, or $\frac{(x-1)^2}{36} + \frac{y^2}{18} = 1$. Hence we conclude that the sought-for projection is an ellipse whose axes are in the same directions with the x- and y-axes, respectively, its centre is found at the point C' (1, 0), and the semi-axes are: $a = 6$, $b = 3\sqrt{2}$.

In Problems 4.1 to 4.12 determine what geometrical images are defined by the given equations:

4.1. $z + 5 = 0$. **4.2.** $x - 2y + z - 1 = 0$.

4.3. $x^2 + y^2 + z^2 = 4$. **4.4.** $(x - 2)^2 + y^2 + (z + 1)^2 = 16$.

4.5. $2x^2 + y^2 + 3z^2 = 0$. **4.6.** $x^2 + 4z^2 = 0$.

4.7. $x^2 + 2y^2 + 2z^2 + 7 = 0$. **4.8.** $x^2 - 4z^2 = 0$.

4.9. $xz = 0$. **4.10.** $xyz = 0$.

4.11. $x^2 - 4x = 0$. **4.12.** $xy - y^2 = 0$.

4.13. Write the equation of a surface such that the difference of the squares of the distances of each of its points to the points F_1 (2, 3, —5) and F_2 (2, —7, —5) is equal to 13.

4.14. Write the equation of the surface the sum of the squares of the distances from each point of which to the points F_1 $(-a, 0, 0)$ and F_2 $(a, 0, 0)$ is equal to a constant number $4a^2$.

4.15. Write the equation of the surface the sum of the distances from each point of which to the points F_1 (0, 0, —4) and F_2 (0, 0, 4) is equal to 10.

4.16. Write the equation of the surface the modulus of the difference of the distances from each point of which to the points F_1 (0, —5, 0) and F_2 (0, 5, 0) is equal to 6.

4.17. Show that each of the following equations defines a sphere; find its centre C and radius R:

(a) $x^2 + y^2 + z^2 - 6z = 0$;

(b) $x^2 + y^2 + z^2 - 4x - 2y + 2z - 19 = 0$.

4.18. Write the equation of the sphere in each of the following cases (designation: C is the centre of the sphere, R is its radius, M, M_1, M_2, M_3 are points on the spheres):

(a) $C\ (-1,\ 2,\ 0),\ R = 2$;

(b) $M\ (2,\ -1,\ -3),\ C\ (3,\ -2,\ 1)$;

(c) $M_1\ (2,\ -3,\ 5)$ and $M_2\ (4,\ 1,\ -3)$ are the end points of the diameter of the sphere;

(d) $C\ (3,\ -5,\ -2)$, the plane $2x - y - 3z + 11 = 0$ touches the sphere;

(e) $M_1\ (3,\ 1,\ -3),\ M_2\ (-2,\ 4,\ 1),\ M_3\ (-5,\ 0,\ 0),\ C \in P$: $2x + y - z + 3 = 0$.

4.19. Write the equation of the sphere whose centre lies on the straight line

$$\begin{cases} 2x + 4y - z - 7 = 0, \\ 4x + 5y + z - 14 = 0 \end{cases}$$

and which touches the planes $x + 2y - 2z - 2 = 0$ and $x + 2y - 2z + 4 = 0$.

4.20. Write the equation of the sphere inscribed in a tetrahedron formed by the planes

$$3x - 2y + 6z - 8 = 0,\ x = 0,\ y = 0,\ z = 0.$$

4.21. Write the parametric equations of the diameter of the sphere $x^2 + y^2 + z^2 - 2x - 6y + z - 11 = 0$ perpendicular to the plane $5x - y + 2z - 17 = 0$.

4.22. On the sphere $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 25$ find the point M_0 nearest to the plane $3x - 4z + 19 = 0$, and compute the distance from this point to the plane.

4.23. Determine the position of the plane relative to the sphere (intersects, touches or passes outside of it) if the plane and the sphere are given by the respective equations:

(a) $z = 3,\ x^2 + y^2 + z^2 - 6x + 2y - 10z + 22 = 0$;

(b) $y = 1,\ x^2 + y^2 + z^2 + 4x - 2y - 6z + 14 = 0$;

(c) $x = 5,\ x^2 + y^2 + z^2 - 2x + 4y - 2z - 4 = 0$.

4.24. Find out what curves are defined by the following equations:

(a) $\begin{cases} x - 5 = 0, \\ z + 2 = 0; \end{cases}$ (b) $\begin{cases} x^2 + y^2 + z^2 = 49, \\ y = 0; \end{cases}$

(c) $\begin{cases} x^2 + y^2 + z^2 = 20, \\ z - 2 = 0; \end{cases}$ (d) $\begin{cases} x^2 + y^2 + z^2 = 49, \\ x^2 + y^2 + z^2 - \\ \quad - 4z - 25 = 0. \end{cases}$

4.25. Find the centre and the radius of the given circle:

(a) $\begin{cases} x^2 + y^2 + z^2 = 10y, \\ x + 2y + 2z - 19 = 0; \end{cases}$

(b) $\begin{cases} (x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 100, \\ 2x - 2y - z + 9 = 0. \end{cases}$

Hint. The centre of a sphere is the projection of the centre of a sphere on a plane.

4.26. Find the projection on the plane $z = 0$ of the section of the sphere $x^2 + y^2 + z^2 = 4(x - 2y - 2z)$ by the plane passing through the centre of the sphere and perpendicular to the straight line $x = 0$, $y + z = 0$.

4.27. The points A (3, —2, 5) and B (—1, 6, —3) are the end points of the diameter of a circle passing through the point C (1, —4, 1). Write the equations of this circle.

4.28. Write the equations of the circle passing through the three points: M_1 (3, —1, —2), M_2 (1, 1, —2), and M_3 (—1, 3, 0).

2. Algebraic Surfaces of the Second Order. *An algebraic surface of the second order* is defined as a surface S whose equation in rectangular Cartesian coordinates has the form

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + Gx + Hy + Iz + K = 0, \quad (6)$$

where not all the coefficients of the second-order terms are zero (otherwise S is an algebraic surface of the first order, i.e. a plane).

It may happen that equation (6) defines a so-called *degenerate* surface (an empty set, a point, a plane, a pair of planes, a straight line). But if a surface is *nondegenerate*, then, by transforming the rectangular Cartesian coordinate system, its equation (6) can be reduced to one of the below indicated forms called *canonical* and deter-

mining the type of surface.

1. *Ellipsoid:* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (Fig. 36).

2. *Hyperboloid*

(a) *of one sheet:* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (Fig. 37, *a*);

(b) *of two sheets:* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ (Fig. 37, *b*).

3. *Cone of the second order:* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ (Fig. 38).

4. *Paraboloid*

(a) *elliptic:* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$ (Fig. 39, *a*);

(b) *hyperbolic:* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$ (Fig. 39, *b*).

5. *Cylinder of the second order*

(a) *elliptic:* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (Fig. 40, *a*);

(b) *hyperbolic:* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (Fig. 40, *b*);

(c) *parabolic:* $y^2=2px,\ p>0$ (Fig. 40, *c*).

The general methods of reducing equation (6) to the canonical form are based on the theory of quadratic forms and are considered in Chapter 4, Section 4.3, subsection 4. In the present subsection we shall study the basic geometrical properties of nondegenerate surfaces of the second order on the basis of their canonical equations.

The *method of sections* is the leading technique for investigating the shape of a surface by its equation.

Example 3. Using the method of sections, investigate the shape and construct the surface given by the equation

$$z=2\left(1-\frac{x^2}{16}-\frac{y^2}{25}\right). \tag{7}$$

A horizontal plane $z = h$ cuts the given surface in a curve Γ_h whose projection on the xy-plane is defined by the equation

$$h=2\left(1-\frac{x^2}{16}-\frac{y^2}{25}\right),$$

or

$$\frac{x^2}{16}+\frac{y^2}{25}=2-h. \tag{8}$$

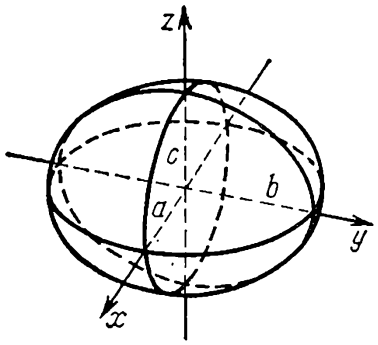

Fig. 36.

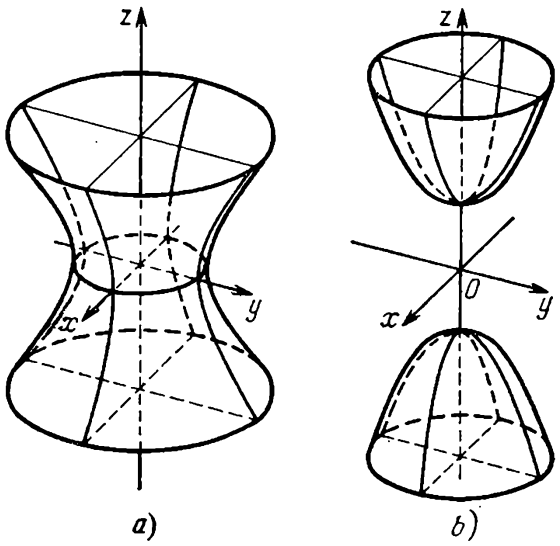

Fig. 37.

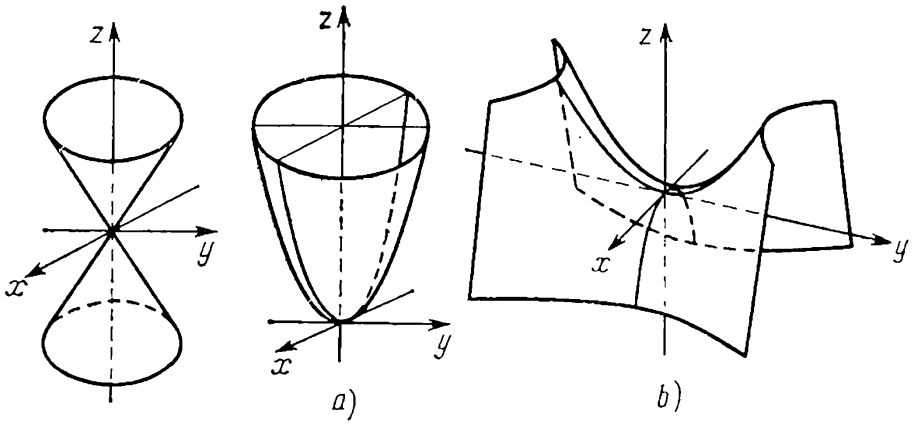

Fig. 38.

Fig. 39.

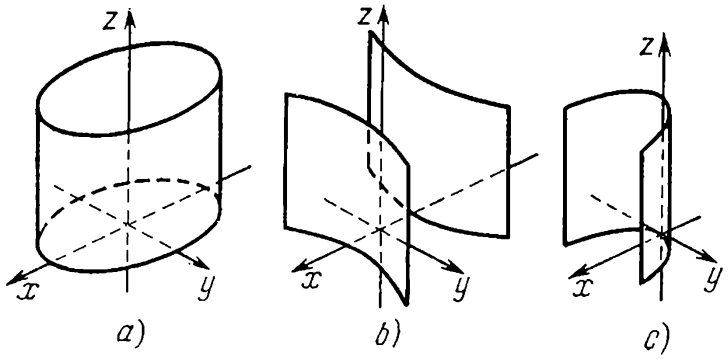

Fig. 40.

For $h > 2$ equation (8) has no solutions with respect to (x, y). This means that the corresponding section is empty, i.e. the surface under investigation is entirely situated below the plane $z = 2$. For $h \leqslant 2$ equation (8) defines an ellipse with semi-axes $a = 4\sqrt{2-h}$ and $b = 5\sqrt{2-h}$, which degenerates into a point $x = y = 0$ for $h = 2$. Note that all the ellipses cut off the given surface by the planes $x = h \leqslant 2$, are similar $\left(\frac{a}{b} = \text{const} = \frac{4}{5}\right)$ their semi-axes infinitely and monotonically increasing with a decrease in h.

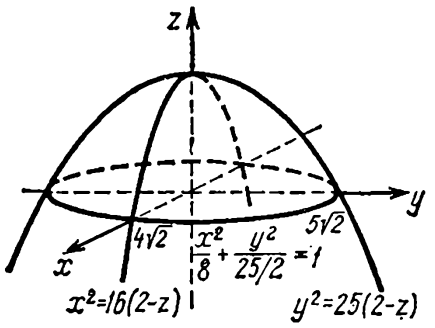

Fig. 41.

The obtained information is sufficient to sketch the surface. A more precise definition of its shape can be obtained by considering the sections by the coordinate planes Oxz and Oyz. The section by the xz-plane $(y = 0)$ yields the curve $x^2 = 16(2 - z)$, i.e. a parabola with the parameter $p = 8$, vertex at the point $x = 0$, $z = 2$ and branches directed towards decreasing values of z. Finally, the section by the yz-plane $(x = 0)$ yields a parabola $y^2 = 25(2 - z)$ with the parameter $p = \frac{25}{2}$, vertex at the point $y = 0$, $z = 2$ and similarly directed branches.

The above investigation enables us now to represent the given surface in more detail (see Fig. 41).

The given surface turns out to be an elliptic paraboloid. The transformation of the coordinates

$$x' = x, \quad y' = y, \quad z' = 2 - z$$

(which is reduced to the shift of the origin to the vertex of the paraboloid—the point (0, 0, 2)—and conversion of the direction of the z-axis) brings its original equation (7) to the canonical form

$$\frac{(x')^2}{16} + \frac{(y')^2}{25} = z'. \tag{9}$$

In Problems 4.29 to 4.40 determine the type of the given surfaces and construct them:

4.29. $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25} = 1$. **4.30.** $\frac{x^2}{16} + \frac{y^2}{4} - \frac{z^2}{36} = 1$.

4.31. $x^2 + y^2 - z^2 = -1$. **4.32.** $x^2 - y^2 = z^2$.

4.33. $x^2 + y^2 = 2az$, $a \neq 0$. **4.34.** $x^2 - y^2 = 2az$, $a \neq 0$.

4.35. $2z = x^2 + \frac{y^2}{2}$. **4.36.** $x^2 = 2az$, $a \neq 0$.

4.37. $z = 2 + x^2 + y^2$. **4.38.** $\frac{x^2}{5} - \frac{y^2}{4} = 6z$.

4.39. $x^2 + y^2 - z^2 = 4$. **4.40.** $x^2 - y^2 + z^2 + 4 = 0$.

4.41*. Prove that the equation $z^2 = xy$ defines a cone with vertex at the origin.

4.42*. Prove that the equation $z = xy$ defines a hyperbolic paraboloid.

4.43. Name and construct the following surfaces:

(a) $x^2 = 2yz$; (b) $z - a = xy$.

4.44. Write the equations of the projections on the coordinate planes of the section of the elliptic paraboloid $y^2 + z^2 = x$ by the plane $x + 2y - z = 0$.

4.45. What curves are defined by the following equations:

(a) $\begin{cases} \frac{x^2}{3} + \frac{y^2}{6} = 2z, \\ 3x - y + 6z - 14 = 0; \end{cases}$ (b) $\begin{cases} \frac{x^2}{4} - \frac{y^2}{3} = 2z, \\ x - 2y + 2 = 0? \end{cases}$

4.46. Find the points of intersection of the given surface and straight line:

(a) $\frac{x^2}{81} + \frac{y^2}{36} + \frac{z^2}{9} = 1$ and $\frac{x-3}{3} = \frac{y-4}{-6} = \frac{z+2}{4}$;

(b) $\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{4} = 1$ and $\frac{x}{4} = \frac{y}{-3} = \frac{z+2}{4}$;

(c) $\frac{x^2}{5} + \frac{y^2}{3} = z$ and $\frac{x+1}{2} = \frac{y-2}{-1} = \frac{z+3}{-2}$.

Hint. Pass to the parametric equations of the straight line.

4.47. Prove that in each of the below indicated cases the given surface and plane have a common point and find its coordinates:

(a) $\frac{x^2}{9} + \frac{z^2}{4} = 2y$, $2x - 2y - z - 10 = 0$;

(b) $\frac{x^2}{3} + \frac{y^2}{4} - \frac{z^2}{25} = -1$, $5x + 2z + 5 = 0$;

(c) $\frac{x^2}{81} + \frac{y^2}{36} - \frac{z^2}{9} = 1$, $4x - 3y + 12z - 54 = 0$.

4.48. Prove that the plane $2x - 12y - z + 16 = 0$ intersects the hyperbolic paraboloid $x^2 - 4y^2 = 2z$ along rulings (i.e. along straight lines lying entirely on this surface). Write the equations of these rulings.

4.49. Prove that the plane $4x - 5y - 10z - 20 = 0$ intersects the hyperboloid of one sheet $\frac{x^2}{25} + \frac{y^2}{16} - \frac{z^2}{4} = 1$ along rulings. Write the equations of these rulings.

3. Classification of Surfaces According to the Type of Transformations of Symmetry. According to the type of symmetry, surfaces are grouped into three classes: cylindrical, conical, and surfaces of revolution.

A *cylindrical surface* (a *cylinder*) is defined as a surface invariant with respect to transformations of translation $T(t\boldsymbol{q})$ defined by any vector collinear with some vector $\boldsymbol{q}(l, m, n)$. It follows from this definition that if the point $M_0(x_0, y_0, z_0)$ belongs to a cylinder S, then the entire straight line $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$ also belongs to this cylinder.

The following terminology is used here: any straight line collinear with the vector $\boldsymbol{q}(l, m, n)$ is called the *axis* of the cylinder S; the straight lines $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$, $M_0(x_0, y_0, z_0) \in S$, belonging entirely to the cylinder are called its *elements* (or *generatrices*); any curve Γ lying on the cylinder and intersecting all of its generatrices (elements) is termed the *directrix* of this cylinder.

Let $\boldsymbol{q}(l, m, n)$ be any arbitrary vector collinear with the axis of the cylinder S and the directrix Γ is given by the equations

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0.$$

Point $M(x, y, z)$ belongs to the cylinder S if and only if there exists a number t such that the point with the coordinates $x + tl$, $y + tm$, $z + tn$ lies on the generatrix Γ, i.e.

$$\begin{aligned} F_1(x + tl, y + tm, z + tn) &= 0, \\ F_2(x + tl, y + tm, z + tn) &= 0. \end{aligned} \tag{10}$$

Eliminating the parameter t from system (10), we obtain a relation of the form $F(x, y, z) = 0$ which is just the equation of the given cylinder.

Example 4. Write the equation of the cylinder whose axis coincides with the coordinate axis Oz (z-axis), and the directrix is given by the equations

$$F(x, y) = 0, \quad z - h = 0.$$

Setting $\boldsymbol{q} = \boldsymbol{k}(0, 0, 1)$, we obtain system (10) in the form $F(x, y) = 0$, $z + t - h = 0$. This result means that the point $M(x, y, z)$ belongs to the cylinder if and only if its coordinates x and y satisfy the equation $F(x, y) = 0$ for an arbitrary value of the z-coordinate. Consequently, the equation $F(x, y) = 0$ describing the projection of the directrix on the xy-plane is just the equation of the given cylinder.

In Problems 4.50 to 4.59 construct the given cylindrical surfaces:

4.50. $y^2+z^2=4$. **4.51.** $\frac{x^2}{16}-\frac{y^2}{9}=1$.

4.52. $x^2+y^2=ax$. **4.53.** $x^2=6z$. **4.54.** $z=4-x^2$.

4.55. $x^2-xy=0$. **4.56.** $x^2-z^2=0$.

4.57. $y^2+2z^2=0$. **4.58.** $xz=4$. **4.59.** $y^2+z^2=-z$.

4.60. Write the equations of the three cylindrical surfaces circumscribed about the sphere $x^2+y^2+z^2-2ax=0$:

(a) the axis of the cylindrical surface is parallel to the x-axis;

(b) the axis of the cylindrical surface is parallel to the y-axis;

(c) the axis of the cylindrical surface is parallel to the z-axis.

4.61. Write the equation of the cylinder projecting the circle

$$\begin{cases} x^2+(y+2)^2+(z-1)^2=25, \\ x^2+y^2+z^2=16 \end{cases}$$

on: (a) xy-plane; (b) xz-plane; (c) yz-plane.

4.62. Write the equation of the projection of the circle

$$\begin{cases} (x+1)^2+(y+2)^2+(z-2)^2=36, \\ x^2+(y+2)^2+(z-1)^2=25 \end{cases}$$

on: (a) xy-plane; (b) xz-plane; (c) yz-plane.

4.63. Write the equation of the surface each point of which is equidistant from the straight line $x=a$, $y=0$ and the yz-plane. Construct the surface.

4.64. Write the equation of the cylinder if:

(a) its axis is collinear with the vector $\boldsymbol{q}$ (1, 2, 3) and the directrix is given by the equations $y^2=4x$, $z=0$;

(b) its axis is collinear with the vector $\boldsymbol{q}$ (1, 1, 1) and the directrix is given by the equations $x^2+y^2=4x$, $z=0$.

4.65. The sphere $x^2+y^2+z^2=4z$ is illuminated by light rays parallel to the straight line $x=0$, $y=z$. Determine the shape of the shadow cast by the sphere on the xy-plane.

4.66. Construct the solid bounded by the surfaces $y^2=x$, $z=0$, $z=4$, $x=4$, and write the equation of the diagonals of the face lying in the plane $x=4$.

A *conical surface* (a *cone*) defined as a surface invariant to homothetic transformations (or transformations of similitude) $H(k, M_0)$ with an arbitrary ratio of similitude k and centre at some point $M_0(x_0, y_0, z_0)$ called the *vertex* of the cone. It follows from this definition that if a point $M_1(x_1, y_1, z_1)$ belongs to the cone, then the entire straight line $\frac{x-x_1}{x_1-x_0} = \frac{y-y_1}{y_1-y_0} = \frac{z-z_1}{z_1-z_0}$, passing through this point and vertex M_0 and called the *generatrix* of the cone, lies entirely on the cone. Any curve Γ lying on the cone and intersecting all of its elements (generatrices) is termed the *directrix* of this cone.

Let there be given a cone S with vertex $M_0(x_0, y_0, z_0)$ and directrix

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0.$$

A point $M(x, y, z)$ belongs to the cone S if and only if there exists a number t such that the point with the coordinates $x + t(x - x_0)$, $y + t(y - y_0)$, $z + t(z - z_0)$ lies on the generatrix Γ, i.e.

$$\begin{cases} F_1(x + t(x - x_0), y + t(y - y_0), z + t(z - z_0)) = 0, \\ F_2(x + t(x - x_0), y + t(y - y_0), z + t(z - z_0)) = 0. \end{cases} \tag{11}$$

Eliminating the parameter t from system (11), we obtain the equation of the cone in the form $F(x, y, z) = 0$.

Example 5. Write the equation of the cone whose vertex is situated at the point $M_0(x_0, y_0, z_0)$ and the directrix is given by the equations

$$F(x, y) = 0, \quad z - h = 0.$$

Under these conditions system (11) takes the form

$$\begin{cases} F(x + t(x - x_0), y + t(y - y_0)) = 0, \\ z + t(z - z_0) - h = 0. \end{cases}$$

From the second equation $t = \frac{h-z}{z-z_0} = \frac{(h-z_0)-(z-z_0)}{z-z_0} = \frac{h-z_0}{z-z_0} - 1$, which, after being substituted into the first equation, yields

$$F\left(x_0 + (h - z_0)\frac{x-x_0}{z-z_0},\ y_0 + (h-z_0)\frac{y-y_0}{z-z_0}\right) = 0. \tag{12}$$

Equation (12) is just the equation of the given cone. In a particular case $x_0 = y_0 = z_0 = 0$ (the vertex of the cone is found at the origin) the equation of the cone takes the form

$$F\left(h\frac{x}{z},\ h\frac{y}{z}\right) = 0. \tag{13}$$

Note that equation (13) is *homogeneous* with respect to x, y, and z (i.e. it remains unchanged when x, y, z are replaced by tx, ty, tz for an arbitrary $t \neq 0$), and equation (12) is homogeneous with respect to $x - x_0$, $y - y_0$, and $z - z_0$.

4.67. Let a function of three variables $F(x, y, z) = 0$ be homogeneous with respect to x, y, and z, i.e.

$$\forall t \neq 0 \quad \exists s \in \mathbb{R} \; (F(tx, ty, tz) = t^s F(x, y, z)).$$

Show that the equation $F(x, y, z) = 0$ defines a cone with vertex at the origin, and for any h the curve

$$F\left(\frac{x}{h}, \frac{y}{h}, 1\right) = 0, \quad z - h = 0$$

is its directrix.

4.68. Write the equation of the cone whose vertex is situated at the origin and the directrix is given by the equations:

(a) $\begin{cases} x^2 + y^2 = a^2, \\ z = h; \end{cases}$ (b) $\begin{cases} x^2 + (y-6)^2 + z^2 = 25, \\ y = 3; \end{cases}$

(c) $\begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \\ x = a; \end{cases}$ (d) $\begin{cases} a^2 - 2z + 1 = 0, \\ y - z + 1 = 0. \end{cases}$

Construct the obtained cones.

4.69. Write the equation of the cone given the coordinates of its vertex M_0 and the equations of its directrix:

(a) $M_0(0, -a, 0)$, $x^2 = 2py$, $z = h$;

(b) $M_0(0, 0, c)$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$;

(c) $M_0(0, -a, 0)$, $x^2 + y^2 + z^2 = a^2$, $y + z = a$;

(d) $M_0(3, -1, -2)$, $x^2 + y^2 - z^2 = 1$, $x - y + z = 0$.

Construct the relevant cones.

4.70. Construct the cone, determine its vertex and directrix in the plane $z = h$ if it is given by the equation:

(a) $x^2 + (y - h)^2 - z^2 = 0$;

(b) $x^2 = 2yz$.

4.71. Write the equation of the circular cone for which the coordinate axes serve as its generatrices (elements).

4.72. Write the equations of the projections of the line of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and cone $x^2 + y^2 - z^2 = 0$ on the coordinate axes:

(a) Oxy, (b) Oxz; (c) Oyz.

4.73. A source of light placed at the point M_0 (5, 0, 0) illuminates the sphere $x^2 + y^2 + z^2 = 9$. Determine the shape of its shadow on the yz-plane.

A *surface of revolution* is defined as a surface invariant with respect to rotations $R(\varphi, u)$ through any angle φ about some fixed axis u. This surface can be obtained by revolving about the axis u the curve obtained in the section of the surface by any plane passing through the axis of symmetry.

Example 6. Write the equation of the surface generated by revolving the curve $F(x, z) = 0$, $y = 0$ about the axis Oz (Fig. 42).

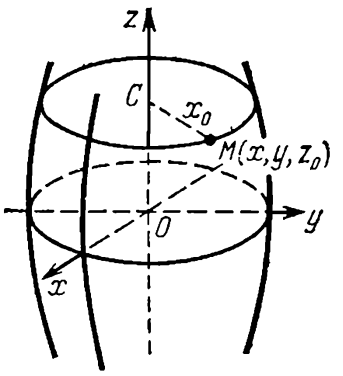

Fig. 42.

An arbitrary plane $z = z_0$ cuts the given surface in a circle of radius x_0 with centre at the point C (0, 0, z_0), and $F(x_0, z_0) = 0$. Therefore for an arbitrary point $M(x, y, z)$ of this circle we have: $z = z_0$ and $\rho(M, Oz) = \sqrt{x^2 + y^2} = x_0$. Substituting these equalities into the relation $F(x_0, z_0) = 0$, we get

$$F(\sqrt{x^2 + y^2}, z) = 0, \tag{14}$$

Equation (14) is just the desired equation of the given surface of revolution.

4.74. Write the equation of the surface generated by revolving the curve $z = x^2$, $y = 0$ about

(a) the z-axis; (b) the x-axis.

Construct both surfaces.

4.75. Write the equation of the surface generated by revolving the straight line $z = y$, $x = 0$ about

(a) the y-axis; (b) the z-axis.

Construct both surfaces.

4.76. Write the equation of the surface generated by revolving about the z-axis:

(a) the curve $z = e^{-x^2}$, $y = 0$; (b) the curve $z = \frac{4}{x^2}$, $y = 0$. Construct both surfaces in the left-handed coordinate system.

4.77. Show that the surface $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$ is a surface of revolution with axis Ox. Write the equation of the curve in the plane $z=0$ which generates this surface when being revolved.

4.78. Show that the surface $\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=1$ is a surface of revolution. Find its axis of symmetry and the equations of some curve which generates this curve when being revolved.

Answers

1.8. $\overline{AB}=\frac{2}{3}\,p-\frac{2}{3}\,q$, $\overline{BC}=\frac{2}{3}\,p+\frac{4}{3}\,q$, $\overline{CA}=-\frac{4}{3}\,p-\frac{2}{3}\,q$. **1.9.** $\overline{MA}=-\frac{1}{2}(a+b)$, $\overline{MB}=\frac{1}{2}(a-b)$, $\overline{MC}=-\overline{MA}$, $\overline{MD}=-\overline{MB}$. **1.10.** $AN=\alpha\beta a+(1-\beta)\,b$. **1.11.** $\overline{CD}=q-p$, $\overline{DE}=-p$, $\overline{EF}=-q$, $\overline{EA}=p-q$, $\overline{AC}=p+q$, $\overline{AD}=2q$, $\overline{AE}=2q-p$. **1.13.** $\overline{MM'}=\frac{1}{3}(\overline{AA'}+\overline{BB'}+\overline{CC'})$. **1.15.** $\overline{AB}=\frac{\lambda a-b}{1+\lambda}$, $\overline{BC}=\frac{a+b}{1+\lambda}$, $\overline{CD}=\frac{\lambda b-a}{1+\lambda}$, $\overline{DA}=-\frac{\lambda}{1+\lambda}(a+b)$. **1.16.** (a) $\alpha=\beta$; (b) $\overline{AB}=\frac{1}{\alpha-\beta}(\beta p-q)$, $\overline{AC}=\frac{1}{\alpha-\beta}(\alpha p-q)$. **1.18.** $\lambda=5$. **1.19.** $S=\frac{2}{5}\,p+\frac{3}{5}\,q+\frac{3}{5}\,r$. **1.20.** $3p-4q-3r-2s=0$. **1.21.** 0. **1.24.** 0, 1, 2. **1.25.** $\lambda=\mu=1$. **1.26.** (a) (−1/2, 1/2, 1/2); (b) (1/3, 1/3, 1/3). **1.27.** (7/10, 3/20, 3/20). **1.28.** (a) (1/2, 0, 1/2); (b) (1, −1/2, 1/2). **1.29.** $\left(1-\frac{1}{\lambda},\ -1\right)$. **1.30.** (a) $\left(\frac{1}{\alpha-1},\ \frac{1}{\beta-1}\right)$. $\overline{AO}=\overline{AM}+\overline{MO}=\alpha\overline{AB}-\overline{OM}=\alpha(\overline{AN}+\overline{NB})-\overline{OM}=\alpha(\overline{AO}+\overline{ON})+\alpha\overline{NB}-\overline{OM}=\alpha\overline{AO}+\alpha\overline{ON}+\alpha\lambda\overline{ON}-\overline{OM}$. Hence we find $\overline{AO}=\frac{\alpha(1+\lambda)}{1-\alpha}\times\overline{ON}-\frac{1}{1-\alpha}\,\overline{OM}$. Reasoning in a similar way, we obtain $\overline{AO}=\frac{\beta(1+\mu)}{1-\beta}\,\overline{OM}-\frac{1}{1-\beta}\,\overline{ON}$. Here $NB=\lambda ON$ and $MC=\mu OM$. By virtue of uniqueness of decomposition in a basis, we then have $\overline{AO}=\frac{1}{\alpha-1}\,\overline{OM}+\frac{1}{\beta-1}\,\overline{ON}$. (b) $\overline{AB}\left(\frac{1}{\alpha-1},\ \frac{1}{\alpha(\beta-1)}\right)$,

$\overline{BC}\left(\frac{1-\beta}{\beta(1-\alpha)}, \frac{\alpha-1}{\alpha(1-\beta)}\right)$, $\overline{CA}\left(\frac{1}{\beta(1-\alpha)}, \frac{1}{1-\beta}\right)$. *Hint.* Take advantage of the result obtained in Problem 1.30 (a). **1.31.** $\overline{AP}\left(\frac{\alpha(1-\gamma)}{1-\alpha\gamma}, \frac{\gamma(1-\alpha)}{1-\alpha\gamma}\right)$, $\overline{BQ}\left(\frac{2\alpha\beta-\alpha-\beta}{1-\alpha\beta}, \frac{\beta(1-\alpha)}{1-\alpha\beta}\right)$, $\overline{CQ}\left(\frac{\beta(1-\gamma)}{1-\beta\gamma}, \frac{2\beta\gamma-\beta-\gamma}{1-\beta\gamma}\right)$. **1.32.** (a) $\overline{AC}\left(\frac{1}{2}(\sqrt{5}+1), 1\right)$, $\overline{AD}\left(1, \frac{1}{2}(\sqrt{5}+1)\right)$. (b) $\overline{BC}\left(\frac{1}{2}(\sqrt{5}-1), 1\right)$, $\overline{CD}\left(\frac{1}{2}(\sqrt{5}+3), \frac{1}{2}(\sqrt{5}+3)\right)$, $\overline{DE}\left(-1, \frac{1}{2}(1-\sqrt{5})\right)$. **1.33.** (a) $\left(\frac{2}{5}, \frac{3}{5}\right)$, (b) $\gamma=-9/5$. **1.34.** $\overline{DM}$ (1/2, 1/2, −1), $\overline{AQ}$ (1/3, 1/3, 1/3). **1.35.** (a) $|\boldsymbol{a}_1| = \sqrt{5}$, $\boldsymbol{a}_{1,0}(-1/\sqrt{5}, 2/\sqrt{5}, 0)$; (b) $\cos\widehat{(\boldsymbol{a}_1, \boldsymbol{j})} = 2/\sqrt{5}$; (c) $X(\boldsymbol{a}) = -19/3$; (d) $\mathrm{pr}_{\boldsymbol{j}}\boldsymbol{a} = Y(\boldsymbol{a}) = 0$. **1.36.** $\boldsymbol{a} = -2\boldsymbol{e}$. **1.37.** $\boldsymbol{a} = -\frac{4}{5}\boldsymbol{e}_1 - \frac{2}{5}\boldsymbol{e}_2$. **1.38.** $\boldsymbol{a} = -2\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3$. **1.39.** (a) $\boldsymbol{a}_0(2/\sqrt{13}, 3/\sqrt{13}, 0)$; (b) $\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} + \boldsymbol{c} = \boldsymbol{d}$ (3, 11/2, 0); (c) $\boldsymbol{a} + \boldsymbol{b} - 2\boldsymbol{c} = -2\boldsymbol{j}$; (d) $\mathrm{pr}_{\boldsymbol{j}}(\boldsymbol{a}-\boldsymbol{b}) = 6$. **1.40.** (6/11, 7/11, −6/11). **1.41.** $\pm 3\sqrt{6}$. **1.42.** $|\boldsymbol{p}| = \sqrt{154}$, $\cos\alpha = \frac{9}{\sqrt{154}}$, $\cos\beta = \frac{8}{\sqrt{154}}$, $\cos\gamma = \frac{3}{\sqrt{154}}$. **1.43.** $\boldsymbol{x} = -5\boldsymbol{i} + 10\boldsymbol{j} + 10\boldsymbol{k}$. **1.44.** $\boldsymbol{x} = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$. **1.45.** $\boldsymbol{x} = \pm 5\boldsymbol{i} + \frac{5}{\sqrt{2}}\boldsymbol{j} - \frac{5}{\sqrt{2}}\boldsymbol{k}$. **1.46.** $\alpha = -1$, $\beta = 4$. **1.47.** $\boldsymbol{x} = \frac{5}{3}(\boldsymbol{i} + 7\boldsymbol{j} + 2\boldsymbol{k})$. *Hint.* $\boldsymbol{x} = \lambda(\boldsymbol{a}_0 + \boldsymbol{b}_0)$, where $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ are the unit vectors of the given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. **1.48.** $\alpha=2$, $\beta=3$, $\gamma=5$. **1.49.** $\left(\frac{ab^2c^2}{a^2b^2+b^2c^2+c^2a^2}, \frac{ba^2c^2}{a^2b^2+b^2c^2+c^2a^2}, \frac{ca^2b^2}{a^2b^2+b^2c^2+c^2a^2}\right)$. **1.50.** (a) (3, −6, 6); (b) (5, 5, 1); (c) $(-5/\sqrt{2}, 7/\sqrt{2}, 5)$; **1.51.** D (9, −5, 6). **1.52.** C (6, −2), D (2, −4). **1.53.** (−6, −2), (2, 8), (10, −6). **1.54.** M_1 (7, 0) and M_2 (−1, 0). **1.55.** M (0, 1, 0). **1.56.** 7. **1.57.** (4, 0) and (5, 2). **1.58.** (−1, 2, 4) and (8, −4, −2). **1.59.** (−19, 10, −17). *Hint.* Decompose the vector $\overline{OD}$ with respect to the basis formed by the vectors $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$. **1.60.** (10, −5, 0). *Hint.* Decompose the vector $\overline{OB}$ relative to the basis formed by the vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\overline{OA}$. **1.62.** $\sqrt{182}/3$. **1.63.** (11/7, 10/7, 18/7). **1.65.** (a) 9;

(b) -61; (c) 13. **1.66.** $\alpha = \pm 3/5$. **1.67.** 15, $\sqrt{593}$. **1.68.** $\frac{2\pi}{3}$. **1.69.** 19/5. **1.70.** 1/2. **1.71.** $\hat{A} = \frac{\pi}{2}$. **1.72.** $(\widehat{e_1, e_2}) = \pi/3$. **1.73.** $\alpha = \arccos \frac{4}{5}$. **1.74.** 5. **1.76.** $\overline{DC} = \frac{|a| - |b|}{|a|} a$, $\overline{CB} = \frac{|b|}{|a|} a - b$, $\overline{AC} = -\frac{|a| - |b|}{|a|} a + b$, $\overline{DB} = a - b$. *Hint.* First find the vector $\overline{AK}$ where K is a point of the base such that $|\overline{AK}| = |\overline{AB}|$. **1.77.** -13. **1.78.** (a) 22; (b) -200; (c) 41; (d) $\sqrt{105}$; (e) 11/3; (f) 22/7; (g) $\cos\alpha = 2/3$, $\cos\beta = -1/3$, $\cos\gamma = -2/3$; (h) $-84/\sqrt{129}$; (i) 11/21. **1.79.** $M_1(1, 0)$ and $M_2(6, 0)$. **1.80.** $|AB| = 5$, $|BC| = 5/\sqrt{2}$, $|AC| = 5$; $\hat{A} = \pi/2$, $\hat{B} = \hat{C} = \pi/4$. **1.81.** (a) $-41/5$; (b) 73/7. **1.83.** $\frac{15}{7\sqrt{85}}$. **1.84.** 4. **1.85.** $-4/5$. **1.87.** $\pi/6$. **1.88.** (1, 1/2, $-1/2$). **1.89.** (-3, 3, 3). **1.90.** $a_j = 2j$, $a_{i,k} = -i + k$. **1.91.** (a) (2/3, 2/3, 2/3); (b) ($-5/3$, 4/3, 1/3). **1.92.** (-2, 0, 2). **1.93.** $-i + \frac{1}{2}j - \frac{1}{2}k$. *Hint.* The vector a_{e_1, e_2}, has the form $a_{e_1, e_2} = \lambda_1 e_1 + \lambda_2 e_2$, where the coefficients λ_1 and λ_2 can be found from the condition of the perpendicularity of the vector $a - a_{e_1, e_2}$ to the plane of the vectors e_1 and e_2. **1.94.** $x' = (x - x_0)\cos\varphi - (y - y_0)\sin\varphi$, $y' = (x - x_0)\sin\varphi + (y - y_0)\cos\varphi$. **1.95.** $X' = X\cos\varphi + Y\sin\varphi$, $Y' = -X\sin\varphi + Y\cos\varphi$, $Z' = -Z$. **1.96.** $(-2, \sqrt{2}, 0)$. **1.97.** $a_1 a_2 = \sum\limits_{i, k=1}^{3} X_i^{(1)} X_k^{(2)} e_2 e_k = 5X_1^{(1)}X_1^{(2)} + 2X_2^{(1)}X_2^{(2)} + 9X_3^{(1)}X_3^{(2)} - 2(X_1^{(1)}X_2^{(2)} + X_2^{(1)}X_1^{(2)}) - 3(X_1^{(1)}X_3^{(2)} + X_3^{(1)}X_1^{(2)}) + 4(X_2^{(1)}X_3^{(2)} + X_3^{(1)}X_2^{(2)})$. **1.98.** (a) $\sqrt{3}$; (b) $3\sqrt{3}$; (c) $10\sqrt{3}$. **1.99.** $[a_1, a_2] = 0$, i.e. $a_1 \parallel a_2$. **1.100.** (a) $2(k - i)$; (b) $2[a, c]$; (c) $[a, c]$; (d) 3. **1.102.** $50\sqrt{2}$. **1.105.** (a) $\frac{1}{3}[a, b]$; (b) $-\frac{1}{3}[a, b]$. **1.106.** (a) (-3; 5, 7); (b) (-6, 10, 14); (c) (-12, 20, 28). **1.107.** $2\sqrt{6}$. **1.108.** 5. **1.109.** $\alpha = -6$, $\beta = 21$. **1.110.** $-\frac{103}{\sqrt{26}}$. **1.111.** $\sqrt{6}$. **1.112.** (-20, 7, -11). **1.113.** (5, 16, 7). **1.114.** $|a| = |b| = |c| = 1$; the vectors are pairwise perpendicular. **1.115.** $-4i + 3j + 4k$. **1.116.** $\sqrt{66}$; $\cos\alpha = 1/\sqrt{66}$, $\cos\beta = -4/\sqrt{66}$, $\cos\gamma = -7/\sqrt{66}$. **1.117.** $\frac{3}{2}\sqrt{2}$. **1.118.** (-6, -24, 8). **1.119.** (7, 5, 1). **1.120.** $a_2 \perp a_1$, an infinite set of solutions. **1.121.** ($-1/2$, 0, 1/2). **1.122.** The sign of the determinant will be reversed; in the case of an oblique basis the formula is incorrect. **1.123.** *Hint.* Compute the coordinates of the both members and get convinced that they are equal. It is

advisable to compute the coordinates in the following special basis: the unit vector i is in the direction of the vector b, the unit vector j lies in the plane of the vectors b and c. 1.124. 24. 1.125. —3/2. 1.126. —7. (a) Left-handed; (b) right-handed; (c) right-handed. 1.127. (a) No; (b) yes. 1.132. 17/2. 1.133. 6. 1.134. $3\sqrt{2}$. 1.135. (a) Yes; (b) no. 1.136. (a) —3; (b) for any λ. 1.138. (a) (0, 0, 0); (b) (0, 1, 0).

2.1. (a) $2(x+1)+2(y-2)=0$. General equation: $x+y-1=0$. Normal equation: $\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y-\frac{1}{\sqrt{2}}=0$; $p=\frac{1}{\sqrt{2}}$; (b) $2(x-2)=0$. General equation: $x-2=0$, the straight line is parallel to the y-axis. Normal equation: $x-2=0$; $p=2$; (c) $2(x-1)-(y-1)=0$. General equation: $2x-y-1=0$. Normal equation: $\frac{2}{\sqrt{5}}x-\frac{1}{\sqrt{5}}y-\frac{1}{\sqrt{5}}=0$; $p=\frac{1}{\sqrt{5}}$. 2.2. (a) $\frac{x+1}{3}=\frac{y-2}{-1}$. General equation: $x+3y-5=0$. Normal equation: $\frac{1}{\sqrt{10}}x-\frac{3}{\sqrt{10}}y-\frac{5}{\sqrt{10}}=0$; $p=\frac{5}{\sqrt{10}}$. (b) $\frac{x-1}{0}=\frac{y-1}{-1}$. General equation: $-x+1=0$, the straight line is parallel to the y-axis. Normal equation: $x-1=0$; $p=1$. (c) $\frac{x+1}{2}=\frac{y-1}{0}$. General equation: $y-1=0$, the straight line is parallel to the x-axis. Normal equation: $y-1=0$; $p=1$. 2.3. (a) $\frac{x-1}{-2}=\frac{y-2}{-2}$. General equation: $x-y+1=0$. Normal equation: $-\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y-\frac{1}{\sqrt{2}}=0$; $p=\frac{1}{\sqrt{2}}$; (b) $\frac{x-1}{0}=\frac{y-1}{-3}$. General equation: $x-1=0$, the straight line is parallel to the y-axis. Normal equation: $x-1=0$; $p=1$; (c) $\frac{x-2}{-2}=\frac{y-2}{0}$. General equation: $y-2=0$, the straight line is parallel to the x-axis. Normal equation: $y-2=0$, $p=2$. 2.4. (a) $\rho(M, L)=\frac{3}{\sqrt{5}}$, L': $\frac{x+1}{-2}=\frac{y-2}{1}$, L'': $-2(x+1)+(y-2)=0$; (b) $\rho(M, L)=1/2$, L': $\frac{x-1}{0}=\frac{y}{2}$, L'': $2y=0$; (c) $\rho(M, L)=0$, L': $\frac{x}{1}=\frac{y+1}{1}$, L'': $x+y+1=0$. 2.5. The lines intersect at the point $M_0(-3/4, -1/2)$; $\cos(\widehat{L_1, L_2})=1/\sqrt{5}$. 2.6. The lines intersect at the point $M_0(1, 0)$; $\cos(\widehat{L_1, L_2})=2/\sqrt{5}$. 2.7. The lines are parallel: $\rho(L_1, L_2)=\sqrt{2}/4$. 2.8. The lines are parallel; $\rho(L_1, L_2)=\sqrt{2}$. 2.9. The lines coincide. 2.10. (a) (AB):

$\frac{x-1}{1}=\frac{y-2}{-4}$, (CD): $\frac{x-6}{-4}=\frac{y-1}{-1}$, $h=\frac{19}{\sqrt{17}}$, $\cos\varphi=$ $=\frac{19}{\sqrt{17\cdot 58}}$, L_1: $\frac{x-1}{\sqrt{26}+5\sqrt{17}}=\frac{y-2}{-4\sqrt{26}-\sqrt{17}}$, L_2: $(\sqrt{26}+$ $+5\sqrt{17})(x-1)+(-4\sqrt{26}-\sqrt{17})(y-2)=0$; (b) (AB): $\frac{x-2}{4}=\frac{y+3}{3}$, (CD): $\frac{x+2}{3}=\frac{y}{-4}$, $h=4$, $\cos\varphi\frac{1}{\sqrt{10}}$, L_1: $\frac{x-2}{4-2\sqrt{5}}=\frac{y+2}{3+\sqrt{5}}$, L_2: $(4-2\sqrt{5})(x-2)+(3+\sqrt{5})\times$ $\times(y+2)=0$. **2.11.** $t=-1/2$. **2.12.** $4/\sqrt{5}$. **2.13.** $x+1=0$, $y-2=0$. **2.14.** $13x+y-11=0$, $15x-y-13=0$. **2.15.** $3x-$ $-y-1=0$, $3x-y-21=0$. **2.17.** $y=2x-6$, $y=-2x+6$. **2.18.** $3x+y\pm 2=0$, $x-3y\pm 6=0$. **2.19.** 20, $-\frac{4}{5}$. **2.20.** $7x-y-25=0$. **2.21.** $3x-5y+47=0$, $3x-5y-11=0$, $3x+5y+17=0$, $3x+5y-41=0$. **2.22.** $3x-2y-12=0$, $3x-8y+24=0$. **2.23.** (a) $3x-2y-7=0$; (b) $x-5y-7/6=$ $=0$. **2.24.** $x-5y+3=0$, or $5x+y-11=0$. **2.25.** B (1, 1), D (−1, 3); (AB): $x-1=0$, (BC): $y-1=0$; (CD): $x+1=0$, (AD): $y-3=0$. **2.26.** $3x-y+1=0$, $x+3y+7=0$, $3x-$ $-y+11=0$. **2.27.** (AB): $3x+4y+1=0$, (BC): $4x-3y-7=$ $=0$, (CD): $3x+4y-24=0$, (AD): $4x-3y-32=0$, (AC): $7x+y-31=0$. **2.28.** $x-2y-7=0$, $x-4y-1=0$, $x-$ $-y+2=0$. **2.29.** *Hint.* The deviations $\delta(M_1, L)$ and $\delta(M_2, L)$ have opposite signs. **2.30.** $4x+y+5=0$ or $y-3=0$. **2.31.** (a) In one angle; (b) in vertical angles. **2.32.** Obtuse. **2.33.** $4x-3y+$ $+10=0$, $7x+y-20=0$, $3x+4y-5=0$. **2.34.** $x-3y-$ $-23=0$, $7x+9y+19=0$, $4x+3y+13=0$. **2.35.** $2x+9y-$ $-65=0$, $6x-7y-25=0$, $18x+13y-41=0$. **2.36.** $x-6y+$ $+17=0$, $8x+3y-17=0$, $7x+9y+17=0$. **2.37.** $4x-3y=$ $=0$, $12x+5y+16=0$. **2.38.** (b) H (3, 4), M (−1/3, 22/3), N (−2, 9), $\lambda=-2/3$. **2.40.** (a) $2x-y+z-2=0$; $1/\sqrt{6}$; (b) $x-y=0$, the plane is parallel to the z-axis and passes through the origin; $1/\sqrt{2}$. **2.41.** (a) $x+y-3=0$; (b) $x+2y-2=0$. **2.42.** (a) $x-2y-z=0$; (b) $-x+y+2z-5=0$. **2.43.** (a) $-x+$ $+2y-3z-3=0$; (b) $2x-2y-z+1=0$. **2.44.** (a) $x+y-$ $-3=0$; (b) $2x-y-1=0$. **2.45.** The planes intersect, $\cos\widehat{(P_1, P_2)}=\frac{1}{2\sqrt{15}}$. **2.46.** The planes are parallel, $\rho(P_1, P_2)=$ $=\frac{3}{2\sqrt{6}}$. **2.47.** The planes intersect, $\cos\widehat{(P_1, P_2)}=1/2$. **2.48.** The planes coincide. **2.49.** 8. **2.50.** $x+y+z-3=0$. **2.51.** $3\sqrt{5}x-$ $-6y-4\sqrt{5}z+12\sqrt{5}=0$, $\frac{4\sqrt{5}}{\sqrt{161}}$. **2.52.** (a) $4x-5y+z-2=0$

and $2x + y - 3z + 8 = 0$; (b) $3x - 6y + 7z - 4 = 0$ and $x + 4y + 3z + 4 = 0$. **2.53.** (a) $4x - y - 2z - 4 = 0$; (b) $20x - 12y + 4z + 13 = 0$. **2.54.** (a) In adjacent angles; (b) in one angle. **2.55.** $x - y + 3z - 2 = 0$, $x - y - 2 = 0$, $5x - 2y + 12z - 10 = 0$, $5x - 2y + 21z - 19 = 0$. **2.56.** $2x - y - z - 2 = 0$. **2.57.** (a) $q = [n_1, n_2] = -3i + 4j + 5k$, the equations in projection form: $\begin{cases} 4x+3y-5=0, \\ 5x+3z-7=0, \\ 5y-4z+1=0; \end{cases}$ (b) $q = [n_1, n_2] = -i - 7j - 5k$, the equations in projection form: $\begin{cases} 7x-y+1=0, \\ 5x-z-1=0, \\ 5y-7z-12=0. \end{cases}$ **2.58.** (a) $\frac{x-2}{2} = \frac{y}{-3} = \frac{z+3}{5}$; (b) $\frac{x-2}{5} = \frac{y}{2} = \frac{z+3}{-1}$; (c) $\frac{x-2}{2} = \frac{y}{0} = \frac{z+3}{0}$; (d) $\frac{x-2}{0} = \frac{y}{0} = \frac{z+3}{1}$; (e) $\frac{x-2}{-4} = \frac{y}{6} = \frac{z+3}{10}$; (f) $\frac{x-2}{1} = \frac{y}{2} = \frac{z+3}{-1/2}$. **2.59.** (a) $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-1}{-2}$; (b) $\frac{x-3}{-2} = \frac{y+1}{1} = \frac{z}{-3}$. **2.60.** (a) $x - 2y + z = 0$; (b) $2x + y - 1 = 0$; (c) $\begin{cases} x-2y+z=0, \\ 2x+y-1=0, \end{cases}$ or $\frac{x}{-1} = \frac{y-1}{2} = \frac{z-2}{5}$; (d) $18/\sqrt{30}$; (e) $M'(3/5, -1/5, -1)$. **2.61.** (a) $1/\sqrt{15}$, $M(1, -6, -4)$, (b) $3x - y + 2z - 1 = 0$; (c) $\begin{cases} x+y-z+1=0, \\ 3x-y+2z=1=0. \end{cases}$ **2.63.** (a) $2x - 16y - 13z + 31 = 0$; (b) $6x - 20y - 11z + 1 = 0$. **2.64.** 3. **2.65.** (a) $6/\sqrt{5}$, (b) 21. **2.66.** 25. **2.67.** $\frac{x+1}{7} = \frac{y-2}{-1} = \frac{z-3}{-5}$. **2.68.** -11. **2.69.** $\frac{x-15}{4} = \frac{y}{-1} = \frac{z-23}{7}$. **2.70.** $\arcsin 3/2\sqrt{7}$. **2.71.** $\frac{x-7}{67} = \frac{y-1}{-28} = \frac{7}{70}$. **2.72.** $\frac{x-3}{5} = \frac{y+2}{-6} = \frac{z+4}{9}$. **2.74.** (b) $4x+3y+12z-93=0$; (c) 13; (d) $\begin{cases} 54x-44y-7z+181=0. \\ -45x-76y+34z+497=0. \end{cases}$ **2.75.** (b) $4x+12y+3z+76=0$; (c) 127/13; (d) $\begin{cases} 53x-7y-44z- \\ -429=0. \\ 105x-23y-48z+ \\ +136=0. \end{cases}$ **2.76.** (b) $6x-3y-2z+2=0$; (c) 7; (d) $\begin{cases} 17x+16y+27z-90=0, \\ 31x+58y+6z-20=0. \end{cases}$

2.77. (a) $\frac{x}{1}=\frac{y}{1}=\frac{z-1}{-1}$, $\frac{x-1}{0}=\frac{y}{1}=\frac{z}{1}$; (b) $1/\sqrt{6}$.

(c) $\begin{cases} y+z-1=0 \\ x+y-2z-1=0 \end{cases}$; (d) $x+y+3z-2=0$. (c) $\arcsin\frac{5}{\sqrt{33}}$.

3.1. See Fig. 43. 3.2. See Fig. 44. 3.3. Straight lines $x=0$ and $x-y=0$. 3.4. Straight lines $y=0$ and $x+y=0$. 3.5. Straight lines $x-y=0$ and $x+y=0$. 3.6. Straight lines $x=0$ and $y=0$. 3.7. Straight lines $y=\pm 3$. 3.8. Straight lines $x=-2$ and $x=3$.

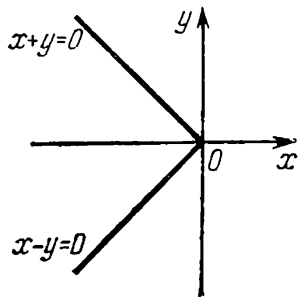

Fig. 43.

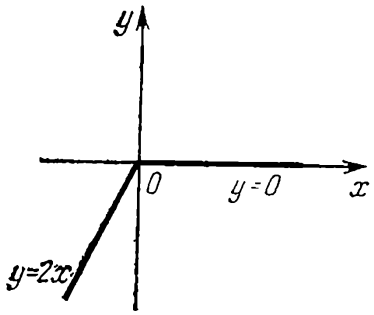

Fig. 44.

3.9. Straight lines $y=0$, $x=2$, and $x=5$. 3.10. A circle of radius $R=2$ centred at the origin. 3.11. A circle of radius $R=1$ centred at the point $C(0, -3)$. 3.12. The origin. 3.13. An empty set. 3.14. Points $(0, \pm 1)$. 3.15. $x-y=0$. 3.16. $4ax\pm c=0$. 3.17. $y=\pm 2x$. 3.18. $x^2+y^2=16$. 3.19. $x^2+y^2=8$. 3.20. $\frac{x^2}{5}+y^2=1$. 3.21. $xy=2$. 3.22. $y=\frac{x^2}{4}-x+2$. 3.23. (a) $C(2, -3)$, $R=4$; (b) $C(4, 0)$, $R=4$; (c) $C(0, -2)$, $R=2$. 3.24. (a) $(x-2)^2+(y+3)^2=49$; (b) $(x+1)^2+(y-2)^2=25$; (c) $(x-1)^2+(y-4)^2=8$; (d) $(x-1)^2+(y+1)^2=4$; (e) $(x-1)^2+(y-1)^2=1$ or $(x-5)^2+(y-5)^2=25$; (f) $(x-2)^2+(y-4)^2=10$; (g) $(x+4)^2+(y+1)^2=25$. *Hint.* Write the equation of the sought-for circle in the form $x^2+y^2+Dx+Ey+F=0$, substitute the coordinates of each point into it and then find D, E, and F. 3.25. $2x-5y+19=0$. 3.26. (a) 7; (b) 2. 3.27. (a) Intersects; (b) touches; (c) passes outside of the circle. 3.28. (a) $a=5$, $b=3$; (b) $F_1(-4, 0)$, $F_2(4, 0)$; (c) $e=\frac{4}{5}$; (d) D_1: $x=-\frac{25}{4}$, D_2: $x=\frac{25}{4}$. 3.29. (a) $\frac{x^2}{9}+\frac{y^2}{4}=1$; (b) $\frac{x^2}{25}+\frac{y^2}{9}=1$; (c) $\frac{x^2}{25}+\frac{y^2}{16}=1$; (d) $\frac{x^2}{169}+\frac{y^2}{25}=1$; (e) $\frac{x^2}{5}+\frac{y^2}{1}=1$; (f) $\frac{x^2}{64}+\frac{y^2}{48}=1$.

3.30. $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$. **3.31.** (a) $C\,(3,\,-1)$, $a=3$, $b=\sqrt{5}$, $e=2/3$, D_1: $2x+3=0$, D_2: $2x-15=0$; (b) $C\,(-1,\,2)$, $a=5$, $b=4$, $e=3/5$, D_1: $3x+28=0$, D_2: $3x-22=0$; (c) $C\,(1,\,-2)$, $a=4$, $b=2\sqrt{3}$, $e=1/2$, D_1: $y+10=0$, D_2: $y-6=0$. **3.34.** $\frac{x^2}{16}+\frac{y^2}{4}=1$, $r_{1,2}=4\pm\sqrt{3}$, $\rho_{1,2}=\frac{2}{\sqrt{3}}(4\pm\sqrt{3})$. **3.35.** $\left(-\frac{15}{4},\ \pm\frac{\sqrt{63}}{4}\right)$. **3.36.** $2x^2-2xy+2y^2-3=0$. **3.37.** $7x^2-2xy+7y^2-46x+2y+71=0$. **3.38.** (a) The straight line intersects the ellipse; (b) passes outside of the ellipse; (c) touches the ellipse. **3.40.** $3x+2y-10=0$ and $3x+2y+10=0$. **3.41.** $x+y-5=0$ and $x+y+5=0$. **3.43.** $x+y-5=0$ and $x+4y-10=0$. **3.44.** $M_0\,(-3,\,2)$, $\sqrt{13}$. **3.46.** $2x+11y-10=0$. *Hint.* Take advantage of the result obtained in Problem 3.45. **3.47.** (a) $a=3$, $b=4$; (b) $F_1\,(-5,\,0)$, $F_2\,(5,\,0)$; (c) $e=\frac{5}{3}$; (d) $y=\pm\frac{4}{3}x$; (e) $x=\pm\frac{9}{5}$. **3.48.** (a) $a=4$, $b=3$; (b) $F_1\,(0,\,-5)$, $F_2\,(0,\,5)$; (c) $e=\frac{5}{4}$; (d) $y=\pm\frac{4}{3}x$; (e) $y=\pm\frac{16}{5}$. **3.49.** (a) $\frac{x^2}{4}-\frac{y^2}{9}=1$; (b) $\frac{x^2}{9}-\frac{y^2}{16}=1$; (c) $\frac{x^2}{4}-\frac{y^2}{5}=1$; (d) $\frac{x^2}{64}-\frac{y^2}{36}=1$; (e) $\frac{x^2}{36}-\frac{y^2}{64}=1$; (f) $\frac{x^2}{4}-\frac{y^2}{5}=1$. **3.50.** $\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$. **3.51.** (a) $C\,(2,\,-3)$, $a=3$, $b=4$, $e=5/3$, equations of asymptotes: $4x-3y-17=0$ and $4x+3y+1=0$, equations of directrices: $5x-1=0$ and $5x-19=0$; (b) $C\,(-5,\,1)$, $a=8$, $b=6$, $e=5/4$, equations of asymptotes: $3x+4y+11=0$ and $3x-4y+19=0$, equations of directrices: $x=-11.4$ and $x=1.4$; (c) $C\,(2,\,-1)$, $a=4$, $b=3$, $e=5/4$, equations of asymptotes: $4x+3y-5=0$ and $4x-3y-11=0$, equations of directrices: $y=-4.2$ and $y=2.2$. **3.54.** $r_1=9/4$, $r_2=41/4$, $\rho\,(M,\,D_1)=9/5$, $\rho\,(M,\,D_2)=41/5$. **3.55.** $(-6,\,\pm4\sqrt{3})$. **3.56.** $7y^2+24xy-144=0$. **3.57.** $7x^2-6xy-y^2+26x-18y-17=0$. **3.58.** $\frac{x^2}{2}-\frac{y^2}{2}=1$, $e=\sqrt{2}$, $F_1\,(-\sqrt{2},\,-\sqrt{2})$, $F_2\,(\sqrt{2},\,\sqrt{2})$, $D_{1,2}$: $x+y\pm\sqrt{2}=0$. **3.59.** $\frac{x_0x}{a^2}-\frac{y_0y}{b^2}=1$. *Hint.* See Problem 3.39. **3.60.** $10x-3y-32=0$, $10x-3y+32=0$. **3.61.** $3x-4y-10=0$, $3x-4y+10=0$. **3.63.** $5x-3y-16=0$, $13x+5y+48=0$. **3.64.** $M_0\,(-6,\,3)$, $\rho=11/\sqrt{13}$. **3.66.** $2x+11y+6=0$. *Hint.* Take advantage of the result obtained in Problem 3.65. **3.67.** (a) $p=3$; (b) $p=5/2$; (c) $p=2$; (d) $p=1/2$. **3.68.** (a) $y^2=-x$; (b) $x^2=-2y$;

(c) $x^2 = -12y$. **3.69.** (a) $(y - y_0)^2 = 2p(x - x_0)$; (b) $(y - y_0)^2 = -2p(x - x_0)$. **3.70.** (a) $A(2, 0)$, $p = 2$; (b) $A(0, 2)$, $p = 1/2$; (c) $A(1, 3)$, $p = 1/8$; (d) $A(6, -1)$, $p = 3$; (e) $A(1, 2)$, $p = 2$; (f) $A(-4, 3)$, $p = 1/4$. **3.72.** 6. **3.73.** (a) $y = \frac{1}{8}x^2 - x + 3$; (b) $x^2 + 2xy + y^2 - 6x + 2y + 9 = 0$. **3.74.** $y_0 y = p(x + x_0)$. **3.75.** $x + y + 2 = 0$. **3.76.** $2x - y - 16 = 0$. **3.77.** $3x - y + 3 = 0$ and $3x - 2y + 12 = 0$. **3.78.** $M_0(9, -24)$, $\rho(M_0, L) = 10$. **3.80.** $y - 18 = 0$. **3.81.** $\tan\varphi = 1$. **3.82.** $r\sin\varphi = 1$. **3.83.** $r\cos\left(\varphi - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. **3.84.** $r = a$. **3.85.** $r^2 = \frac{a^2}{\cos 2\varphi}$. **3.86.** $r = a\cos\varphi$. **3.87.** Circle $x^2 + y^2 = 25$. **3.88.** Straight line $y = -x$. **3.89.** Straight line $x = 2$. **3.90.** Straight line $y = 1$. **3.91.** Straight line $x - y - 1 = 0$. **3.92.** Straight line $x + y - 2 = 0$. **3.93.** Circle $(x - a)^2 + y^2 = a^2$. **3.94.** Circle $x^2 + (y - a)^2 = a^2$. **3.95.** Pair of rays $x = \pm 2y$, $y \geqslant 0$. **3.96.** Family of concentric circles of radii $r_n = (-1)^n \frac{\pi}{6} + \pi n$, $n = 0, 1, 2, \ldots$. **3.97.** Hyperbola $xy = a^2$. **3.98.** Lemniscate of Bernoulli, $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. **3.99.** (a) $r\cos\varphi = 3$; (b) $\varphi = \pi/3$; (c) $\tan\varphi = 1$. **3.100.** (a) $r = 10\cos\varphi$; (b) $r = \pm 6\sin\varphi$. **3.101.** (a) $C(2, 0)$, $R = 2$; (b) $C(3/2, \pi/2)$, $R = 3/2$; (c) $C(5/2, -\pi/2)$, $R = 5/2$; (d) $C(3, \pi/3)$, $R = 3$; (e) $C(4, 5\pi/6)$, $R = 4$; (f) $C(4, -\pi/6)$, $R = 4$. **3.102.** $r^2 - 2r_0 r \times \cos(\varphi - \varphi_0) = R^2 - r_0^2$. **3.103.** (a) $r = \frac{16}{5 - 3\cos\varphi}$; (b) $r = \frac{16}{5 + 3\cos\varphi}$. **3.104.** (a) $r = -\frac{9}{4 - 5\cos\varphi}$; (b) $r = \frac{9}{4 - 5\cos\varphi}$. **3.105.** $r = \frac{3}{1 - \cos\varphi}$. **3.106.** (a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$; (b) $\frac{x^2}{16} - \frac{y^2}{9} = 1$; (c) $y^2 = 6x$. **3.107.** $r^2 = \frac{b^2}{1 - e^2\cos^2\varphi}$. **3.108.** $r^2 = \frac{b^2}{e^2\cos^2\varphi - 1}$. **3.109.** $r = \frac{2p\cos\varphi}{\sin^2\varphi}$. **3.110.** (a) $x = t$, $y = t + 1$, $t \in [-1, +\infty)$; (b) $x = t - 1$, $y = t$, $t \in [0, +\infty)$; (c) $x = -1 + \frac{\sqrt{2}}{2}t$, $y = \frac{\sqrt{2}}{2}t$, $t \in [0, +\infty)$; (d) $x = \frac{\sqrt{2}}{2}\frac{\cos t}{\cos\left(t - \frac{3\pi}{4}\right)}$, $y = \frac{\sqrt{2}}{2}\frac{\sin t}{\cos\left(t - \frac{3\pi}{4}\right)}$, $t \in \left[-\pi, \frac{\pi}{4}\right)$. **3.111.** (a) $x = 1 + \frac{1}{\sqrt{5}}t$, $y = 1 + \frac{2}{\sqrt{5}}t$, $t \in [0, \sqrt{5}]$; (b) $x = 2 - \frac{1}{\sqrt{5}}t$, $y = 3 - \frac{2}{\sqrt{5}}t$, $t \in [0, \sqrt{5}]$.

3.112. $x = x_0 + R\cos t$, $y = y_0 + R\sin t$, $t \in [0, 2\pi)$. **3.113.** (a) $x = R(1 + \cos 2t)$, $y = R\sin 2t$, $t \in [-\pi/2, \pi/2)$; (b) $x = R(1 + \cos t)$, $y = R\sin t$, $t \in [0, 2\pi)$. **3.114.** Straight line $x + 2y - 3 = 0$. **3.115.** Parabola $y^2 = x$. **3.116.** Circle $(x + 1)^2 + (y - 3)^2 = 4$. **3.117.** Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **3.118.** Right-hand branch of the hyperbola $\frac{(x-1)^2}{4} - \frac{(y+1)^2}{1} = 1$. **3.119.** Right-hand branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **3.120.** Circle $(x - R)^2 + y^2 = R^2$. **3.121.** Circle $x^2 + (y - R)^2 = R^2$. **3.122.** Upper branch of the parabola $y^2 = 2px$. **3.123.** $x = \frac{ab\cos t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$, $y = \frac{ab\sin t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$, $t \in [0, 2\pi)$. **3.124.** $x = \frac{ab\cos t}{\sqrt{b^2\cos^2 t - a^2\sin^2 t}}$, $y = \frac{ab\sin t}{\sqrt{b^2\cos^2 t - a^2\sin^2 t}}$, where $t \in \left(-\operatorname{arc\,tan}\frac{b}{a}, \operatorname{arc\,tan}\frac{b}{a}\right)$ for the right-hand branch and $t \in \left(\pi - \operatorname{arc\,tan}\frac{b}{a}, \pi + \operatorname{arc\,tan}\frac{b}{a}\right)$ for the left-hand branch. **3.125.** (a) $x = \frac{t^2}{2p}$, $y = t$, $t \in (-\infty, +\infty)$; (b) $x = 2p\cot^2 t$, $y = 2p\cot t$, where $t \in (0, \pi/2)$ for the upper branch and $t \in (3\pi/2, 2\pi)$ for the lower branch; (c) $x = \frac{p}{2}\cot^2\frac{t}{2}$, $y = p\cot\frac{t}{2}$, $t \in (0, 2\pi)$.

4.1. Plane $z = -5$ parallel to the xy-plane. **4.2.** Plane with the normal vector n (1, −2, 1). **4.3.** Sphere of radius $R = 2$ centred at the origin. **4.4.** Sphere of radius $R = 4$ centred at the point C (2, 0, −1). **4.5.** Origin. **4.6.** y-axis. **4.7.** Empty set. **4.8.** Pair of intersecting planes $x - 2z = 0$ and $x + 2z = 0$ parallel to the y-axis. **4.9.** Pair of coordinates planes: Oyz and Oxy. **4.10.** A triple of coordinate planes. **4.11.** A pair of planes $x = 0$ and $x = 4$. **4.12.** A pair of planes $y = 0$ and $y = x$. **4.13.** $20y + 53 = 0$. **4.14.** $x^2 + y^2 + z^2 = a^2$. **4.15.** $\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{25} = 1$. **4.16.** $\frac{x^2}{16} - \frac{y^2}{9} + \frac{z^2}{16} = -1$. **4.17.** (a) C (0, 0, 3), $R = 3$; (b) C (2, 1, −1), $R = 5$. **4.18.** (a) $(x + 1)^2 + (y - 2)^2 + z^2 = 4$; (b) $(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = 18$; (c) $(x - 3)^2 + (y + 1)^2 + (z - 1)^2 = 21$; (d) $(x - 3)^2 + (y + 5)^2 + (z + 2)^2 = 56$; (e) $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 49$. **4.19.** $(x + 1)^2 + (y - 3)^2 + (z - 3)^2 = 1$. **4.20.** $\left(x - \frac{4}{9}\right)^2 + \left(y + \frac{4}{9}\right)^2 + \left(z - \frac{4}{9}\right)^2 = \frac{64}{729}$. **4.21.** $x = -1 + 5t$, $y = 3 - t$, $z = -\frac{1}{2} + 2t$.

4.22. M_0 $(-2, -2, 7)$, $\rho = 3$. **4.23.** (a) Intersects; (b) touches; (c) passes outside of the sphere. **4.24.** (a) A straight line passing through the point $(5, 0, -2)$ parallel to the y-axis; (b) a circle of radius $R = 7$ in the xz-plane with centre at the origin; (c) a circle of radius $R = 4$ lying in the plane $z = 2$ and centred at the point C $(0, 0, 2)$; (d) a circle of radius $R = \sqrt{13}$ lying in the plane $z = 6$ and centred at the point C $(0, 0, 6)$. **4.25.** (a) C $(1, 7, 2)$, $R = 4$; (b) C $(-1, 2, 3)$, $R = 8$. **4.26.** Ellipse $\dfrac{(x-2)^2}{36} + \dfrac{(y+4)^2}{18} = 1$.

4.27. $\begin{cases} (x-1)^2 + (y-2)^2 + (z-1)^2 = 36. \\ 2x - z - 1 = 0. \end{cases}$

4.28. $\begin{cases} (x-2)^2 + y^2 + (z-3)^2 = 27, \\ x + y - 2 = 0. \end{cases}$

4.29. Ellipsoid. **4.30.** Hyperboloid of one sheet. **4.31.** Hyperboloid of revolution of two sheets. **4.32.** Cone. **4.33.** Paraboloid of revolution. **4.34.** Hyperbolic paraboloid. **4.35.** Elliptic paraboloid. **4.36.** Parabolic cylinder. **4.37.** Paraboloid of revolution with vertex $(0, 0, 2)$. **4.38.** Hyperbolic paraboloid. **4.39.** Hyperboloid of revolution of one sheet. **4.40.** Hyperboloid of revolution of two sheets. **4.41.** *Hint.* Take advantage of the homogeneity of the equation. **4.42.** *Hint.* Pass to a new coordinate system by rotating the x- and y-axes about the z-axis through an angle of $\pi/4$. **4.43.** (a) Cone of the second order with vertex at the origin (see Problem 4.41); (b) hyperbolic paraboloid (see Problem 4.42). **4.44.** On the xy-plane: $x^2 + 4xy + 5y^2 - x = 0$; on the xz-plane: $x^2 - 2xz + 5z^2 - 4x = 0$; on the yz-plane: $y^2 + z^2 + 2y - z = 0$. **4.45.** (a) Ellipse; (b) parabola. **4.46.** (a) M_1 $(3, 4, -2)$ and M_2 $(6, -2, 2)$; (b) M $(4, -3, 2)$ — the straight line is tangent to the surface; (c) the straight line and the surface have no points in common. **4.47.** (a) M $(9, 5, -2)$; (b) M $(3, 0, -10)$; (c) M $(6, -2, 2)$. **4.48.** $\begin{cases} 2x - 12y - z + 16 = 0, \\ x - 2y + 4 = 0, \end{cases}$ $\begin{cases} 2x - 12y - z + 16 = 0, \\ x + 2y - 8 = 0. \end{cases}$ **4.49.** $\begin{cases} y + 2z = 0, \\ x - 5 = 0, \end{cases}$ $\begin{cases} 2x - 5z = 0, \\ y + 4 = 0. \end{cases}$

4.60. (a) $y^2 + z^2 = a^2$; (b) $x^2 + z^2 = 2ax$; (c) $x^2 + y^2 = 2ax$. **4.61.** (a) $x^2 + 5y^2 - 8y - 12 = 0$; (b) $4x^2 + 5z^2 + 4z - 60 = 0$; (c) $2y - z - 2 = 0$. **4.62.** (a) $8x^2 + 4y^2 - 36x + 16y - 3 = 0$, $z = 0$; (b) $2x - 2z - 7 = 0$, $y = 0$; (c) $4y^2 + 8z^2 + 16y + 36z - 31 = 0$, $x = 0$. **4.63.** $y^2 = 2ax - x^2$. **4.64.** (a) $(3y - 2z)^2 = 12(3x - z)$; (b) $(x - z)^2 + (y - z)^2 = 4(x - z)$. **4.65.** The equation of the projecting cylinder: $2x^2 + (y - z + 2)^2 = 8$; the shape of the shadow is an ellipse: $\dfrac{x^2}{4} + \dfrac{(y+2)^2}{8} = 1$. **4.66.** $x = 4$, $z \pm y = 2$. **4.68.** (a) $\dfrac{x^2+y^2}{a^2} = \dfrac{z^2}{h^2}$; (b) $9(x^2 + z^2) = 16y^2$; (c) $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 0$; (d) $x^2 +$

$+ y^2 - z^2 = 0$. **4.69.** (a) $h^2x^2 = 2pz\,(h\,(y + a) - az)$; (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{(z-c)^2}{c^2} = 0$; (c) $x^2 + z^2 = z\,(y + a)$; (d) $3x^2 - 5y^2 + 7z^2 - 6xy + 10xz - 2yz - 4x + 4y - 4z + 4 = 0$. **4.70.** (a) The vertex: $(0, h, 0)$, the directrix is a circle $x^2 + (y - h)^2 = h^2$, $z = h$; (b) the vertex: $(0, 0, 0)$, the directrix is a parabola: $x^2 = 2hy$, $z = h$. **4.71.** $xy + xz + yz = 0$, the axis of the cone passes through the first and seventh octants $xy + xz - yz = 0$, the axis of the cone passes through the second and eighth octants; $xy - xz - yz = 0$, the axis of the cone passes through the third and fifth octants; $xy - xz + yz = 0$, the axis of the cone passes through the fourth and sixth octants. **4.72.** (a) Circle $x^2 + y^2 = (a/\sqrt{2})^2$; (b) line segments $z = \pm a/\sqrt{2}$, $-a/\sqrt{2} \leqslant x \leqslant a/\sqrt{2}$; (c) line segments $z = \pm a/\sqrt{2}$, $-a/\sqrt{2} \leqslant y \leqslant a/\sqrt{2}$. **4.73.** Equation of the projecting cone: $9x^2 - 16y^2 - 16z^2 - 90x + 225 = 0$, the shadow has the form of a circle: $y^2 + z^2 = (15/4)^2$. **4.74.** (a) $z = x^2 + y^2$; (b) $\sqrt{y^2 + z^2} = x^2$. **4.75.** (a) $x^2 + z^2 = y^2$; (b) $z^2 = x^2 + y^2$. **4.76.** (a) $z = e^{-(x^2+y^2)}$; (b) $z = \frac{4}{x^2+y^2}$. **4.77.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **4.78.** The given surface is generated by revolving the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$, $y = 0$ about the z-axis.

Chapter 3

DETERMINANTS AND MATRICES. SYSTEMS OF LINEAR EQUATIONS

SEC. 3.1. DETERMINANTS

1. Determinants of the Second and Third Orders. A square array

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

formed of four real (or complex) numbers is called a *square matrix of the second order. The determinant of the second order* corresponding

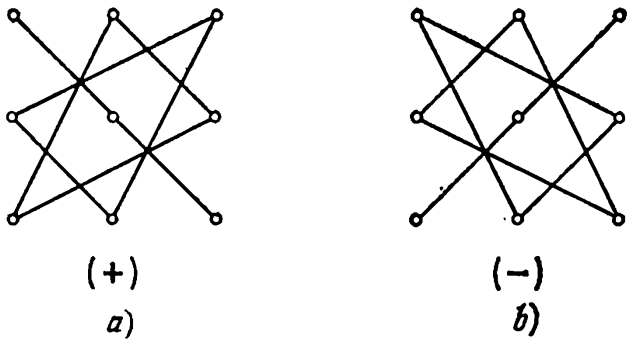

Fig. 45.

to matrix A (or simply, *the determinant of matrix* A) is defined as the number

$$\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}.$$

Analogously, if

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is a square matrix of the third order, then the *determinant of the third order* corresponding to this matrix is the number

$$\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{12}a_{23}a_{31}-$$
$$-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}. \quad (1)$$

Determinants of the third order are usually computed using the following rule (*Sarrus' rule*): one of the three terms standing in the right-hand side of (1) with the plus sign is the product of the elements forming the *principal* (or *leading*) diagonal of matrix A, either of the two others is the product of the elements lying on the parallel to this diagonal and an element from the opposite corner of the matrix (Fig. 45, *a*), and the terms contained in (1) with the minus sign are constructed in the same way, but with respect to the second (*auxiliary*) diagonal (Fig. 45, *b*).

Compute the following determinants of the second order (Problems 1.1 to 1.7):

1.1. $\begin{vmatrix} -1 & 4 \\ -5 & 2 \end{vmatrix}$. **1.2.** $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$.

1.3. $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$. **1.4.** $\begin{vmatrix} a+bi & b \\ 2a & a-bi \end{vmatrix}$.

1.5. $\begin{vmatrix} \cos\alpha + i\sin\alpha & 1 \\ 1 & \cos\alpha - i\sin\alpha \end{vmatrix}$.

1.6. $\begin{vmatrix} 2\sin\varphi\cos\varphi & 2\sin^2 - 1 \\ 2\cos^2\varphi - 1 & 2\sin\varphi\cos\varphi \end{vmatrix}$.

1.7. $\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ -\frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix}$.

Solve the following equations (Problems 1.8 and 1.9):

1.8. $\begin{vmatrix} x & x+1 \\ -4 & x+1 \end{vmatrix} = 0$. **1.9.** $\begin{vmatrix} \cos 8x & -\sin 5x \\ \sin 8x & \cos 5x \end{vmatrix} = 0$.

1.10*. Prove that for real a, b, c, d the roots of the equation $\begin{vmatrix} a-x & c+di \\ c-di & b-x \end{vmatrix} = 0$ are real.

1.11. Prove that for a determinant of the second order to be equal to zero, it is necessary and sufficient that its rows be proportional (that is, the elements of one row are obtained from the corresponding elements of the other row by multiplying by one and the same number). The same is true for columns.

Compute the following determinants of the third order (Problems 1.12 to 1.18):

1.12. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. **1.13.** $\begin{vmatrix} 3 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix}$.

1.14. $\begin{vmatrix} a+x & x & x \\ x & b+x & x \\ x & x & c+x \end{vmatrix}$. **1.15.** $\begin{vmatrix} \alpha^2+1 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2+1 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2+1 \end{vmatrix}$.

1.16. $\begin{vmatrix} \sin\alpha & \cos\alpha & 1 \\ \sin\beta & \cos\beta & 1 \\ \sin\gamma & \cos\gamma & 1 \end{vmatrix}$.

1.17. $\begin{vmatrix} 1 & 1 & \varepsilon \\ 1 & 1 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon & 1 \end{vmatrix}$, where $\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. **1.18.** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{vmatrix}$, where $\varepsilon = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$.

Solve the following equations (Problems 1.19 and 1.20):

1.19. $\begin{vmatrix} 3 & x & -x \\ 2 & -1 & 3 \\ x+10 & 1 & 1 \end{vmatrix} = 0.$

1.20. $\begin{vmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{vmatrix} = 0.$

Solve the following inequalities (Problems 1.21 and 1.22):

1.21. $\begin{vmatrix} 3 & -1 & 1 \\ 1 & x & -2 \\ -1 & 2 & -1 \end{vmatrix} < 0.$ **1.22.** $\begin{vmatrix} 2 & x+2 & -1 \\ 1 & 1 & -2 \\ 5 & -3 & x \end{vmatrix} > 0.$

1.23. Prove the following properties of the determinant of the third order, using its definition:

(a) if each of the rows of the matrix of a determinant is substituted by a column of the same position number, that is, if the matrix is *transposed*, then the magnitude of the determinant remains unchanged;

(b) if all elements of a row (column) are multiplied by one and the same number, then the determinant is multiplied by this number;

(c) if any two rows (columns) are interchanged, the absolute value of a determinant remains unaltered, while the sign is reversed. In particular, a determinant with two identical rows (columns) is equal to zero;

(d) if every element of some row (column) is the sum of two terms, then the determinant is equal to the sum of two determinants: one containing only the first term in place of each sum, the other only the second term (the remaining elements of both determinants are the same as in the given determinant);

(e) if one row (column) of a determinant is a linear combination of the remaining rows (columns), then the determinant is equal to zero.

Using the properties of the determinant of the third order mentioned in Problem 1.21, prove the following identities without expanding the given determinants (Problems 1.24 to 1.26):

1.24. $$\begin{vmatrix} a_1+b_1x & a_1-b_1x & c_1 \\ a_2+b_2x & a_2-b_2x & c_2 \\ a_3+b_3x & a_3-b_3x & c_3 \end{vmatrix} = -2x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

1.25. $$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

1.26*. $$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

In Problems 1.27 to 1.30 compute the given determinants, using the properties of a third-order determinant listed in Problem 1.23.

1.27. $$\begin{vmatrix} x+y & z & 1 \\ y+z & x & 1 \\ z+x & y & 1 \end{vmatrix}.$$

1.28. $$\begin{vmatrix} (a+1)^2 & a^2+1 & a \\ (b+1)^2 & b^2+1 & b \\ (c+1)^2 & c^2+1 & c \end{vmatrix}.$$

1.29. $$\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}.$$

1.30. $$\begin{vmatrix} \sin^2\alpha & 1 & \cos^2\alpha \\ \sin^2\beta & 1 & \cos^2\beta \\ \sin^2\gamma & 1 & \cos^2\gamma \end{vmatrix}.$$

1.31. Check to show that the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

is divisible by $x - y$, $y - z$, and $z - x$.

1.32. Check to see that the determinant

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

is divisible by $x + y$, and by $x^2 - xy + y^2$.

1.33. Construct the graph of the function

$$y = \frac{1}{b-a} \begin{vmatrix} x & x^2 & 1 \\ a & a^2 & 1 \\ b & b^2 & 1 \end{vmatrix} \quad (a \neq b).$$

2. Determinants of the *n*th Order. Any one-to-one mapping of the n set $\{1, 2, \ldots, n\}$ of the first n natural numbers on itself is called a *substitution of the nth order.*

Any substitution can be written in the form

$$\pi = \begin{pmatrix} i_1 & i_2 & \ldots & i_n \\ \alpha_{i_1} & \alpha_{i_2} & \ldots & \alpha_{i_n} \end{pmatrix}, \tag{2}$$

where $\alpha_{i_k} = \pi(i_k)$ is the image of the element $i_k \in \{1, 2, \ldots, n\}$ when mapping π. For a fixed substitution π there exists many ways of notation of form (2) which differ in numbering the elements of the upper row. In particular, notation of the form

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix} \tag{3}$$

is called *canonical.*

A pair of elements (i, j) is said to form an *inversion* in the substitution π, if $i < j$, but $\alpha_i > \alpha_j$. The number $s(\pi)$ of all inverse pairs determines the parity of a substitution: a substitution is said to be *even* if $s(\pi)$ is an even number, and to be *odd* if $s(\pi)$ is an odd number.

Example 1. Determine whether the substitution

$$\pi = \begin{pmatrix} 1 & 3 & 5 & 2 & 4 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$$

is even or odd.

Let us rewrite the given substitution in canonical form (3)

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}$$

and find the number of inversions. Since the latter form the pairs (1, 4), (2, 3), (2, 4), (3, 4), then $s(\pi) = 4$ and, hence, π is an even substitution.

The *determinant of the nth order* corresponding to the square matrix

$$= \begin{pmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots\ldots\ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{pmatrix}$$

(or *the determinant of matrix A*) is defined as the number

$$\det A = \begin{vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots\ldots\ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{vmatrix} = \sum_{\pi} (-1)^{s(\pi)} a_{1,\,\pi(1)} \ldots a_{n,\,\pi(n)},$$

where the sum is carried out over all π-substitutions of the nth order.

Determinants of the nth order possess properties which are analogous to properties (a) to (e) from Problem 1.21.

1.34. Find the π-substitution on the set $\{1, \ldots, 6\}$ if $\pi(k)$ is the remainder from the division of the number $3k$ by 7. Determine whether it is even or odd.

1.35. Find the π-substitution on the set $\{1, \ldots, 8\}$ if $\pi(k)$ is the remainder from the division of the number $5k$ by 9. Determine whether it is even or odd.

In Problems 1.36 to 1.39 determine the parity of the indicated substitutions.

1.36. $\begin{pmatrix} 5 & 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$.

1.37. $\begin{pmatrix} 3 & 4 & 6 & 5 & 3 & 1 \\ 1 & 4 & 3 & 2 & 5 & 6 \end{pmatrix}$.

1.38. $\begin{pmatrix} 2n & 2n-1 & \ldots & 4 & 3 & 2 & 1 \\ 2n-1 & 2n & \ldots & 3 & 4 & 1 & 2 \end{pmatrix}$.

1.39.

$$\begin{pmatrix} n & n-1 & \ldots & n-k+1 & n-k & n-k-1 & \ldots & 2 & 1 \\ k & k-1 & \ldots & 1 & n & n-1 & \ldots & k+2 & k+1 \end{pmatrix}.$$

Find out which of the below given products enter into determinants of appropriate orders and with what signs (Problems 1.40 to 1.43).

1.40. $a_{43}a_{21}a_{35}a_{12}a_{54}$. **1.41.** $a_{61}a_{23}a_{45}a_{36}a_{12}a_{54}$.

1.42. $a_{27}a_{36}a_{51}a_{74}a_{25}a_{43}a_{62}$. **1.43.** $a_{33}a_{16}a_{72}a_{27}a_{55}a_{61}a_{44}$.

1.44. Choose the values of i and k so that the product $a_{62}a_{i5}a_{33}a_{k4}a_{46}a_{21}$ enters into some determinant with the minus sign.

1.45. Choose the values of i and k so that the product $a_{47}a_{63}a_{1i}a_{55}a_{7k}a_{24}a_{31}$ would enter into some determinant with the plus sign.

1.46. In the determinant

$$\begin{vmatrix} 5x & 1 & 2 & 3 \\ x & x & 1 & 2 \\ 1 & 2 & x & 3 \\ x & 1 & 2 & 2x \end{vmatrix}$$

find the terms containing x^4 and x^3.

Using only the definition, compute the following determinants:

1.47.
$$\begin{vmatrix} 0 & \dots & 0 & 0 & a_{1,n} \\ 0 & \dots & 0 & a_{2,n-1} & a_{2,n} \\ 0 & \dots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,n-2} & a_{n,n-1} & a_{n,n} \end{vmatrix}.$$

1.48.
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix}.$$

1.49. How will the determinant change if:

(a) the last row is added to each of its rows, but the last one;

(b) from each of its rows, but the last one, all subsequent rows are subtracted;

(c) from each of its rows, but the last one, the subsequent row is subtracted, and the first original row is subtracted from the last row;

(d) its matrix is "turned through an angle of 90° about the centre";

(e) the first column is rearranged to occupy the last place and all the remaining columns are moved to the left without changing their arrangement?

3. Basic Methods of Computing Determinants of *n*th Order.

(a) *Reduction of the Order of a Determinant.* This method is based on the following relation (i fixed):

$$\det A = \sum_{k=1}^{n} a_{ik} A^{(i, k)}, \tag{4}$$

where

$$A^{(i, k)} = (-1)^{i+k} \begin{vmatrix} a_{11} & \dots & a_{1, k-1} & a_{1, k+1} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i-1, 1} & \dots & a_{i-1, k-1} & a_{i-1, k+1} & \dots & a_{i-1, n} \\ a_{i+1, 1} & \dots & a_{i+1, k-1} & a_{i+1, k+1} & \dots & a_{i+1, n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{n, k-1} & a_{n, k+1} & \dots & a_{nn} \end{vmatrix} \tag{5}$$

is called the *cofactor* of the element a_{ik} and represents a determinant of the $(n-1)$th order.

Relation (4) is called the *expansion of the determinant in terms of the* ith *row. An expansion of the determinant in terms of a column* is defined in an analogous manner. Prior to using this method, it is useful, applying the basic properties of the determinant, to reduce to zero all the elements of some of its rows (columns) except one element.

Example 2. Evaluate the determinant

$$D = \begin{vmatrix} 8 & 7 & 2 & 10 \\ -8 & 2 & 7 & 10 \\ 4 & 4 & 4 & 5 \\ 0 & 4 & -3 & 2 \end{vmatrix}.$$

Subtracting the doubled third row from the first row and adding it to the second row, and expanding the determinant thus obtained in terms of the first column, we have

$$D = \begin{vmatrix} 0 & -1 & -6 & 0 \\ 0 & 10 & 15 & 20 \\ 4 & 4 & 4 & 5 \\ 0 & 4 & -3 & 2 \end{vmatrix} = (-1)^{1+3} \cdot 4 \cdot \begin{vmatrix} -1 & -6 & 0 \\ 10 & 15 & 20 \\ 4 & -3 & 2 \end{vmatrix}.$$

Then, once again, we reduce to zero all the elements of the first column, except for the element in the upper left corner, and compute

the determinant of the second order:

$$D=4\begin{vmatrix}-1 & -6 & 0\\ 0 & -45 & 20\\ 0 & -27 & 2\end{vmatrix}=4\cdot(-1)^{1+1}\cdot(-1)\begin{vmatrix}-45 & 20\\ -27 & 2\end{vmatrix}$$

$$=-4(-90+540)=-1800.$$

(b) *Reduction to the Triangular Form.* This method consists in the following: all the elements lying on one side of one of the diagonals of a determinant are transformed to become equal to zero.

Example 3. Evaluate the determinant

$$D=\begin{vmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 2 & 2\\ 1 & 1 & -1 & 3\\ 1 & 1 & 1 & -1\end{vmatrix}.$$

Subtracting the first row from all the remaining ones, we obtain

$$D=\begin{vmatrix}1 & 1 & 1 & 1\\ 0 & -2 & 1 & 1\\ 0 & 0 & -2 & 2\\ 0 & 0 & 0 & -2\end{vmatrix}=-8.$$

(c) *Method of Recursion Relations.* This method enables us to express a given determinant (transforming and expanding it in terms of a row or a column) in terms of determinants of the same form, but of a lower order. The obtained equality is called the recursion (recurrence) relation.

Example 4. Evaluate the *Vandermonde determinant*

$$D_n=\begin{vmatrix}1 & 1 & 1 & \dots & 1\\ a_1 & a_2 & a_3 & \dots & a_n\\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2\\ \dots & \dots & \dots & \dots & \dots\\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & a_n^{n-1}\end{vmatrix}.$$

Let us show that for any n ($n \geqslant 2$) the Vandermonde determinant is equal to the product of all possible differences $a_i - a_j$, $1 \leqslant j < i \leqslant n$. We shall carry out the proof by induction, using the method of recursion relations.

Indeed, for $n = 2$ we have

$$D_2=\begin{vmatrix}1 & 1\\ a_1 & a_2\end{vmatrix}=a_2-a_1.$$

Let our assertion be proved for Vandermonde determinants of order $(n-1)$, i.e.

$$D_{n-1}=\prod_{1\leqslant j<i\leqslant n-1}(a_i-a_j).$$

Let us transform the determinant D_n in the following way: subtract the $(n-1)$th row multiplied by a_1 from the last (nth) row, and, in general, subtract in succession the $(k-1)$th row multiplied by a_1 from the kth row. We obtain

$$D_n=\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & a_2-a_1 & a_3-a_1 & \dots & a_n-a_1 \\ 0 & a_2^2-a_1a_2 & a_3^2-a_1a_3 & \dots & a_n^2-a_1a_n \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a_2^{n-1}-a_1a_2^{n-2} & a_3^{n-1}-a_1a_3^{n-2} & \dots & a_n^{n-1}-a_1a_n^{n-2} \end{vmatrix}.$$

Let us expand the last determinant in terms of the first column and take the common factors of all the columns outside the sign of the determinant. The determinant takes the form

$$D_n=(a_2-a_1)(a_3-a_1)\dots(a_n-a_1)\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_2 & a_3 & a_4 & \dots & a_n \\ a_2^2 & a_3^2 & a_4^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ a_2^{n-2} & a_3^{n-2} & a_4^{n-2} & \dots & a_n^{n-2} \end{vmatrix}$$

$$=(a_2-a_1)(a_3-a_1)\dots(a_n-a_1)\,D_{n-1}.$$

We have thus obtained a recursion relation. Using the assumption of induction, we finally derive:

$$D_n=(a_2-a_1)(a_3-a_1)\dots(a_n-a_1)\prod_{2\leqslant j<i\leqslant n}(a_i-a_j)$$

$$=\prod_{1\leqslant j<i\leqslant n}(a_i-a_j).$$

1.50. Compute the given determinants, expanding them in terms of the third row and second column, respectively:

(a) $\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ a & b & c & d \\ 3 & -1 & 4 & 3 \end{vmatrix}$; (b) $\begin{vmatrix} 5 & a & 2 & -1 \\ 4 & b & 4 & -3 \\ 2 & c & 3 & -2 \\ 4 & d & 5 & -4 \end{vmatrix}$;

(c) $\begin{vmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{vmatrix}$.

In Problems 1.51 to 1.60 evaluate the given determinants:

1.51. $\begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 2 & 3 \\ 3 & 1 & 6 & 1 \end{vmatrix}$.

1.52. $\begin{vmatrix} 2 & 3 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 2 & 1 & 0 \\ 2 & 3 & 0 & -5 \end{vmatrix}$.

1.53. $\begin{vmatrix} 3 & -1 & 4 & 2 \\ 5 & 2 & 0 & 1 \\ 0 & 2 & 1 & -3 \\ 6 & -2 & 9 & 8 \end{vmatrix}$.

1.54. $\begin{vmatrix} \sqrt{2} & \sqrt{3} & \sqrt{5} & \sqrt{3} \\ \sqrt{6} & \sqrt{21} & \sqrt{10} & -2\sqrt{3} \\ \sqrt{10} & 2\sqrt{15} & 5 & \sqrt{6} \\ 2 & 2\sqrt{6} & \sqrt{10} & \sqrt{15} \end{vmatrix}$.

1.55. $\begin{vmatrix} 0 & -a & -b & -d \\ a & 0 & -c & -e \\ b & c & 0 & 0 \\ d & e & 0 & 0 \end{vmatrix}$.

1.56. $\begin{vmatrix} 0 & b & c & d \\ b & 0 & d & c \\ c & d & 0 & b \\ d & c & b & 0 \end{vmatrix}$.

1.57. $\begin{vmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{vmatrix}$.

1.58. $\begin{vmatrix} 5 & 6 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{vmatrix}$.

1.59. $\begin{vmatrix} x & 0 & -1 & 1 & 0 \\ 1 & x & -1 & 1 & 0 \\ 1 & 0 & x-1 & 0 & 1 \\ 0 & 1 & -1 & x & 1 \\ 0 & 1 & -1 & 0 & x \end{vmatrix}$.

1.60. $\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 & 5x_4 \\ 1 & 4x & 9x^2 & 16x^3 & 25x^4 \\ 1 & y & y^2 & y^3 & y^4 \\ 1 & 2y & 4y^2 & 4y^3 & 5y^4 \end{vmatrix}$.

Evaluate the given determinants of order n by reducing them to the triangular form:

1.61. $\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix}$.

1.62. $\begin{vmatrix} 3 & 2 & 2 & \dots & 2 \\ 2 & 3 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & 3 \end{vmatrix}$.

1.63. Compute the determinant whose elements are specified by the conditions $a_{ij} = \min(i, j)$.

1.64. Evaluate the determinant whose elements are defined by the conditions $a_{ij} = \max(i, j)$.

In Problems 1.65 to 1.68 compute the given determinants of order n, using the method of recurrence relations:

1.65. $\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & a_1 & 0 & \dots & 0 \\ 1 & 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & a_n \end{vmatrix}$.

1.66. $\begin{vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 \end{vmatrix}$.

1.67. $\begin{vmatrix} \alpha & \alpha\beta & 0 & \dots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{vmatrix}$.

1.68. $\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & \dots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & 1 & \alpha+\beta \end{vmatrix}$.

1.69. Evaluate the determinant

$$\begin{vmatrix} a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & a_n^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \\ a_1 & a_2 & a_3 & \dots & a_n \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix}.$$

1.70. Prove that for any determinant the following relation is fulfilled:

$$\sum_{j=1}^{n} a_{ij} A^{(k,j)} = \begin{cases} \det A, & k = i, \\ 0, & k \neq i, \end{cases}$$

where $A^{(k,j)}$ is the cofactor of the element a_{kj} (see (5)).

SEC. 3.2.
MATRICES

1. Operations on Matrices. *A matrix of order* (or *dimension*) $m \times n$ or $(m \times n)$-*matrix* is defined as a rectangular array of numbers a_{ij}, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

consisting of m rows and n columns.

The *sum* $A + B$ of $(m \times n)$-*matrices* $A = (a_{ij})$ and $B = (b_{ij})$ is the matrix $C = (c_{ij})$ of the same order each element of which is equal to the sum of the corresponding elements of matrices A and B:

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.$$

The *product* αA *of a matrix* $A = (a_{ij})$ *by a number* α (real or complex) is defined as the matrix $B = (b_{ij})$ obtained from matrix A by multiplying all of its elements by α:

$$b_{ij} = \alpha a_{ij}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.$$

The *product* AB *of an* $(m \times n)$-*matrix* $A = (a_{ij})$ *by* $(n \times k)$ *matrix* $B = (b_{ij})$ is the $(m \times k)$-matrix $C = (c_{ij})$ whose element d_{ij} in row i and column j is equal to the sum of products of the corresponding elements in row i of matrix A and column j of matrix B:

$$c_{ij} = \sum_{\nu=1}^{n} a_{i\nu} b_{\nu j}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots k.$$

2.1. Prove the following properties of algebraic operations on matrices:

(a) $A + B = B + A$, $A + (B + C) = (A + B) + C$;

(b) $(\alpha + \beta) A = \alpha A + \beta A$, $\alpha (A + B) = \alpha A + \alpha B$, $(\alpha\beta) A = \alpha (\beta A)$;

(c) $A (BC) = (AB) C$, $A (B + C) + AB + AC$.

Compute the given linear combinations of matrices A and B:

2.2. $3A + 2B$, $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & -4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 2 & 2 \end{pmatrix}$.

2.3. $(1 + i) A + (1 - i) B$, $A = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, $B = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$.

Compute (Problems 2.4 to 2.15):

2.4. $\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$.

2.5. $\begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} 9 & -6 \\ 6 & -4 \end{pmatrix}$.

2.6. $\begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} -28 & 93 \\ 38 & -126 \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$.

2.7. $\begin{pmatrix} 1 & -3 & 2 \\ 3 & -4 & 1 \\ 2 & -5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 & 6 \\ 1 & 2 & 5 \\ 1 & 3 & 2 \end{pmatrix}$.

2.8. $\begin{pmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{pmatrix}$.

2.9. $\begin{pmatrix} 5 & 0 & 2 & 3 \\ 4 & 1 & 5 & 3 \\ 3 & 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -2 \\ 7 \\ 4 \end{pmatrix}$.

2.10. $(4 \quad 0 \quad -2 \quad 3 \quad 1) \begin{pmatrix} 3 \\ 1 \\ -1 \\ 5 \\ 2 \end{pmatrix}$.

2.11. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

2.12. $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}^3$. **2.13.** $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^n$, $a \in \mathbb{R}$.

2.14. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n$, $\lambda \in \mathbb{R}$. **2.15.** $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n$.

Evaluate the polynomial $f(A)$ of matrix A (Problems 2.16 to 2.18):

2.16. $f(x) = 3x^2 - 4$, $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

2.17. $f(x) = x^2 - 3x + 1$, $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

2.18. $f(x) = 3x^2 - 2x + 5$, $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 1 \\ 3 & -5 & 2 \end{pmatrix}$.

In Problems 2.19 to 2.21 compute $AB - BA$:

2.19. $A = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$.

2.20. $A = \begin{pmatrix} 1 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$.

2.21. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 5 & 3 \\ 0 & 7 & 5 \\ 0 & 0 & 7 \end{pmatrix}$.

Matrices A and B are said to be *commutative* if $AB = BA$.

In Problems 2.22 to 2.24 find all the matrices which are commutative with the given one:

2.22. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. **2.23.** $\begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix}$. **2.24.** $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

2.25. Find all the matrices of the second order whose squares are equal to the zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

2.26. Find all the matrices of the second order whose squares are equal to the identity matrix $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2.27. How will the product AB of matrices A and B change if:

(a) the ith and jth rows of matrix A are interchanged;

(b) the jth row of matrix A multiplied by the number α is added to its ith row;

(c) the ith and jth columns of matrix B are interchanged;

(d) the jth column of matrix B multiplied by the number α is added to its ith column?

The matrix A^{T} is called a *transpose* of matrix A if the condition $a_{ij}^{\mathsf{T}} = a_{ji}$ is fulfilled for all i, j, where a_{ij} and a_{ij}^{T} are elements of matrices A_{ij} and A_{ij}^{T}, respectively.

2.28. Prove the following relationships:

(a) $(A^{\mathsf{T}})^{\mathsf{T}} = A$; (b) $(A + B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}}$;

(c) $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$.

A square matrix B is called *symmetric* if $B^{\mathsf{T}} = B$. A square matrix C is said to be *skew-symmetric* if $C^{\mathsf{T}} = -C$.

In Problems 2.29 and 2.30 compute AA^{T}:

2.29. $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 4 & -1 & 5 & -1 \end{pmatrix}$. **2.30.** $\begin{pmatrix} -1 & 1 & -1 & 1 & -1 \\ 2 & 0 & 2 & 0 & 2 \\ 0 & -2 & 0 & -2 & 0 \end{pmatrix}$.

2.31. Prove that any matrix A can be represented (in the unique way!) in the form $A = B + C$, where B is a symmetric and C is a skew-symmetric matrices.

2. Inverse of a Matrix. A square matrix A is called *singular* if its determinant is equal to zero; it is termed *nonsingular* if otherwise. If A is a nonsingular matrix, then there exists a unique matrix A^{-1}

such that

$$AA^{-1} = A^{-1}A = E,$$

where E is the identity matrix (i.e. a matrix whose elements forming the principal diagonal are unities, all the remaining elements being equal to zero). The matrix A^{-1} is defined as an *inverse* of matrix A.

Consider the basic methods for computing the inverse of a matrix.

(a) *The method of an adjoint matrix.* The matrix $A^{\vee}$ whose elements are the cofactors of the corresponding elements of matrix A (see formula (5) from Sec. 1) is said to be adjoint to matrix A. Thus

$$A^{\vee} = \begin{pmatrix} A^{(1,1)} & A^{(2,1)} & \dots & A^{(n,1)} \\ A^{(1,2)} & A^{(2,2)} & \dots & A^{(n,2)} \\ \dots\dots\dots & & & \dots \\ A^{(1,n)} & A^{(2,n)} & \dots & A^{(n,n)} \end{pmatrix}.$$

The following equality holds:

$$(A^{\vee})\ A = A\ (A^{\vee}) = \det A \cdot E.$$

Hence it follows that if A is a nonsingular matrix, then

$$A^{-1} = \frac{1}{\det A}(A^{\vee}).$$

Example 1. Using the method of an adjoint matrix, find A^{-1} if

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & -2 & 5 \end{pmatrix}.$$

We have $\det A = -4$. Let us find the cofactors of the corresponding elements of matrix A:

$$A^{(1,1)} = \begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix} = 4, \qquad A^{(2,1)} = -\begin{vmatrix} 2 & -1 \\ -2 & 5 \end{vmatrix} = -8,$$

$$A^{(3,1)} = \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = 4,$$

$$A^{(1,2)} = -\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = -7, \qquad A^{(2,2)} = \begin{vmatrix} 1 & -1 \\ 4 & 5 \end{vmatrix} = 9,$$

$$A^{(3,2)} = -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = -5,$$

$$A^{(1,3)} = \begin{vmatrix} 3 & 0 \\ 4 & -2 \end{vmatrix} = -6, \qquad A^{(2,3)} = -\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = 10,$$

$$A^{(3,3)} = \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = -6.$$

Therefore,

$$(A^{\vee}) = \begin{pmatrix} 4 & -8 & 4 \\ -7 & 9 & -5 \\ -6 & 10 & -6 \end{pmatrix} \text{ and } A^{-1} = -\frac{1}{4}(A^{\vee})$$

$$= \begin{pmatrix} -1 & 2 & -1 \\ 7/4 & -9/4 & 5/4 \\ 3/2 & -5/2 & 3/2 \end{pmatrix}.$$

(b) *The method of elementary transformations.* The following are known as the elementary transformations of a matrix:

(1) interchange of rows (columns);

(2) multiplication of a row (column) by a number different from zero;

(3) addition to the elements of a row (column) of the corresponding elements of another row (column) multiplied by some number.

For the given matrix A of order n let us construct a rectangular matrix $\Gamma_A = (A \mid E)$ of dimension $n \times 2n$, by attaching to A on its right an identity matrix. Then, using the elementary transformations on rows, we reduce the matrix Γ_A to the form $(E \mid B)$ which is always possible if A is nonsingular. Then $B = A^{-1}$.

Example 2. Using the method of elementary transformations, find A^{-1} for

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}.$$

Let us form Γ_A:

$$\Gamma_A = \left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 4 & 5 & 2 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right).$$

Denoting the rows of matrix Γ_A by γ_1, γ_2, γ_3, let us carry out the following transformations:

$$\gamma_1' = \frac{1}{3}\gamma_1, \qquad \gamma_1'' = \gamma_1' - \frac{2}{7}\gamma_2', \quad \gamma_1''' = \gamma_1'' - \frac{1}{24}\gamma_3'',$$

$$\gamma_2' = \gamma_2 - \frac{4}{3}\gamma_1, \quad \gamma_2'' = \frac{3}{7}\gamma_2', \qquad \gamma_2''' = \gamma_2'' - \frac{1}{12}\gamma_3'',$$

$$\gamma_3' = \gamma_3 - \frac{2}{3}\gamma_1, \quad \gamma_3'' = \gamma_3' + \frac{1}{7}\gamma_2', \quad \gamma_3''' = \frac{7}{24}\gamma_3''.$$

As a result, we obtain in succession

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 4 & 5 & 2 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 7/3 & 2/3 & -4/3 & 1 & 0 \\ 0 & -1/3 & 10/3 & -2/3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1/7 & 5/7 & -2/7 & 0 \\ 0 & 1 & 2/7 & -4/7 & 3/7 & 0 \\ 0 & 0 & 24/7 & -6/7 & 1/7 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -7/24 & -1/24 \\ 0 & 1 & 0 & -1/2 & 5/12 & -1/12 \\ 0 & 0 & 1 & -1/4 & 1/24 & 7/24 \end{array}\right).$$

Consequently,

$$A^{-1} = \begin{pmatrix} 3/4 & -7/24 & -1/24 \\ -1/2 & 5/12 & -1/12 \\ -1/4 & 1/24 & 7/24 \end{pmatrix}.$$

In Problems 2.32 to 2.39, using the method of an adjoint matrix, find the inverse of the given matrix:

2.32. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. **2.33.** $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$. **2.34.** $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

2.35. $\begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{pmatrix}$. **2.36.** $\begin{pmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{pmatrix}$.

2.37. $\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$. **2.38.** $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

2.39. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3/\sqrt{5} & 1/\sqrt{5} & -1/\sqrt{5} & -3/\sqrt{5} \\ 1 & -1 & -1 & 1 \\ 1/\sqrt{5} & -3/\sqrt{5} & 3/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}$.

In Problems 2.40 to 2.45, applying the method of elementary transformations, find the inverse of the given matrix:

2.40. $\begin{pmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{pmatrix}$. **2.41.** $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

2.42. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

2.43. $\begin{pmatrix} 3 & 3 & -4 & -3 \\ 0 & 6 & 1 & 1 \\ 5 & 4 & 2 & 1 \\ 2 & 3 & 3 & 2 \end{pmatrix}$. **2.44.** $\begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 1 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$.

2.45. $\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 3 & 1 & 4 \\ 2 & 7 & 6 & -1 \\ 1 & 2 & 2 & -1 \end{pmatrix}$. **2.46.** $\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 3 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & n \end{pmatrix}$.

In Problems 2.47 to 2.51 solve the given matrix equations:

2.47. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$.

2.48. $X \cdot \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}$.

2.49. $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix}$.

2.50. $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & -3 & 0 \\ 10 & 2 & 7 \\ 10 & 7 & 8 \end{pmatrix}$.

2.51. $X \cdot \begin{pmatrix} 5 & 3 & 1 \\ 1 & -3 & -2 \\ -5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -8 & 3 & 0 \\ -5 & 9 & 0 \\ -2 & 15 & 0 \end{pmatrix}$.

2.52. Prove the following equalities:

(a) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$; (b) $(AB)^{-1} = B^{-1}A^{-1}$;

(c) $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.

In Problems 2.53 to 2.55 evaluate $g(A)$ of matrix A:

2.53. $g(x) = x^2 - 3x + \frac{2}{x} - \frac{1}{x^2}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

2.54. $g(x) = x - \frac{8}{x} + \frac{16}{x^2}$, $A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$.

2.55. $g(x) = \frac{1}{x^2-1} - \frac{1}{x^2+1}$, $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

SEC. 3.3.
SPACE OF ARITHMETICAL VECTORS. RANK OF A MATRIX

1. Arithmetical Vectors. Any ordered set of n real (complex) numbers is called a *real (complex) arithmetical vector* and is denoted as

$$x = (x_1, x_2, \ldots, x_n).$$

The numbers $x_1, x_2, \ldots, x_n$ are termed the *components* of the arithmetical vector x.

We define here the following operations on arithmetical vectors.

Addition: if

$$x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n),$$

then

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n). \tag{1}$$

Multiplication by a number: if λ is a number (real or complex) and $x = (x_1, x_2, \ldots, x_n)$ is an arithmetical vector, then

$$\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n). \tag{2}$$

The set of all real (complex) arithmetical n-component vectors with the above introduced operations of addition (1) and multiplication by a number (2) is called the *space of arithmetical vectors* (real or complex, respectively). Henceforward, if not stated otherwise, we shall consider the real space of arithmetical vectors denoted by the symbol $\mathbb{R}^n$.

A system of arithmetical vectors $\{x_1, \ldots, x_s\}$ is said to be *linearly dependent* if there are numbers $\lambda_1, \ldots, \lambda_s$, not all equal to zero, such that $\lambda_1 x_1 + \ldots + \lambda_s x_s = \mathbf{0}$ (where $\mathbf{0} = (0, 0, \ldots, 0)$ is a null vector). If otherwise, then the system is called *linearly independent*.

Let Q be an arbitrary set of arithmetical vectors. A system of vectors $\mathfrak{B} = (e_1, \ldots, e_s)$ is said to be a *basis* in Q if the following conditions are fulfilled:

(a) $e_k \in Q$, $k = 1, 2, \ldots, s$;

(b) the system $\mathfrak{B} = (e_1, \ldots, e_s)$ is linearly independent;

(c) for any vector $x \in Q$ there are numbers $\lambda_1, \ldots, \lambda_s$ such that

$$x = \sum_{k=1}^{s} \lambda_k e_k. \tag{3}$$

Formula (3) is known as the *decomposition* of vector x by the basis $\mathfrak{B}$. The coefficients $\lambda_1, \ldots, \lambda_s$ are defined identically by the vector x and are called the *coordinates* of this vector in basis $\mathfrak{B}$.

The following statements hold:

(1) Any system of vectors $Q \subset \mathbb{R}^n$ has at least one basis; and it turns out that all bases of this system consist of the same number of vectors called the *rank* of system Q and denoted rank Q or $r(Q)$.

(2) The rank of the entire space $\mathbb{R}^n$ is equal to n and is said to be the *dimension* of this space, and the following system can be taken as the basis of $\mathbb{R}^n$:

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0), \\ e_2 &= (0, 1, 0, \ldots, 0), \\ e_3 &= (0, 0, 1, \ldots, 0), \\ &\ldots\ldots\ldots\ldots \\ e_n &= (0, 0, 0, \ldots, 1). \end{aligned} \tag{4}$$

This basis is used to be called *canonical*.

Let us fix an arbitrary basis $\mathfrak{B} = (e_1, \ldots, e_n)$ in space $\mathbb{R}^n$. Then any vector x can be associated one-to-one with the column of its coordinates in this basis, i.e.

$$x = x_1 e_1 + \ldots + x_n e_n \Leftrightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}.$$

Remark. One should distinguish between the components of a vector and its coordinates in some basis. We denote them in the same way, although we should bear in mind that the coordinates of a vector coincide with its components only in a canonical basis.

This is how linear operations (1) and (2) on arithmetical vectors look like in the coordinate form:

$$z = x + y \Leftrightarrow Z = X + Y \quad (\Leftrightarrow z_k = x_k + y_k, \quad k = 1, 2, \ldots, n),$$
$$y = \lambda x \Leftrightarrow Y = \lambda \cdot X \quad (\Leftrightarrow y_k = \lambda x_k, \quad k = 1, 2, \ldots, n).$$

3.1. Prove that linear operations (1) and (2) possess the following properties:

(1a) $x + y = y + x$;

(1b) $(x + y) + z = x + (y + z)$;

(1c) $x + 0 = x$;

(1d) $\forall x, y \exists! z\ (x = y + z)$ (vector z is called the *difference* between the vectors x and y and is denoted as $z = x - y$);

(2a) $\lambda(\mu x) = (\lambda\mu)\, x$ for any numbers λ and μ;

(2b) $1 \cdot x = x$;

(3a) $\lambda(x + y) = \lambda x + \lambda y$;

(3b) $(\lambda + \mu)\, x = \lambda x + \mu x$.

Given the arithmetical vectors: $a_1 = (4, 1, 3, -2)$, $a_2 = (1, 2, -3, 2)$, $a_3 = (16, 9, 1, -3)$, $a_4 = (0, 1, 2, 3)$, $a_5 = (1, -1, 15, 0)$. Find the following linear combinations (Problems 3.2 to 3.5):

3.2. $3a_1 + 5a_2 - a_3$. **3.3.** $a_1 + 2a_2 - a_4 - 2a_5$.

3.4. $2a_1 + 4a_3 - 2a_5$. **3.5.** $(1/2)\, a_1 + 3a_3 - (1/2)\, a_4 + a_5$.

Given the same arithmetical vectors a_1, a_2, a_3, a_4, a_5. Find the vector x from the following equations (Problems 3.6 to 3.9):

3.6. $2x + a_1 - 2a_2 - a_5 = 0$. **3.7.** $a_1 - 3a_5 + x + a_3 = 0$.

3.8. $2(a_1 - x) + 5(a_4 + x) = 0$.

3.9. $3(a_3 + 2x) - 2(a_5 - x) = 0$.

3.10. Prove that any system of vectors:

(a) containing two equal vectors;

(b) containing two vectors differing in numerical factor;

(c) containing a null vector;

(d) containing a linearly dependent subsystem

is linearly dependent.

Find out whether the following systems of arithmetical vectors are linearly dependent or linearly independent (Problems 3.11 to 3.15):

3.11. $x_1 = (-3, 1, 5)$, $x_2 = (6, -3, 15)$.

3.12. $x_1 = (1, 2, 3, 0)$, $x_2 = (2, 4, 6, 0)$.

3.13. $x_1 = (2, -3, 1)$, $x_2 = (3, -1, 5)$, $x_3 = (1, -4, 3)$.

3.14. $x_1 = (1, i, 2 - i, 3 + i)$, $x_2 = (1 - i, 1 + i, 1 - 3i, 4 - 2i)$.

3.15*. Show that the system of arithmetical vectors $e_1 = (1, 1, 1, 1, 1)$, $e_2 = (0, 1, 1, 1, 1)$, $e_3 = (0, 0, 1, 1, 1)$, $e_4 = (0, 0, 0, 1, 1)$, $e_5 = (0, 0, 0, 0, 1)$ forms a basis in $\mathbb{R}^5$.

In Problems 3.16 to 3.18 find the coordinates of the given vector x in the basis $\mathfrak{B} = (e_1, \ldots, e_5)$ from Problem 3.15.

3.16.** $x = (1, 0, 1, 0, 1)$. **3.17.** $x = (2, -1, 2, -1, 2)$.

3.18. $x = (5, 4, 3, 2, 1)$.

3.19. Prove that if the vectors a_1, a_2, a_3 are linearly dependent and the vector a_3 is not expressible linearly in terms of the vectors a_1 and a_2, then the vectors a_1 and a_2 differ only by a numerical factor.

3.20. Prove that if the vectors $a_1, a_2, \ldots, a_n$ are linearly independent and the vectors $a_1, a_2, \ldots, a_k, b$ are linearly dependent, then the vector b is linearly expressible in terms of the vectors $a_1, a_2, \ldots, a_k$.

3.21. Prove that an ordered system of vectors $a_1, a_2, \ldots, a_k$, containing no null vector, is linearly independent if and only if none of these vectors is linearly expressible in terms of the previous ones.

2. Rank of a Matrix. Let k rows and k columns ($k \leqslant \min(m, n)$) be arbitrarily chosen in matrix A of dimension $m \times n$. The elements lying at the intersection of the chosen rows and columns form a square matrix of order k whose determinant is called the *minor* of the kth *order* of matrix A.

The greatest order r of the nonzero minors of matrix A is called its *rank*, and any nonzero minor of order r is termed the *basis minor*.

The rows (columns) of matrix A of dimension $m \times n$ can be regarded as a system of arithmetical vectors from $\mathbb{R}^n$ ($\mathbb{R}^m$).

Theorem on the basis minor. *The rank of matrix is equal to the rank of the system of its rows (columns); the system of rows (columns) of the matrix containing the basis minor forms a basis in the system of all the rows (columns) of this matrix.*

Let us consider the basic methods for computing the rank of a matrix.

(a) *The method of bordering minors.* Let in a matrix there be found a nonzero minor M of order k. We consider only those minors of order $k + 1$ which contain (border) minor M: if all of them are zero, then the rank of the matrix is equal to k. Otherwise, among the bordering minors there can be found a nonzero minor of order $k + 1$ and the entire procedure is repeated.

Example 1. Find the rank of the matrix

$$A=\left(\begin{array}{ccc:cc} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ \hdashline 0 & 1 & -1 & 3 & 1 \\ \hdashline 4 & -7 & 4 & -4 & 5 \end{array}\right).$$

We fix a nonzero minor of the second order:

$$M_2=\begin{vmatrix} -4 & 3 \\ -2 & 1 \end{vmatrix} \neq 0.$$

The third-order minor

$$M_3=\begin{vmatrix} 2 & -4 & 3 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{vmatrix},$$

bordering the minor M_2 is also different from zero. However, both minors of the fourth order bordering M_3 are equal to zero:

$$\left|\begin{array}{ccc:c} 2 & -4 & 3 & 1 \\ 1 & -2 & 1 & -4 \\ 0 & 1 & -1 & \\ \hdashline 4 & -7 & 4 & -4 \end{array}\right|=0, \quad \left|\begin{array}{ccc:c} 2 & -4 & 3 & 0 \\ 1 & -2 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ \hdashline 4 & -7 & 4 & 5 \end{array}\right|=0.$$

Therefore, the rank of A is equal to 3. As to the basis minor, M_3 for instance, can be regarded as such.

(b) *The method of elementary transformations.* This method is based on the fact that the elementary transformations (see Sec. 3.2, subsection 2) do not change the rank of a matrix (see Problem 3.24). Using these transformations, we can reduce a matrix to such a form when all of its elements, except $a_{11}, a_{22}, \ldots, a_{rr}$ ($r \leqslant \min(m, n)$), are equal to zero. Consequently, the rank of the matrix is equal to r.

Example 2. Find the rank of the matrix

$$A=\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \\ 0 & 5 & -10 \\ 2 & 3 & 0 \end{pmatrix}.$$

Carrying out elementary transformations in succession, we shall have

$$\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \\ 0 & 5 & -10 \\ 2 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & -5 \\ 2 & 3 & 0 \\ 3 & 1 & 7 \\ 0 & 5 & -10 \\ 0 & 2 & -4 \end{pmatrix} \to$$

$$\to\begin{pmatrix}1 & 4 & -5\\ 0 & -5 & 10\\ 0 & -11 & 22\\ 0 & 5 & -10\\ 0 & 2 & -4\end{pmatrix}\to\begin{pmatrix}1 & 4 & -5\\ 0 & 1 & -2\\ 0 & 1 & -2\\ 0 & 1 & -2\\ 0 & 1 & -2\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The rank of the last matrix is equal to two, hence, the original matrix has the same rank.

In Problems 3.22 to 3.24 find the rank of the given matrix using the method of bordering minors.

3.22. $\begin{pmatrix}2 & -1 & 3 & -2 & 4\\ 4 & -2 & 5 & 1 & 7\\ 2 & -1 & 1 & 8 & 2\end{pmatrix}$. **3.23.** $\begin{pmatrix}1 & 3 & 5 & -1\\ 2 & -1 & -3 & 4\\ 5 & 1 & -1 & 7\\ 7 & 7 & 9 & 1\end{pmatrix}$.

3.24. $\begin{pmatrix}3 & -1 & 3 & 2 & 5\\ 5 & -3 & 2 & 3 & 4\\ 1 & -3 & -5 & 0 & -7\\ 7 & -5 & 1 & 4 & 1\end{pmatrix}$. **3.25.** $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 6\\ 4 & 5 & 6 & 7\end{pmatrix}$.

3.26. $\begin{pmatrix}1 & 2 & 3 & 0 & -1\\ 0 & 1 & 1 & 1 & 0\\ 1 & 3 & 4 & 1 & -1\end{pmatrix}$. **3.27.** $\begin{pmatrix}1+i & 1-i & 2+3i\\ i & 1 & 2\\ 1-i & -1-i & 3-2i\\ 4 & -4i & 10+2i\end{pmatrix}$.

What is the rank of matrix A for different values of λ (Problems 3.28 and 3.31)?

3.28. $A=\begin{pmatrix}3 & 1 & 1 & 4\\ \lambda & 4 & 10 & 1\\ 1 & 7 & 17 & 3\\ 2 & 2 & 4 & 3\end{pmatrix}$. **3.29.** $A=\begin{pmatrix}1 & \lambda & -1 & 2\\ 2 & -1 & \lambda & 5\\ 1 & 10 & -6 & \lambda\end{pmatrix}$.

3.30. $\begin{pmatrix}-1 & 3 & 3 & -4\\ 4 & -7 & -2 & 1\\ -3 & 5 & 1 & 0\\ -2 & 3 & 0 & 1\end{pmatrix}$. **3.31.** $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\\ 10 & 11 & 12\end{pmatrix}$.

3.32. Show that elementary transformations do not change the rank of a matrix.

In Problems 3.33 to 3.36 determine the rank of the matrix, using the method of elementary transformations.

3.33. $\begin{pmatrix} 25 & 31 & 17 & 43 \\ 75 & 94 & 53 & 132 \\ 75 & 94 & 54 & 134 \\ 25 & 32 & 20 & 48 \end{pmatrix}$.

3.34. $\begin{pmatrix} 47 & -67 & 35 & 201 & 155 \\ 26 & 98 & 23 & -294 & 86 \\ 16 & -428 & 1 & 1284 & 52 \end{pmatrix}$.

3.35. $\begin{pmatrix} 24 & 19 & 36 & 72 & -38 \\ 49 & 40 & 73 & 147 & -80 \\ 73 & 59 & 98 & 219 & -118 \\ 47 & 36 & 71 & 141 & -72 \end{pmatrix}$.

3.36. $\begin{pmatrix} 17 & -28 & 45 & 11 & 39 \\ 24 & -37 & 61 & 13 & 50 \\ 25 & -7 & 32 & -18 & -11 \\ 31 & 12 & 19 & -43 & -55 \\ 42 & 13 & 29 & -55 & -68 \end{pmatrix}$

In Problems 3.37 to 3.42 compute the rank of the given matrix.

3.37. $\begin{pmatrix} 1 & 3 & -1 & 6 \\ 7 & 1 & -3 & 10 \\ 17 & 1 & -7 & 22 \\ 3 & 4 & -2 & 10 \end{pmatrix}$.

3.38. $\begin{pmatrix} 0 & 1 & 10 & 3 \\ 2 & 0 & 4 & -1 \\ 16 & 4 & 52 & 9 \\ 8 & -1 & 6 & -7 \end{pmatrix}$.

3.39. $\begin{Vmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{Vmatrix}$.

3.40. $\begin{pmatrix} 0 & 1 & 0 & 4 & 3 & 1 \\ 0 & 1 & 3 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ -1 & 2 & -1 & -1 & -1 & 1 \end{pmatrix}$.

3.41. $\begin{pmatrix} 2 & 2 & 1 & 5 & -1 \\ 1 & 0 & 4 & -2 & 1 \\ 2 & 1 & 5 & 0 & 1 \\ -1 & -2 & 2 & -6 & 1 \\ -3 & -1 & -8 & 1 & -1 \\ 1 & 2 & -3 & 7 & -2 \end{pmatrix}$.

3.42. $\begin{pmatrix} 0 & -1 & 0 & 1 & 3 & 1 \\ 1 & 0 & -2 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -3 & -1 & 2 & 2 & 0 \\ 3 & 2 & -1 & -1 & -1 & -1 \end{pmatrix}$

3.43. Prove that if the product of two matrices AB is defined, then rank $(AB) \leqslant \min \{\text{rank } A, \text{ rank } B\}$.

3.44. Let A be a nonsingular matrix, and matrices B and C be such that AB and CA are defined. Prove that rank $(AB) = \text{rank } B$ and rank $(CA) = \text{rank } C$.

3.45. Prove that if the sum of matrices $A + B$ is defined, then

$$\text{rank } (A + B) \leqslant \text{rank } A + \text{rank } B.$$

The concept of the rank of a matrix is used for investigating a system of arithmetical vectors for linear dependence.

Example 3. Find out whether the system of arithmetical vectors $a_1 = (2, -3, 1)$, $a_2 = (3, -1, 5)$, $a_3 = (1, -5, -3)$ is linearly dependent or linearly independent. Find its rank and some basis.

Let us write matrix A whose column vectors are a_1, a_2, a_3:

$$A = (a_1, a_2, a_3) = \left(\begin{array}{cc|c} 2 & 3 & 1 \\ -3 & -1 & -5 \\ \hline 1 & 5 & -3 \end{array}\right).$$

As is easily seen, the rank of A is equal to 2. Consequently, the original system of arithmetical vectors is linearly dependent, and its rank is also equal to 2 (by the theorem on a basis minor). The second-order minor

$$M_2 = \begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix} = 7$$

is different from zero and therefore can be taken for a basis minor. Hence it follows that the arithmetical vectors a_1 and a_2 form the basis of the original system.

In Problems 3.46 and 3.47 find out whether the indicated systems of vectors are linearly dependent or linearly independent:

3.46. $x_1 = (1, 1, 1, 1)$, $x_2 = (1, -1, -1, 1)$, $x_3 = (1, -1, 1, -1)$, $x_4 = (1, 1, -1, -1)$.

3.47. $x_1 = (4, -5, 2, 6)$, $x_2 = (2, -2, 1, 3)$, $x_3 = (6, -3, 3, 9)$, $x_4 = (4, -1, 5, 6)$.

In Problems 3.48 and 3.49 find the rank of the given system of vectors.

3.48. $a_1 = (1, -1, 0, 0)$, $a_2 = (0, 1, -1, 0)$, $a_3 = (1, 0, -1, 1)$, $a_4 = (0, 0, 0, 1)$, $a_5 = (3, -5, 2, -3)$.

3.49. $a_1 = (1, i, -1, -i, 1)$, $a_2 = (1, -i, -1, i, 1)$, $a_3 = (1, -1, 1, -1, 1)$, $a_4 = (3, -1, -1, -1, 3)$.

In Problems 3.50 to 3.52 find all values of λ for which the vector x is linearly expressed in terms of vectors a_1, a_2, a_3:

3.50. $a_1 = (2, 3, 5)$, $a_2 = (3, 7, 8)$, $a_3 = (1, -6, 1)$, $x = (7, -2, \lambda)$.

3.51. $a_1 = (3, 2, 5)$, $a_2 = (2, 4, 7)$, $a_3 = (5, 6, \lambda)$, $x = (1, 3, 5)$.

3.52. $a_1 = (3, 2, 6)$, $a_2 = (7, 3, 9)$, $a_3 = (5, 1, 3)$, $x = (\lambda, 2, 5)$.

In Problems 3.53 to 3.55 find the rank and one of the bases of the given system of vectors:

3.53. $a_1 = (5, 2, -3, 1)$, $a_2 = (4, 1, -2, 3)$, $a_3 = (1, 1, -1, -2)$, $a_4 = (3, 4, -1, 2)$.

3.54. $a_1 = (2, -1, 3, 5)$, $a_2 = (4, -3, 1, 3)$, $a_3 = (3, -2, 3, 4)$, $a_4 = (4, -1, 15, 17)$, $a_5 = (7, -6, -7, 0)$.

3.55. $a_1 = (1, 2, 3, -4)$, $a_2 = (2, 3, -4, 1)$, $a_3 = (2, -5, 8, -3)$, $a_4 = (5, 26, -9, -12)$, $a_5 = (3, -4, 1 2)$.

In Problems 3.56 to 3.58 find the rank and all the bases of the given systems of vectors.

3.56. $a_1 = (1, 2, 0, 0)$, $a_2 = (1, 2, 3, 4)$, $a_3 = (3, 6, 0, 0)$.

3.57. $a_1 = (1, 2, 3, 4)$, $a_2 = (2, 3, 4, 5)$, $a_3 = (3, 4, 5, 6)$, $a_4 = (4, 5, 6, 7)$.

3.58. $a_1 = (2, 1, -3, 1)$, $a_2 = (4, 2, -6, 2)$, $a_3 = (6, 3, -9, 3)$, $a_4 = (1, 1, 1, 1)$.

SEC. 3.4.
SYSTEMS OF LINEAR EQUATIONS

1. Cramer's Rule. Let there be given a system of n linear equations in n unknowns of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \cdots\cdots\cdots\

$$X = A^{-1}B = -\frac{1}{2}\begin{pmatrix} -5 & 5 & 3 \\ 1 & -1 & -1 \\ 11 & -13 & -7 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$$

i.e. $x_1 = 2$, $x_2 = -1$, $x_3 = 1$.

Applying Cramer's rule, solve the following systems (Problems 4.1 to 4.8):

4.1. $3x - 5y = 13$,
$2x - 7y = 81$.

4.2. $3x - 4y = 1$,
$3x + 4y = 18$.

4.3. $2ax - 3by = 0$,
$3ax - 6by = ab$.

4.4. $7x + 2y + 3z = 15$,
$5x - 3y + 2z = 15$,
$10x - 11y + 5z = 36$.

4.5. $2x + y = 5$,
$x + 3z = 16$,
$5y - z = 10$.

4.6. $x + y - 2z = 6$,
$2x + 3y - 7z = 16$,
$5x + 2y + z = 16$.

4.7.

$$\begin{aligned} 4x_1 + 4x_2 + 5x_3 + 5x_4 &= 0, \\ 2x_1 + 3x_3 - x_4 &= 10, \\ x_1 + x_2 - 5x_3 &= -10, \\ 3x_2 + 2x_3 &= 1. \end{aligned}$$

4.8.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 4, \\ 3x_1 + 3x_2 + 3x_3 + 2x_4 &= 6, \\ 3x_1 - x_2 - x_3 - 2x_4 &= 6, \\ 3x_1 - x_2 + 3x_3 - x_4 &= 6. \end{aligned}$$

4.9*. Prove that for any distinct numbers x_1, x_2, x_3 and any numbers y_1, y_2, y_3 there exists only one polynomial $y = f(x)$ of degree $\leqslant 2$ for which

$$f(x_i) = y_i, \quad i = 1, 2, 3.$$

When is the degree of this polynomial less than 2 (equal to 1, equal to zero)?

In Problems 4.10 and 4.11 find the polynomial $f(x)$ from the given conditions:

4.10. $f(1) = -1$, $f(-1) = 9$, $f(2) = -3$.

4.11. $f_j(x_i) = \delta_{ij}$, $i, j = 1, 2, 3$, $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

In Problems 4.12 to 4.17 solve the given systems of equations:

4.12. $\begin{aligned} 5x_1 + 8x_2 + x_3 &= 2, \\ 3x_1 - 2x_2 + 6x_3 &= -7, \\ 2x_1 + x_2 - x_3 &= -5. \end{aligned}$

4.13. $\begin{aligned} 2x_1 - 3x_2 + x_3 &= -7, \\ x_1 + 4x_2 + 2x_3 &= -1, \\ x_1 - 4x_2 &= -5. \end{aligned}$

4.14. $\begin{aligned} 2x_1 + 2x_2 - x_3 + x_4 &= 4, \\ 4x_1 + 3x_2 - x_3 + 2x_4 &= 6, \\ 8x_1 + 5x_2 - 3x_3 + 4x_4 &= 12, \\ 3x_1 + 3x_2 - 2x_3 + 2x_4 &= 6. \end{aligned}$

4.15. $\begin{aligned} 2x_1 + 3x_2 + 11x_3 + 5x_4 &= 2, \\ x_1 + x_2 + 5x_3 + 2x_4 &= 1, \\ 2x_1 + x_2 + 3x_3 + 2x_4 &= -3, \\ x_1 + x_2 + 3x_3 + 4x_4 &= -3. \end{aligned}$

4.16. $\begin{aligned} 2x_1 + 5x_2 + 4x_3 + x_4 - 20 &= 0, \\ x_1 + 3x_2 + 2x_3 + x_4 - 11 &= 0, \\ 2x_1 + 10x_2 + 9x_3 + 9x_4 - 40 &= 0, \\ 3x_1 + 8x_2 + 9x_3 + 2x_4 - 37 &= 0. \end{aligned}$

4.17. $\begin{aligned} 3x_1 + 4x_2 + x_3 + 2x_4 + 3 &= 0, \\ 3x_1 + 5x_2 + 3x_3 + 5x_4 + 6 &= 0, \\ 6x_1 + 8x_2 + x_3 + 5x_4 + 8 &= 0, \\ 3x_1 + 5x_2 + 3x_3 + 7x_4 + 8 &= 0. \end{aligned}$

2. Solution of Arbitrary Systems. Let there be given a system of m linear equations in n unknowns in the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \cdots\cdots\cdots\

If $B = 0$, then the system is said to be *homogeneous*, otherwise it is called *nonhomogeneous*.

A *solution* of system (2) is understood as any n-component column vector X turning matrix equation (3) into an equality (the arithmetical vector $x \in \mathbb{R}^n$ corresponding to the solution X will also be called a solution of system (2)). A system is said to be *compatible* if it has at least one solution, otherwise it is called *incompatible*. Two systems are termed *equivalent* if the sets of their solutions coincide.

The Kronecker-Capelli theorem. *For system* (2) *to be compatible, it is necessary and sufficient that*

$$\operatorname{rank} A = \operatorname{rank} \overline{A}, \tag{4}$$

where $\overline{A} = (A \mid B)$ *is the* ***augmented matrix*** *of the system.*

Let $\operatorname{rank} A = \operatorname{rank} \overline{A} = r$, i.e. the system is compatible. Without loss of generality, we shall assume that the basis minor is found in the first r ($1 \leqslant r \leqslant \min(m, n)$) rows and columns of matrix A. Rejecting the last $m - r$ equations of system (2), let us write the *shortened* system

$$\begin{aligned} &a_{11}x_1 + \ldots + a_{1r}x_r + a_{1,r+1}x_{r+1} + \ldots + a_{1n}x_n = b_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{r1}x_1 + \ldots + a_{rr}x_r + a_{r,r+1}x_{r+1} + \ldots + a_{rn}x_n = b_r, \end{aligned} \tag{5}$$

which is equivalent to the original system. Let us call $x_1, \ldots, x_r$ *basis* unknowns and $x_{r+1}, \ldots, x_n$ *free* unknowns. Transposing the terms containing free unknowns to the right side of equation (5), we obtain a system with respect to the basis unknowns:

$$\begin{aligned} &a_{11}x_1 + \ldots + a_{1r}x_r = b_1 - a_{1,\,r+1}x_{r+1} - \ldots - a_{1n}x_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{r1}x_1 + \ldots + a_{rr}x_r = b_r - a_{r,\,r+1}x_{r+1} - \ldots - a_{rn}x_n, \end{aligned}$$

which for each set of values of the free unknowns $x_{r+1} = c_1, \ldots, x_n = c_{n-r}$ has the unique solution $x_1(c_1, \ldots, c_{n-r}), \ldots, x_r(c_1, \ldots, c_{n-r})$ found by Cramer's rule. The corresponding solution of the shortened, and, consequently, also the original systems has the form

$$X(c_1, \ldots, c_{n-r}) = \begin{pmatrix} x_1(c_1, \ldots, c_{n-r}) \\ \vdots \\ x_r(c_1, \ldots, c_{n-r}) \\ c_1 \\ \vdots \\ c_{n-r} \end{pmatrix}. \tag{6}$$

Formula (6) expressing an arbitrary solution of the system in the form of a vector function in $n - r$ free unknowns is called the *general solution* of system (2).

Example 2. Find out whether the given system is compatible or not and find its general solution

$$\begin{aligned} 2x_1 + x_2 - x_3 - 3x_4 &= 2, \\ 4x_1 + x_3 - 7x_4 &= 3, \\ 2x_2 - 3x_3 + x_4 &= 1, \\ 2x_1 + 3x_2 - 4x_3 - 2x_4 &= 3. \end{aligned}$$

Let us write the basic and augmented matrices of the system:

$$A = \left(\begin{array}{cc:cc} 2 & 1 & -1 & -3 \\ 4 & 0 & 1 & -7 \\ \hdashline 0 & 2 & -3 & 1 \\ 2 & 3 & -4 & -2 \end{array}\right), \quad \bar{A} = \left(\begin{array}{cccc|c} 2 & 1 & -1 & -3 & 2 \\ 4 & 0 & 1 & -7 & 3 \\ 0 & 2 & -3 & 1 & 1 \\ 2 & 3 & -4 & -2 & 3 \end{array}\right).$$

Since rank A = rank $\bar{A}$ = 2 (check this!), the original system is compatible.

Let us choose the minor $M_2 = \begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix}$ as a basis one. Then x_1, x_2 are basis unknowns, x_3, x_4 are free unknowns, and the shortened system has the form

$$\begin{aligned} 2x_1 + x_2 &= 2 + x_3 + 3x_4, \\ 4x_1 &= 3 - x_3 + 7x_4. \end{aligned}$$

Setting $x_3 = c_1$, $x_4 = c_2$ and solving the shortened system with respect to the basis unknowns, we get

$$\begin{aligned} x_1 &= \frac{3}{4} - \frac{1}{4}c_1 + \frac{7}{4}c_2, \\ x_2 &= \frac{1}{2} + \frac{3}{2}c_1 - \frac{1}{2}c_2. \end{aligned}$$

Consequently, the general solution of the general system has the form

$$X(c_1, c_2) = \begin{pmatrix} \frac{3}{4} - \frac{1}{4}c_1 + \frac{7}{4}c_2 \\ \frac{1}{2} + \frac{3}{2}c_1 - \frac{1}{2}c_2 \\ c_1 \\ c_2 \end{pmatrix}.$$

In Problems 4.18 to 4.31 investigate the given system for compatibility and find its general solution.

4.18. $x - \sqrt{3}y = 1$, $\sqrt{3}x - 3y = \sqrt{3}$.

4.19. $\sqrt{5}x - 5y = \sqrt{5}$, $x - \sqrt{5}y = 5$.

4.20.
$$\begin{aligned} 2x - y + z &= -2, \\ x + 2y + 3z &= -1, \\ x - 3y - 2z &= 3. \end{aligned}$$

4.21.
$$\begin{aligned} x + 2y - 4z &= 1, \\ 2x + y - 5z &= -1, \\ x - y - z &= -2. \end{aligned}$$

4.22.
$$\begin{aligned} 3x_1 - 2x_2 - 5x_3 + x_4 &= 3, \\ 2x_1 - 3x_2 + x_3 + 5x_4 &= -3, \\ x_1 + 2x_2 - 4x_4 &= -3, \\ x_1 - x_2 - 4x_3 + 9x_4 &= 22. \end{aligned}$$

4.23.
$$\begin{aligned} x_1 + x_2 - 6x_3 - 4x_4 &= 6, \\ 3x_1 - x_2 - 6x_3 - 4x_4 &= 2, \\ 2x_1 + 3x_2 + 9x_3 + 2x_4 &= 6, \\ 3x_1 + 2x_2 + 3x_3 + 8x_4 &= -7. \end{aligned}$$

4.24.
$$\begin{aligned} 2x_1 + 7x_2 + 3x_3 + x_4 &= 6, \\ 3x_1 + 5x_2 + 2x_3 + 2x_4 &= 4, \\ 9x_1 + 4x_2 + x_3 + 7x_4 &= 2. \end{aligned}$$

4.25.
$$\begin{aligned} 3x_1 - 5x_2 + 2x_3 + 4x_4 &= 2, \\ 7x_1 - 4x_2 + x_3 + 3x_4 &= 5, \\ 5x_1 + 7x_2 - 4x_3 - 6x_4 &= 3. \end{aligned}$$

4.26.
$$\begin{aligned} 9x_1 - 3x_2 + 5x_3 + 6x_4 &= 4, \\ 6x_1 - 2x_2 + 3x_3 + 4x_4 &= 5, \\ 3x_1 - x_2 + 3x_3 + 14x_4 &= -8. \end{aligned}$$

4.27.
$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 + 2x_4 &= 2, \\ 2x_1 + 3x_2 + 2x_3 + 5x_4 &= 3, \\ 9x_1 + x_2 + 4x_3 - 5x_4 &= 1, \\ 2x_1 + 2x_2 + 3x_3 + 4x_4 &= 5, \\ 7x_1 + x_2 + 6x_3 - x_4 &= 7. \end{aligned}$$

4.28.
$$\begin{aligned} x_1 + x_2 + 3x_3 - 2x_4 + 3x_5 &= 1, \\ 2x_1 + 2x_2 + 4x_3 - x_4 + 3x_5 &= 2, \\ 3x_1 + 3x_2 + 5x_3 - 2x_4 + 3x_5 &= 1, \\ 2x_1 + 2x_2 + 8x_3 - 3x_4 + 9x_5 &= 2. \end{aligned}$$

4.29.
$$\begin{aligned} 2x_1 - x_2 + x_3 + 2x_4 + 3x_5 &= 2, \\ 6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 &= 3, \\ 6x_1 - 3x_2 + 4x_3 + 8x_4 + 13x_5 &= 9, \\ 4x_1 - 2x_2 + x_3 + x_4 + 2x_5 &= 1. \end{aligned}$$

4.30.
$$\begin{aligned} 12x_1 + 14x_2 - 15x_3 + 24x_4 + 27x_5 &= 5,\\ 16x_1 + 18x_2 - 22x_3 + 29x_4 + 37x_5 &= 8,\\ 18x_1 + 20x_2 - 21x_3 + 32x_4 + 41x_5 &= 9,\\ 10x_1 + 12x_2 - 16x_3 + 20x_4 + 23x_5 &= 4. \end{aligned}$$

4.31.
$$\begin{aligned} 24x_1 + 14x_2 + 30x_3 + 40x_4 + 41x_5 &= 28,\\ 36x_1 + 21x_2 + 45x_3 + 61x_4 + 62x_5 &= 43,\\ 48x_1 + 28x_2 + 60x_3 + 82x_4 + 83x_5 &= 58,\\ 60x_1 + 35x_2 + 75x_3 + 99x_4 + 102x_5 &= 69. \end{aligned}$$

In Problems 4.32 to 4.35 investigate the given system for compatibility and find its general solution depending on the value of the parameter λ:

4.32.
$$\begin{aligned} 5x_1 - 3x_2 + 2x_3 + 4x_4 &= 3,\\ 4x_1 - 2x_2 + 3x_3 + 7x_4 &= 1,\\ 8x_1 - 6x_2 - x_3 - 5x_4 &= 9,\\ 7x_1 - 3x_2 + 7x_3 + 17x_4 &= \lambda, \end{aligned}$$

4.33.
$$\begin{aligned} \lambda x_1 + x_2 + x_3 + x_4 &= 1,\\ x_1 + \lambda x_2 + x_3 + x_4 &= 1,\\ x_1 + x_2 + \lambda x_3 + x_4 &= 1,\\ x_1 + x_2 + x_3 + \lambda x_4 &= 1. \end{aligned}$$

4.34.
$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 4x_4 &= 5,\\ 4x_1 - 2x_2 + 5x_3 + 6x_4 &= 7,\\ 6x_1 - 3x_2 + 7x_3 + 8x_4 &= 9,\\ \lambda x_1 - 4x_2 + 9x_3 + 10x_4 &= 11. \end{aligned}$$

4.35.
$$\begin{aligned} (1+\lambda)x_1 + x_2 + x_3 &= 1,\\ x_1 + (1+\lambda)x_2 + x_3 &= 1,\\ x_1 + x_2 + (1+\lambda)x_3 &= 1. \end{aligned}$$

3. **Homogeneous Systems.** The homogeneous system $AX = O$ is always compatible, since it has the *trivial* solution $X = O$. For a nontrivial solution of homogeneous system to exist, it is necessary and sufficient that $r = \operatorname{rank} A < n$ (for $m = n$ this condition means that $\det A = 0$).

Let $Q \subset \mathbb{R}^n$ be the set of all solutions of homogeneous system. Any basis in the set Q consists of $n - r$ vectors $e_1, \ldots, e_{n-r}$. The system of column vectors $E_1, \ldots, E_{n-r}$ corresponding to it in the canonical basis (see (4) from Sec. 3.3) is called a *fundamental system of solutions*. The general solution of homogeneous system has the form

$$X = c_1E_1 + \ldots + c_{n-r}E_{n-r},$$

where $c_1, \ldots, c_{n-r}$ are arbitrary constants.

The basic solutions $E_1, \ldots, E_{n-r}$ can be obtained applying the method described in the preceding subsection if the free unknowns are assigned in turn the value 1 and the remaining unknowns are set to be equal to zero.

Example 3. Find the fundamental system of solutions and the general solution of the following homogeneous system of equations:

$$\begin{aligned} 3x_1 + x_2 - 8x_3 + 2x_4 + x_5 &= 0,\\ 2x_1 - 2x_2 - 3x_3 - 7x_4 + 2x_5 &= 0,\\ x_1 + 11x_2 - 12x_3 + 34x_4 - 5x_5 &= 0,\\ x_1 - 5x_2 + 2x_3 - 16x_4 + 3x_5 &= 0. \end{aligned}$$

The matrix of the coefficients

$$A = \left(\begin{array}{cc:ccc} 3 & 1 & -8 & 2 & 1 \\ 2 & -2 & -3 & -7 & 2 \\ \hdashline 1 & 11 & -12 & 34 & -5 \\ 1 & -5 & 2 & -16 & 3 \end{array}\right)$$

has the rank $r=2$ (this should be checked!). Let us take $M_2 = \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} \neq 0$ as a basis minor. The shortened system has the form

$$\begin{aligned} 3x_1 + x_2 &= 8x_3 - 2x_4 - x_5,\\ 2x_1 - 2x_2 &= 3x_3 + 7x_4 - 2x_5, \end{aligned}$$

whence, setting $x_3 = c_1$, $x_4 = c_2$, $x_5 = c_3$, we find

$$x_1 = -\frac{19}{8}c_1 - \frac{3}{8}c_2 + \frac{1}{2}c_3, \quad x_2 = -\frac{7}{8}c_1 + \frac{25}{8}c_2 - \frac{1}{2}c_3.$$

The general solution of the system

$$X(c_1, c_2, c_3) = \begin{pmatrix} -\frac{19}{8}c_1 - \frac{3}{8}c_2 + \frac{1}{2}c_3 \\ -\frac{7}{8}c_1 + \frac{25}{8}c_2 - \frac{1}{2}c_3 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

From the general solution we find the fundamental system of solutions

$$E_1 = X(1, 0, 0) = \begin{pmatrix} -\frac{19}{8} \\ -\frac{7}{8} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad E_2 = X(0, 1, 0) = \begin{pmatrix} -\frac{3}{8} \\ \frac{25}{8} \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$E_3 = X(0, 0, 1) = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using the fundamental system, we can rewrite the general solution as follows:

$$X(c_1, c_2, c_3) = c_1E_1 + c_2E_2 + c_3E_3.$$

4.36. Prove that any linear combination of solutions of a homogeneous system of equations is also its solution.

In Problems 4.37 to 4.46 find the fundamental system of solutions and the general solution of the given system:

4.37.
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0, \\ 2x_1 + 9x_2 - 3x_3 &= 0. \end{aligned}$$

4.38.
$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= 0, \\ -2x_1 + 4x_2 + 6x_3 &= 0. \end{aligned}$$

4.39.
$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 0, \\ 2x_1 + 5x_2 + 3x_3 &= 0, \\ 3x_1 + 4x_2 + 2x_3 &= 0. \end{aligned}$$

4.40.
$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 0, \\ x_1 + x_2 + x_3 &= 0, \\ 3x_1 - 2x_2 + 2x_3 &= 0. \end{aligned}$$

4.41.
$$\begin{aligned} x_1 + 2x_2 + 4x_3 - 3x_4 &= 0, \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 &= 0, \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 &= 0, \\ 3x_1 + 8x_2 + 24x_3 - 19x_4 &= 0. \end{aligned}$$

4.42.
$$\begin{aligned} 2x_1 - 4x_2 + 5x_3 + 3x_4 &= 0, \\ 3x_1 - 6x_2 + 4x_3 + 2x_4 &= 0, \\ 4x_1 - 8x_2 + 17x_3 + 11x_4 &= 0. \end{aligned}$$

4.43.
$$\begin{aligned} 3x_1 + 2x_2 + x_3 + 3x_4 + 5x_5 &= 0, \\ 6x_1 + 4x_2 + 3x_3 + 5x_4 + 7x_5 &= 0, \\ 9x_1 + 6x_2 + 5x_3 + 7x_4 + 9x_5 &= 0, \\ 3x_1 + 2x_2 + 4x_4 + 8x_5 &= 0. \end{aligned}$$

4.44.
$$\begin{aligned} x_1 + x_3 + x_5 &= 0, \\ x_2 - x_4 + x_6 &= 0, \\ x_1 - x_2 + x_5 - x_6 &= 0, \\ x_2 + x_3 + x_6 &= 0, \\ x_1 - x_4 + x_5 &= 0. \end{aligned}$$

4.45. $\begin{aligned} 5x_1 + 6x_2 - 2x_3 + 7x_4 + 4x_5 &= 0,\\ 2x_1 + 3x_2 - x_3 + 4x_4 + 2x_5 &= 0,\\ 7x_1 + 9x_2 - 3x_3 + 5x_4 + 6x_5 &= 0,\\ 5x_1 + 9x_2 - 3x_3 + x_4 + 6x_5 &= 0. \end{aligned}$

4.46. $\begin{aligned} 3x_1 + 4x_2 + x_3 + 2x_4 + 3x_5 &= 0,\\ 5x_1 + 7x_2 + x_3 + 3x_4 + 4x_5 &= 0,\\ 4x_1 + 5x_2 + 2x_3 + x_4 + 5x_5 &= 0,\\ 7x_1 + 10x_2 + x_3 + 6x_4 + 5x_5 &= 0. \end{aligned}$

4.47*. Find out whether or not the rows of either of the matrices

$$A = \begin{pmatrix} 30 & -24 & 43 & 50 & -5 \\ 9 & -15 & 8 & 5 & 2 \\ 4 & 2 & 9 & -20 & -30 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 & 9 & -20 & -3 \\ 1 & -11 & 2 & 13 & 4 \\ 9 & -15 & 8 & 5 & 2 \end{pmatrix}$$

form a fundamental system of solutions for the system of equations

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 + x_4 + 6x_5 &= 0,\\ 5x_1 + 9x_2 + 7x_3 + 4x_4 + 7x_5 &= 0,\\ 4x_1 + 3x_2 - x_3 - x_4 + 11x_5 &= 0,\\ x_1 + 6x_2 + 8x_3 + 5x_4 - 4x_5 &= 0. \end{aligned}$$

In Problems 4.48 and 4.49 determine the values of the parameter a for which the given system has nontrivial solutions and find these solutions.

4.48. $\begin{aligned} a^2x_1 + 3x_2 + 2x_3 &= 0,\\ ax_1 - x_2 + x_3 &= 0,\\ 8x_1 + x_2 + 4x_3 &= 0. \end{aligned}$

4.49. $\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0,\\ 4x_1 - x_2 + 7x_3 &= 0,\\ x_1 + ax_2 + 2x_3 &= 0. \end{aligned}$

If we are given a nonhomogeneous system $AX = B$, then its general solution can be found as the sum of the general solution of the corresponding homogeneous system $AX = 0$ and an arbitrary particular solution of the nonhomogeneous system.

In Problems 4.50 to 4.53 find the general solution of the given nonhomogeneous system, using the fundamental system of solutions of the corresponding homogeneous system.

4.50. $\begin{aligned} 2x_1 + x_2 - x_3 - x_4 + x_5 &= 1,\\ x_1 - x_2 + x_3 + x_4 - 2x_5 &= 0,\\ 3x_1 + 3x_2 - 3x_3 - 3x_4 + 4x_5 &= 2,\\ 4x_1 + 5x_2 - 5x_3 - 5x_4 + 7x_5 &= 3. \end{aligned}$

4.51. $\begin{aligned} 2x_1 - 2x_2 + x_3 - x_4 + x_5 &= 1,\\ x_1 + 2x_2 - x_3 + x_4 - 2x_5 &= 1,\\ 4x_1 - 10x_2 + 5x_3 - 5x_4 + 7x_5 &= 1,\\ 2x_1 + 14x_2 + 7x_3 - 7x_4 + 11x_5 &= 1. \end{aligned}$

4.52. $\begin{aligned} x_1 - x_2 + x_3 - x_4 + x_5 - x_6 &= 1,\\ 2x_1 + 2x_2 + 2x_3 + x_4 - x_5 + x_6 &= 1. \end{aligned}$

4.53. $\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 0,\\ x_1 - 2x_2 - 3x_3 - 4x_4 - 5x_5 &= 2,\\ 2x_2 + 3x_3 + 4x_4 + 5x_5 &= -1. \end{aligned}$

4. Jordan-Gauss Method of Sequential Eliminations. With the aid of elementary transformations on rows and interchange of columns the augmented matrix of system (2) can be reduced to the form

$$\left(\begin{array}{cccccccc|c} 1 & 0 & \dots & 0 & a'_{1,r+1} & \dots & a'_{1n} & b'_1 \\ 0 & 1 & \dots & 0 & a'_{2,r+1} & \dots & a'_{2n} & b'_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a'_{r,r+1} & \dots & a'_{rn} & b'_r \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & b'_{r+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & b'_m \end{array}\right). \tag{7}$$

Matrix (7) is the augmented matrix of the system

$$\begin{aligned} x_1 + a'_{1,r+1}x_{r+1} + \dots + a'_{1n}x_n &= b'_1,\\ x_2 + a'_{2,r+1}x_{r+1} + \dots + a'_{2n}x_n &= b'_2,\\ \dots\dots\dots\

and formulas (8) yield an essentially explicit expression for the basic unknowns $x_1, \ldots, x_r$ in terms of the free unknowns $x_{r+1}, \ldots, x_n$.

Example 4. Using the Jordan-Gauss method, find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 + x_4 &= -3, \\ 3x_1 - x_2 - 2x_3 &= 1, \\ 2x_1 + x_2 - 2x_3 - x_4 &= 4. \\ x_1 + 3x_2 - 2x_3 - 2x_4 &= 7. \end{aligned}$$

Performing elementary transformations on the rows of the augmented matrix, we obtain

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & -3 \\ 3 & -1 & -2 & 0 & 1 \\ 2 & 1 & -2 & -1 & 4 \\ 1 & 3 & -2 & -2 & 7 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & -3 \\ 0 & 5 & -2 & -3 & 10 \\ 0 & 5 & -2 & -3 & 10 \\ 0 & 5 & -2 & -3 & 10 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccccc} 1 & 0 & -\frac{4}{5} & -\frac{1}{5} & 1 \\ 0 & 1 & -\frac{2}{5} & -\frac{3}{5} & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The first two rows of the last matrix form the augmented matrix of the system

$$\begin{aligned} x_1 - \frac{4}{5}x_3 - \frac{1}{5}x_4 &= 1, \\ x_2 - \frac{2}{5}x_3 - \frac{3}{5}x_4 &= 2, \end{aligned}$$

which is equivalent to the original one. Regarding x_1, x_2 as basic unknowns and x_3, x_4 as free ones, we get the general solution in the form

$$X(c_1, c_2) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + \frac{4}{5}c_1 + \frac{1}{5}c_2 \\ 2 + \frac{2}{5}c_1 + \frac{3}{5}c_2 \\ c_1 \\ c_2 \end{pmatrix}.$$

In Problems 4.54 to 4.58, applying the Jordan-Gauss method, investigate the given system for compatibility and find its general solution.

4.54. $\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 0, \\ 7x_1 + 14x_2 + 20x_3 + 27x_4 &= 0, \\ 5x_1 + 10x_2 + 16x_3 + 19x_4 &= -2, \\ 3x_1 + 5x_2 + 6x_3 + 13x_4 &= 5. \end{aligned}$

4.55. $\begin{aligned} x_1 + x_2 &= 1, \\ x_1 + x_2 + x_3 &= 4, \\ x_2 + x_3 + x_4 &= -3, \\ x_3 + x_4 + x_5 &= 2, \\ x_4 + x_5 &= -1. \end{aligned}$

4.56. $\begin{aligned} 105x_1 - 175x_2 - 315x_3 + 245x_4 &= 84, \\ 90x_1 - 150x_2 - 270x_3 + 210x_4 &= 72, \\ 75x_1 - 125x_2 - 225x_3 + 175x_4 &= 59. \end{aligned}$

4.57.

$$\begin{aligned} 8x_1 + 12x_2 &= 20, \\ 14x_1 + 21x_2 &= 35, \\ 9x_3 + 11x_4 &= 0, \\ 16x_3 + 20x_4 &= 0, \\ 10x_5 + 12x_6 &= 22, \\ 15x_5 + 18x_6 &= 33. \end{aligned}$$

4.58.

$$\begin{aligned} 7x_1 - 5x_2 - 2x_3 - 4x_4 &= 8, \\ -3x_1 + 2x_2 + x_3 + 2x_4 &= -3, \\ 2x_1 - x_2 - x_3 - 2x_4 &= 1, \\ -x_1 + x_3 + 24x_4 &= 1, \\ -x_2 + x_3 + 2x_4 &= 3. \end{aligned}$$

SEC. 3.5.
SOME COMPUTATIONAL PROBLEMS OF LINEAR ALGEBRA

1. Operations on Matrices.

5.1. Given matrices:

$$A = \begin{pmatrix} 1.6 & 3.2 & 0 & 8 \\ -1.6 & 3.2 & 1.6 & 6.4 \end{pmatrix}, \quad B = \begin{pmatrix} 0.625 & -3.125 \\ 1.25 & 0 \\ 0.625 & 1.25 \\ -0.625 & 0.625 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.625 & -3.125 & 0.625 \\ 1.25 & 0 & -1.25 \\ 0.625 & 1.25 & 0 \\ -0.625 & 0.625 & 3.125 \end{pmatrix}.$$

Find AB, BA, and AC.

5.2. Given matrices

$$A=\begin{pmatrix} 2.5 & -1.25 & 3.75 & -5 \\ 4.125 & -2.75 & 5.5 & -4.125 \\ 8.125 & -4.875 & -3.25 & 1.625 \\ 5.25 & -5.25 & -1.75 & 3.5 \end{pmatrix},$$

$$B=\begin{pmatrix} 2.8 & 6.4 & 3.6 & 1.8 \\ 2 & 5.6 & 2.4 & 1 \\ 1.2 & 3.2 & 3 & 1.2 \\ 0.8 & 0.8 & 0.6 & 0.4 \end{pmatrix}.$$

Find the matrices AB, BA and $A^{\top}B^{\top}$.

The method of conversion of a matrix with the aid of elementary transformations (considered in Sec. 3.2, subsection 2) can be described in the following way:

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & b_{11} & b_{12} & \dots & b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} & b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_{n1} & b_{n2} & \dots & b_{nn} \end{array}\right)$$

$$\dots\to\left(\begin{array}{cccccccc|cccc} 1 & 0 & \dots & 0 & a^{(k)}_{1,k+1} & & a^{(k)}_{1n} & & b^{(k)}_{11} & b^{(k)}_{12} & \dots & b^{(k)}_{1n} \\ 0 & 1 & \dots & 0 & a^{(k)}_{2,k+1} & & a^{(k)}_{2n} & & b^{(k)}_{21} & b^{(k)}_{22} & \dots & b^{(k)}_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a^{(k)}_{k,k+1} & \dots & a^{(k)}_{kn} & & b^{(k)}_{k1} & b^{(k)}_{k2} & \dots & b^{(k)}_{kn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a^{(k)}_{n,k+1} & \dots & a^{(k)}_{nn} & & b^{(k)}_{n1} & b^{(k)}_{n2} & \dots & b^{(k)}_{nn} \end{array}\right)$$

$$\dots\to\left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & b^{(n)}_{11} & b^{(n)}_{12} & \dots & b^{(n)}_{1n} \\ 0 & 1 & \dots & 0 & b^{(n)}_{21} & b^{(n)}_{22} & \dots & b^{(n)}_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & b^{(n)}_{n1} & b^{(n)}_{n2} & \dots & b^{(n)}_{nn} \end{array}\right),$$

where $(b_{ij}) = E$ is an identity matrix, and the elements of the matrices $(a^{(k)}_{ij})$ and $(b^{(k)}_{ij})$ are related as follows:

$$a^{(k)}_{kj}=\frac{a^{(k-1)}_{kj}}{a^{(k-1)}_{kk}}, \quad k=1, 2, \dots, n, \; j=k+1, \dots, n,$$

$$b^{(k)}_{kj}=\frac{b^{(k-1)}_{kj}}{a^{(k-1)}_{kk}}, \quad k=1, 2, \dots, n, \; j=1, 2, \dots, n,$$

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{kj}^{(k-1)}}{a_{kk}^{(k-1)}} a_{ik}^{(k-1)}, \quad i = 1, 2, \ldots, k-1, k+1, \ldots, n,$$

$$j = k+1, \ldots, n, \; a_{ik}^{(0)} = a_{ik},$$

$$b_{ij}^{(k)} = b_{ij}^{(k-1)} - \frac{b_{kj}^{(k-1)}}{a_{kk}^{(k-1)}} a_{ik}^{(k-1)}, \quad i = 1, 2, \ldots, k-1, k+1, \ldots, n,$$

$$j = 1, 2, \ldots, n, \; b_{ij}^{(0)} = b_{ij}.$$

In Problems 5.3 to 5.6 find A^{-1} for the given matrix A:

5.3. $A = \begin{pmatrix} 0.24 & 0.84 & 0.68 & 0.88 \\ -0.84 & 0.24 & -0.88 & 0.68 \\ -0.68 & 0.88 & 0.24 & -0.84 \\ -0.88 & -0.68 & 0.84 & 0.24 \end{pmatrix}$.

5.4. $A = \begin{pmatrix} -0.96875 & 0.0625 & 0.125 & 0.25 & 0.5 \\ 0.0625 & -1.875 & 0.25 & 0.5 & 1 \\ 0.125 & 0.25 & -3.5 & 1 & 2 \\ 0.25 & 0.5 & 1 & -6 & 4 \\ 0.5 & 1 & 2 & 4 & 8 \end{pmatrix}$.

5.5. $A = \begin{pmatrix} 26.4 & 35.2 & 9.9 & 13.2 \\ 44 & 61.6 & 16.5 & 23.1 \\ 16.5 & 22 & 6.6 & 8.8 \\ 27.5 & 38.5 & 11 & 15.4 \end{pmatrix}$.

5.6. $A = \begin{pmatrix} 1.2 & 2.4 & -1.2 & -2.4 \\ 3.6 & 9.6 & 0 & -4.8 \\ 2.4 & 2.4 & -4.8 & -3.6 \\ 3.6 & 9.6 & 1.2 & -7.2 \end{pmatrix}$.

2. Computation of Determinants. Let us represent the determinant

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \quad \text{for } a_{11} \neq 0$$

in the form

$$\Delta_n = a_{11}\begin{vmatrix} 1 & 0 & \dots & 0 \\ \frac{a_{21}}{a_{11}} & a_{22}^{(1)} & \dots & a_{2n}^{(1)} \\ \dots & \dots & \dots & \dots \\ \frac{a_{n1}}{a_{11}} & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} \end{vmatrix}, \quad \text{or} \quad \Delta_n = a_{11}\cdot\Delta_{n-1}.$$

The elements of the determinants Δ_n, Δ_{n-1}, ..., Δ_1, where

$$\Delta_{n-k} = \begin{vmatrix} a_{k+1,\,k+1}^{(k)} & \dots & a_{k+1,\,n}^{(k)} \\ \dots & \dots & \dots \\ a_{n,\,k+1}^{(k)} & \dots & a_{n,\,n}^{(k)} \end{vmatrix},$$

are related as follows:

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}a_{kj}^{(k-1)}}{a_{kk}^{(k-1)}}, \quad i = k+1, \dots, n, \quad j = k+1, \dots, n,$$

$$a_{ij}^{(0)} = a_{ij}, \quad \Delta_1 = a_{nn}^{(n-1)},$$

and $a_{kk}^{(k-1)} \neq 0$ for all k for $\Delta_{n-k+1} \neq 0$. Therefore

$$\Delta_n = a_{11}^{(0)}\cdot a_{22}^{(1)} \dots a_{kk}^{(k-1)} \dots a_{n}^{(n-1)}.$$

If $a_{kk}^{(k-1)} = 0$, then, taking into consideration the change of sign, we have to interchange the first and some other row of the determinant Δ_{n-k+1} so that the upper left element (called the *leading* one) is not equal to zero. To make the computations more accurate, the leading element should be chosen to be the greatest (by modulus) among all the elements of the first column of each of the determinants Δ_{n-k}, $k = 0, 1, \dots, n-2$.

Example 1. Evaluate the determinant

$$\Delta = \begin{vmatrix} 5 & 1 & -2 & 1 \\ 1 & 3 & 1 & 4 \\ -2 & 6 & 0 & 5 \\ 5 & 1 & 2 & 0 \end{vmatrix}.$$

$$\Delta = 5\cdot\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 2.8 & 1.4 & 3.8 \\ -0.4 & 6.4 & -0.8 & 5.4 \\ 1 & 0 & 4 & -1 \end{vmatrix} - 5\cdot\begin{vmatrix} 2.8 & 1.4 & 3.8 \\ 6.4 & -0.8 & 5.4 \\ 0 & 4 & -1 \end{vmatrix}$$

$$= -5\cdot\begin{vmatrix} 6.4 & -0.8 & 5.4 \\ 2.8 & 1.4 & 3.8 \\ 0 & 4 & -1 \end{vmatrix} = -5\cdot 6.4\cdot\begin{vmatrix} 1 & 0 & 0 \\ 0.4375 & 1.75 & 1.4375 \\ 0 & 4 & -1 \end{vmatrix}$$

$$= -5\cdot 6.4\cdot \begin{vmatrix} 1.75 & 1.4375 \\ 4 & -1 \end{vmatrix} = 5\cdot 6.4\cdot \begin{vmatrix} 4 & -1 \\ 1.75 & 1.4375 \end{vmatrix}$$

$$= 5\cdot 6.4\cdot 4\cdot \begin{vmatrix} 1 & 0 \\ 0.4375 & 1.875 \end{vmatrix} = 5\cdot 6.4\cdot 4\cdot 1.875 = 240.$$

Compute the following determinants (Problems 5.7 to 5.11):

5.7. $\begin{vmatrix} 1.6 & 8.1 & 1464.1 & 62.5 & 240.1 \\ 0.8 & 2.7 & 133.1 & 12.5 & 34.3 \\ 0.8 & 1.8 & 24.2 & 5 & 9.8 \\ 1 & 1.5 & 5.5 & 2.5 & 3.5 \\ 1.3 & 2.3 & 24.7 & 5.5 & 10.3 \end{vmatrix}$.

5.8. $\begin{vmatrix} 5 & 7.5 & 17.5 & 2.5 & 22.5 & 27.5 \\ 3.2 & -3.3 & 10.7 & 1.1 & 5.9 & 12.1 \\ 17.5 & 10 & 22.5 & -2.5 & 27.5 & -12.5 \\ 8.7 & 4.4 & 8.9 & -1.1 & 13.1 & -5.5 \\ 22.5 & -10 & 27.5 & 2.5 & 32.5 & 5 \\ 6.9 & -3.4 & 9.1 & 1.1 & 9.3 & 11.2 \end{vmatrix}$.

5.9. $\begin{vmatrix} 2.15 & 1.14 & 1.23 & 1.48 & 1.05 \\ 4.3 & 1.71 & 2.87 & 3.7 & 2.73 \\ 6.45 & 2.85 & 4.51 & 5.92 & 4.41 \\ 4.3 & -3.99 & 2.87 & 2.59 & 0.42 \\ 2.15 & 2.28 & 2.05 & 1.11 & 2.1 \end{vmatrix}$.

5.10. $\begin{vmatrix} 1.697 & 1.5588 & 2.2361 & 1.3856 \\ 2.9394 & 4.1243 & 3.1623 & -2.7713 \\ 3.7947 & 6.9714 & 5 & 1.9596 \\ 2.4 & 4.4091 & 3.1623 & 3.0984 \end{vmatrix}$.

5.11. $\begin{vmatrix} 1.575 & 2.4 & -0.5 & -0.75 \\ 2.1 & -2.4 & 1.5 & 1.2 \\ 1.75 & -1.6 & 1.33 & 0.7 \\ 0.84 & -0.96 & 0.5 & 0.36 \end{vmatrix}$.

3. Systems of Linear Equations. In the case of the system

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

the *Jordan-Gauss method* (see Sec. 3.4, subsection 4) consists in successive elimination of the unknowns. On eliminating the $(k-1)$th unknown there remain the equations

$$\sum_{j=k}^{n} a_{ij}^{(k)} x_j = b_i^{(k)}, \quad i = k,\ k+1,\ \ldots,\ n, \tag{2}$$

where

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)} a_{kj}^{(k)}}{a_{kk}^{(k)}}, \quad k = 0,\ 1,\ \ldots,\ n-1, \quad a_{ij}^{(0)} = a_{ij},$$

$$b_i^{(k+1)} = b_i^{(k)} - \frac{a_{ik}^{(k)} b_k^{(k)}}{a_{kk}^{(k)}}, \quad b_i^{(0)} = b_i.$$

The computations become more accurate when the leading elements $a_{kk}^{(k)}$ have the greatest modulus in the first column of the matrix of system (2).

For $k = n$ in system (2) there remains one equation from which x_n is computed. This completes the direct step of computation. The reverse step consists in successive finding x_k by already known x_{k+1}, ..., x_n, $k = n-1,\ n-2,\ \ldots,\ 1$.

Example 2. Using the Jordan-Gauss method, solve the system of equations

$$\begin{aligned} 5x_1 - x_2 + x_3 &= 9.13, \\ 2x_1 - x_2 - 5x_3 &= 25, \\ -x_1 + 4x_2 - x_3 &= 43. \end{aligned}$$

Successively eliminating x_1 and x_2 and choosing the greatest (by modulus) as the leading elements in the corresponding columns, we obtain

$$\begin{aligned} x_1 - 0.2x_2 + 0.2x_3 &= 1.826, \\ -0.6x_2 + 4.6x_3 &= 21.348, \\ 3.8x_2 - 0.8x_3 &= 44.826, \\ x_2 - 0.2105x_3 &= 11.7963, \quad (*) \\ 4.4737x_3 &= 28.4258, \end{aligned} \qquad \begin{aligned} x_1 - 0.2x_2 + 0.2x_3 &= 1.826, \quad (*) \\ 3.8x_2 - 0.8x_3 &= 44.826, \\ -0.6x_2 + 4.6x_3 &= 21.348, \\ x_3 &= 6.3540. \end{aligned}$$

From the starred equations we find first $x_2 = 13.1338$, and then $x_1 = 3.1820$.

Solve the following systems (Problems 5.12 to 5.14):

5.12.
$$\begin{aligned} 3.2x_1 + 5.4x_2 + 4.2x_3 + 2.2x_4 &= 2.6, \\ 2.1x_1 + 3.2x_2 + 3.1x_3 + 1.1x_4 &= 4.8, \\ 1.2x_1 + 0.4x_2 - 0.8x_3 - 0.8x_4 &= 3.6, \\ 4.7x_1 + 10.4x_2 + 9.7x_3 + 9.7x_4 &= -8.4. \end{aligned}$$

5.13. $6.087x_1 - 3.913x_2 + 7.547x_3 + 1.734x_4 = 3.21,$
$1.739x_1 + 0.869x_2 + 1.887x_3 + 0.73x_4 = 6.35,$
$2.174x_1 - 1.305x_2 + 2.83x_3 + 1.04x_4 = 1.5,$
$4.5x_1 - 1.305x_2 + 1.887x_3 + 0.514x_4 = -1.27.$

5.14. $2.67x_1 + 5.1x_2 + 3.31x_3 + 5.64x_4 + 4.76x_5 = 6.19,$
$4.44x_1 + 7.5x_2 + 4.67x_3 + 5.7x_4 + 6.14x_5 = 6.95,$
$5.33x_1 + 9.8x_2 + 8.67x_3 + 4.8x_4 + 7.33x_5 = 12.2,$
$3.56x_1 + 5.3x_2 + 4.15x_3 + 3.69x_4 + 3.25x_5 = 5.97,$
$1.78x_1 + 4.17x_2 + 2.67x_3 + 4.69x_4 + 3.75x_5 = 4.42.$

The iteration method. If for system (1) the inequalities

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n, \tag{3}$$

are satisfied, then its solution $X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$ satisfies the relation $X = \lim_{k \to \infty} X^{(k)}$, i.e. $x_i = \lim_{k \to \infty} x_i^{(k)}$, $i = 1, \ldots, n$, where the components of the column vectors $X^{(k)}$ are determined by the equalities

$$x_i^{(0)} = \beta_i,$$

$$x_i^{(k+1)} = \beta_i + \sum_{j=1}^{n} \alpha_{ij} x_j^{(k)}, \quad \alpha_{ii} = 0, \quad i = 1, 2, \ldots, n, \quad k = 0, 1, \ldots,$$

in which $\beta_i = b_i/a_{ii}$, $\alpha_{ij} = -a_{ij}/a_{ii}$.

Example 3. Applying the iteration method, solve the system

$$5x_1 + 0.12x_2 + 0.09x_3 = 10,$$
$$0.08x_1 + 4x_2 - 0.15x_3 = 20,$$
$$0.18x_1 - 0.06x_2 + 3x_3 = -4.5.$$

The given system satisfies conditions (3), and the principal diagonal is formed by the greatest (by modulus) elements of the rows. Let us reduce the system to the normal form:

$$x_1 = 2 - 0.024x_2 - 0.018x_3,$$
$$x_2 = 5 - 0.02x_1 + 0.03x_3,$$
$$x_3 = -1.5 - 0.06x_1 + 0.02x_2.$$

Let us choose the zero approximation $X^{(0)} = \begin{pmatrix} 2 \\ 5 \\ -1.5 \end{pmatrix}$ and find $X^{(1)}$, $X^{(2)}$, $X^{(3)}$:

$$x_1^{(1)} = 2 - 0.024 \cdot 5 - 0.018 \cdot (-1.5) = 1.907,$$
$$x_2^{(1)} = 5 - 0.02 \cdot 2 + 0.03 \cdot (-1.5) = 4.915,$$
$$x_3^{(1)} = -1.5 - 0.06 \cdot 2 + 0.02 \cdot 5 = -1.52;$$
$$x_1^{(2)} = 2 - 0.024 \cdot 4.915 - 0.018 \cdot (-1.52) = 1.90940,$$
$$x_2^{(2)} = 5 - 0.02 \cdot 1.907 + 0.03 \cdot (-1.52) = 4.91626,$$
$$x_3^{(2)} = -1.5 - 0.06 \cdot 1.907 + 0.02 \cdot 4.915 = -1.51612;$$
$$x_1^{(3)} = 2 - 0.024 \cdot 4.91626 - 0.018 \cdot (-1.51612) = 1.9092999,$$
$$x_2^{(3)} = 5 - 0.02 \cdot 1.90940 + 0.03 \cdot (-1.51612) = 4.9163284,$$
$$x_3^{(3)} = -1.5 - 0.06 \cdot 1.90940 + 0.02 \cdot 4.91626 = -1.5162388.$$

We see that $X^{(2)}$ and $X^{(3)}$ have the same three digits after the decimal point, therefore, with an accuracy of 0.001, the solution of the system is the vector

$$X = \begin{pmatrix} 1.909 \\ 4.916 \\ -1.516 \end{pmatrix}.$$

Using the iteration method, solve the following systems (Problems 5.15 and 5.16):

5.15.
$$4.1x_1 + 0.1x_2 + 0.2x_3 + 0.2x_4 = 21.14,$$
$$0.3x_1 + 5.3x_2 + 0.9x_3 - 0.1x_4 = -17.82,$$
$$0.2x_1 + 0.3x_2 + 3.2x_3 + 0.2x_4 = 9.02,$$
$$0.1x_1 + 0.1x_2 + 0.2x_3 - 9.1x_4 = 17.08.$$

5.16.
$$2.4x_1 + 0.2x_2 - 0.3x_3 - 1.1x_4 + 5.8x_5 = 23.84,$$
$$0.3x_1 + 0.1x_2 + 1.1x_3 + 10.2x_4 + x_5 = 38.85,$$
$$0.5x_1 - 6.2x_2 + 0.1x_3 + 1.5x_4 - 1.2x_5 = 17.23,$$
$$0.1x_1 + 2.1x_2 + 5.1x_3 + 0.2x_4 - 0.3x_5 = 6.56,$$
$$2.5x_1 + 0.1x_2 + 0.2x_3 + 0.3x_4 + 0.4x_5 = 6.63.$$

Answers

1.1. 18. **1.2.** $4ab$. **1.3.** 1. **1.4.** $(a-b)^2$. **1.5.** 0. **1.6.** 1. **1.7.** 1. **1.8.** $x_1 = -4$, $x_2 = -1$. **1.9.** $x = \frac{\pi}{6} + \frac{\pi k}{3}$, $k \in \mathbb{Z}$. **1.10.** *Hint.* Show that the discriminant of the corresponding quadratic trinomial is nonneg-

ative. **1.12.** 0. **1.13.** 0. **1.14.** $abc + x(ab + bc + ca)$. **1.15.** $\alpha^2 + \beta^2 + \gamma^2 + 1$. **1.16.** $\sin(\alpha - \beta) + \sin(\beta - \gamma) + \sin(\gamma - \alpha)$. **1.17.** -3. **1.18.** $3\sqrt{3}i$. **1.19.** $-4 \pm \sqrt{22}$. **1.20.** $(-\infty, +\infty)$. **1.21.** $(4, +\infty)$. **1.22.** $(-6, -4)$. **1.26.** *Hint.* Show that the last column of the original determinant can be represented in the form $\begin{pmatrix} a^3 \\ b^3 \\ c^3 \end{pmatrix} = (a+b+c)\begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} - (ab+ac+bc)\begin{pmatrix} a \\ b \\ c \end{pmatrix} + abc\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and take advantage of this representation. **1.27.** 0. **1.28.** 0. **1.29.** 0. **130.** 0. **1.33.** Parabola $y = (x-a)(x-b)$. **1.34.** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 1 & 4 \end{pmatrix}$, odd. **1.35.** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 \end{pmatrix}$, even. **1.36.** Odd. **1.37.** Odd. **1.38.** The parity of the substitution coincides with that of n. **1.39.** If n is odd then the substitution is even for any k; if n is even, then the parity of the substitution coincides with that of k. **1.40.** Enters with the minus sign. **1.41.** Enters with the plus sign. **1.42.** Does not enter. **1.43.** Enters with the plus sign. **1.44.** $i = 5$, $k = 1$. **1.45.** $i = 6$, $k = 2$. **1.46.** $10x^4 - 5x^3$. **1.47.** $(-1)^{\frac{n(n-1)}{2}} \cdot a_{1n}a_{2,n-1} \ldots a_{n1}$. **1.48.** 0. **1.49.** (a) Remains unchanged; (b) remains unchanged; (c) vanishes; (d) becomes multiplied by $(-1)^{\frac{n(n-1)}{2}}$; (e) becomes multiplied by $(-1)^{n-1}$. **1.50.** (a) $8a + 15b + 12c - 19d$; (b) $2a - 8b + c + 5d$; (c) $2a - b - c - d$. **1.51.** 0. **1.52.** 48. **1.53.** 223. **1.54.** $9\sqrt{10}(\sqrt{3} - \sqrt{2})$. **1.55.** $(be - cd)^2$. **1.56.** $(b + c + d)(b + c - d)$. **1.57.** 394. **1.58.** 665. **1.59.** $(x+1) \times (x^2 - x + 1)^2$. **1.60.** $2x^3y(x-y)^6$. **1.61.** $n!$ **1.62.** $2n+1$. **1.63.** 1. **1.64.** $(-1)^{n-1} \cdot n$. **1.65.** $-a_1a_2 \ldots a_n\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$. **1.66.** $n+1$. **1.67.** α^n. **1.68.** $\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$. **1.69.** $(-1)^{\frac{n(n-1)}{2}} \times \prod_{1 \leqslant k < i \leqslant n} (a_i - a_k)$.

2.2. $\begin{pmatrix} 2 & 5 & -3 \\ -6 & 7 & -8 \end{pmatrix}$. **2.3.** $\begin{pmatrix} 2+2i & 0 \\ 0 & 2-2i \end{pmatrix}$.

2.4. $\begin{pmatrix} 5 & 2 \\ 7 & 0 \end{pmatrix}$. **2.5.** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. **2.6.** $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. **2.7.** $\begin{pmatrix} 1 & 5 & -5 \\ 3 & 10 & 0 \\ 2 & 9 & -7 \end{pmatrix}$.

2.8. $\begin{pmatrix} 11 & -22 & 29 \\ 9 & -27 & 32 \\ 13 & -17 & 26 \end{pmatrix}$. **2.9.** $\begin{pmatrix} 56 \\ 69 \\ 17 \end{pmatrix}$. **2.10.** (31). **2.11.** $\begin{pmatrix} 5 \\ 15 \\ 25 \\ 35 \end{pmatrix}$.

2.12. $\begin{pmatrix} 13 & -14 \\ 21 & -22 \end{pmatrix}$. **2.13.** $\begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$. **2.14.** $\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

2.15. $\begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$. **2.16.** $\begin{pmatrix} 8 & 15 \\ 0 & 23 \end{pmatrix}$. **2.17.** $\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$.

2.18. $\begin{pmatrix} 21 & -23 & 15 \\ -13 & 34 & 10 \\ -9 & 22 & 25 \end{pmatrix}$. **2.19.** $\begin{pmatrix} 4 & -8 \\ 12 & -4 \end{pmatrix}$. **2.20.** $\begin{pmatrix} 4 & 1 & 9 \\ -2 & -6 & 3 \\ -8 & -9 & 2 \end{pmatrix}$.

2.21. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. **2.22.** $\begin{pmatrix} a & 2b \\ 3b & a+3b \end{pmatrix}$, where a and b are arbitrary numbers.

2.23. $\begin{pmatrix} a & 3b \\ -5b & a+9b \end{pmatrix}$, where a and b are arbitrary numbers.

2.24. $\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$, where a, b, c are arbitrary numbers. **2.25.** $\begin{pmatrix} a & b \\ a & -a \end{pmatrix}$, where a, b, c are arbitrary numbers satisfying the relationship $a^2 + bc = 0$. **2.26.** $\pm E$; $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $a^2+bc=1$. **2.27.** (a) The ith and jth rows of the product will be interchanged; (b) the jth row multiplied by α will be added to the ith row; (c) the ith and jth columns will be interchanged; (d) the jth column multiplied by α will be added to the ith column of the product. **2.29.** $\begin{pmatrix} 15 & 4 \\ 4 & 43 \end{pmatrix}$.

2.30. $\begin{pmatrix} 5 & -6 & -4 \\ -6 & 12 & 0 \\ -4 & 0 & 8 \end{pmatrix}$. **2.32.** $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$. **2.33.** $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$.

2.34. $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$. **2.35.** $\begin{pmatrix} 1 & -1 & 1 \\ -38 & 41 & -34 \\ 27 & -29 & 24 \end{pmatrix}$.

2.36. $\begin{pmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{pmatrix}$. **2.37.** $\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

2.38. $\begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/4 & -1/2 & 0 \\ 1/4 & -1/4 & 0 & 1/2 \\ 1/4 & -1/4 & 0 & -1/2 \end{pmatrix}$. **2.39.** $\begin{pmatrix} 1/4 & 3/20 & 1/4 & 1/20 \\ 1/4 & 1/20 & -1/4 & -3/20 \\ 1/4 & -1/20 & -1/4 & 3/20 \\ 1/4 & -3/20 & 1/4 & -1/20 \end{pmatrix}$.

2.40. $\begin{pmatrix} -7/3 & 2 & -1/3 \\ 5/3 & -1 & -1/3 \\ -2 & 1 & 1 \end{pmatrix}$. **2.41.** $\begin{pmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 1/9 & -2/9 \\ 2/9 & -2/9 & 1/9 \end{pmatrix}$.

2.42. $\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$. **2.43.** $\begin{pmatrix} -7 & 5 & 12 & -19 \\ 3 & -2 & -5 & 8 \\ -41 & -30 & -69 & 111 \\ -59 & 43 & 99 & -159 \end{pmatrix}$.

2.44. $\begin{pmatrix} 1 & -1 & 1 & -1 & \dots & (-1)^{n-1} \\ 0 & 1 & -1 & 1 & \dots & (-1)^{n-2} \\ 0 & 0 & 1 & -1 & \dots & (-1)^{n-3} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

2.45. $\begin{pmatrix} -1/6 & 1/2 & -7/6 & 10/3 \\ -7/6 & -1/2 & 5/6 & -5/3 \\ 3/2 & 1/2 & -1/2 & 1 \\ 1/2 & 1/2 & -1/2 & 1 \end{pmatrix}$.

2.46. $\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & -1/n \\ 0 & 1/2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1/3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1/n \end{pmatrix}$. **2.47.** $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$.

2.48. $\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$. **2.49.** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. **2.50.** $\begin{pmatrix} 6 & 4 & 5 \\ 2 & 1 & 2 \\ 3 & 3 & 3 \end{pmatrix}$. **2.51.** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

3.2. (1, 4, −7, 7). **3.3.** (4, 6, −35, −1). **3.4.** (70, 40, −20, −16). **3.5.** $\left(51, 26, 18\frac{1}{2}, -11\frac{1}{2}\right)$. **3.6.** (−1/2, 1, 3, 3). **3.7.** (−17, −13, 41, 5). **3.8.** (−8/5, −7/3, −16/3, −11/3). **3.9.** (−23/4, −29/8, 27/8, 9/8). **3.11.** Linearly independent. **3.12.** Linearly dependent. **3.13.** Linearly independent. **3.14.** Linearly dependent. **3.15.** *Hint.* Rewriting the vector equality $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 = 0$ component-wise, show that the system of four equations (with respect to α_1, α_2, α_3, α_4) thus obtained has the unique solution $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. **3.16.** We set $x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 = x$ and rewrite this equality component-wise: $x_1 = 1$, $x_1 + x_2 = 0$, $x_1 + x_2 + x_3 = 1$, $x_1 + x_2 + x_3 + x_4 = 0$, $x_1 + x_2 + x_3 + x_4 + x_5 = 1$. Solving this system, we find $x_1 = 1$, $x_2 = -1$, $x_3 = 1$, $x_4 = -1$, $x_5 = 1$. Hence, $x = e_1 - e_2 + e_3 - e_4 + e_5$. **3.17.** $2e_1 - 3e_2 + 3e_3 - 3e_4 + 3e_5$. **3.18.** $5e_1 + e_2 + e_3 + e_4 + e_5$. **3.22.** 2. **3.23.** 3. **3.24.** 3. **3.25.** 2. **3.26.** 2. **3.27.** 2. **3.28.** $r = 2$, if $\lambda = 0$, and $r = 3$, if $\lambda \neq 0$. **3.29.** $r = 3$ for any λ. **3.30.** 2. **3.31.** 2. **3.33.** 3. **3.34.** 2. **3.35.** 3. **3.36.** 2. **3.37.** 2. **3.38.** 3. **3.39.** 5. **3.40.** 4. **3.41.** 3. **3.42.** 6. **3.46.** Linearly independent. **3.47.** Linearly dependent. **3.48.** 3. **3.49.** 3. **3.50.** $\lambda =$

$= 15$. **3.51.** $\lambda \neq 12$. **3.52.** For no λ. **3.53.** $r = 3$; $\mathfrak{B} = (a_2, a_3, a_4)$. **3.54.** $r = 3$; $\mathfrak{B} = (a_1, a_2, a_5)$. **3.55.** $r = 3$, $\mathfrak{B} = (a_1, a_2, a_4)$. **3.56.** $r = 2$; $\mathfrak{B}_1 = (a_1, a_2)$, $\mathfrak{B}_2 = (a_2, a_3)$. **3.57.** $r = 2$; $\mathfrak{B}_1 = (a_1, a_2)$, $\mathfrak{B}_2 = (a_1, a_3)$, $\mathfrak{B}_3 = (a_1, a_4)$. **3.58.** $r = 2$; $\mathfrak{B}_1 = (a_1, a_4)$, $\mathfrak{B}_2 = (a_2, a_4)$, $\mathfrak{B}_3 = (a_3, a_4)$.

4.1. $x = 16$, $y = 7$. **4.2.** $x = 2$, $y = 3$. **4.3.** $x = -b$, $y = -\frac{2}{3}a$. **4.4.** $x = 2$, $y = -1$, $z = 1$. **4.5.** $x = 1$, $y = 3$, $z = 5$. **4.6.** $x = 3$, $y = 1$, $z = -1$. **4.7.** $x_1 = 1$, $x_2 = -1$, $x_3 = 2$, $x_4 = -2$. **4.8.** $x_1 = -2$, $x_2 = x_3 = x_4 = 0$. **4.9.** The degree of the polynomial is less than two if the relation $k = (y_3 - y_2)(x_2 - x_1) = (y_2 - y_1)(x_3 - x_2)$; if $k \neq 0$, then the degree is equal to unity; and if $k = 0$, it is zero. *Hint.* Prove that the determinant of the system of equations $y_i = ax_i^2 + bx_i + c$, $i = 1, 2, 3$ (with unknowns a, b, c) is nonzero. **4.10.** $f(x) = x^2 - 5x + 3$. **4.11.** $f_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$; $f_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$; $f_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$. **4.12.** $x_1 = -3$, $x_2 = 2$, $x_3 = 1$. **4.13.** $x_1 = -1$, $x_2 = 1$, $x_3 = -2$. **4.14.** $x_1 = 1$, $x_2 = 1$, $x_3 = -1$, $x_4 = -1$. **4.15.** $x_1 = -2$, $x_2 = 0$, $x_3 = 1$, $x_4 = -1$. **4.16.** $x_1 = 1$, $x_2 = 2$, $x_3 = 2$, $x_4 = 0$. **4.17.** $x_1 = 2$, $x_2 = -2$, $x_3 = 1$, $x_4 = -1$. **4.18.** $(1 + \sqrt{3}c_1, c_1)^T$. **4.19.** The system is incompatible. **4.20.** The system is incompatible. **4.21.** $(-1 + 2c_1, 1 + c_1, c_1)^T$. **4.22.** $(-1, 3, -2, 2)^T$. **4.23.** $(0, 2, 1/3, -3/2)^T$. **4.24.** $\left(-\frac{2}{11} + \frac{1}{11}c_1 - \frac{9}{11}c_2, \frac{10}{11} - \frac{5}{11}c_1 + \frac{1}{11}c_2, c_1, c_2\right)^T$. **4.25.** The system is incompatible. **4.26.** $(c_1, -13 + 3c_1, -7, 0)^T$. **4.27.** $\left(-\frac{6}{7} + \frac{8}{7}c_1, \frac{1}{7} - \frac{13}{7}c_1, \frac{15}{7} - \frac{6}{7}c_1, c_1\right)^T$. **4.28.** The system is incompatible. **4.29.** $(c_1, c_2, 5 - 8c_1 + 4c_2, -3, 1 + 2c_1 - c_2)^T$. **4.30.** $\left(\frac{20}{9} + c_1 - \frac{53}{9}c_2, -\frac{5}{3} - \frac{5}{2}c_1 + \frac{5}{6}c_2, -\frac{1}{9} + \frac{2}{9}c_2, c_1, c_2\right)^T$. **4.31.** $\left(-\frac{1}{2} - \frac{7}{12}c_1 - \frac{5}{4}c_2 - \frac{7}{8}c_3, c_1, c_2, 1 - \frac{1}{2}c_3, c_3\right)^T$. **4.32.** If $\lambda \neq 0$, then the system is incompatible; $\lambda = 0$, then $X = \left(-\frac{3}{2} - \frac{5}{2}c_1 - \frac{13}{2}c_2, -\frac{7}{2} - \frac{7}{2}c_1 - \frac{19}{2}c_2, c_1, c_2\right)^T$. **4.33.** If $(\lambda - 1) \times (\lambda + 3) \neq 0$, then $X = \frac{1}{\lambda+3}(1, 1, 1, 1)^T$; if $\lambda = -3$, then the system is incompatible. If $\lambda = 1$, then $X = (1 - c_1 - c_2 - c_3, c_1, c_2, c_3)^T$. **4.34.** If $\lambda = 8$, then $X = (c_1, 4 + 2c_1 - 2c_2, 3 - 2c_2, c_2)^T$; if $\lambda \neq 8$, then $X = (0, 4 - 2c_1, 3 - 2c_1, c_1)^T$. **4.35.** If $\lambda(\lambda + 3) \neq 0$,

then $X = \frac{1}{\lambda+3}(1, 1, 1)^T$; if $\lambda = -3$, then the system is incompatible; if $\lambda = 0$, then $X = (1 - c_1 - c_2, c_1, c_2)^T$. **4.37.** c_1E_1, $E_1 = (3, 1, 5)^T$. **4.38.** $c_1E_1 + c_2E_2$, $E_1 = (2, 1, 0)^T$, $E_2 = (3, 0, 1)^T$. **4.39.** The system has only a trivial solution. **4.40.** c_1E_1, $E_1 = (4, 1, -5)^T$. **4.41.** $c_1E_1 + c_2E_2$, $E_1 = (8, -6, 1, 0)^T$, $E_2 = (-7, 5, 0, 1)^T$. **4.42.** $c_1E_1 + c_2E_2$, $E_1 = (1, 0, -5/2, 7/2)^T$, $E_2 = (0, 1, 5, -7)^T$. **4.43.** $c_1E_1 + c_2E_2 + c_3E_3$, $E_1 = (1, 0, 0, -9/4, 3/4)^T$, $E_2 = (0, 1, 0, -3/2, 1/2)^T$, $E_3 = (0, 0, 1, -2, 1)^T$. **4.44.** $c_1E_1 + c_2E_2 + c_3E_3$, $E_1 = (1, 1, -1, 1, 0, 0)^T$, $E_2 = (-1, 0, 0, 0, 1, 0)^T$, $E_3 = (0, -1, 0, 0, 0, 1)^T$. **4.45.** $c_1E_1 + c_2E_2$, $E_1 = (0, 1/3, 1, 0, 0)^T$, $E_2 = (0, -2/3, 0, 0, 1)^T$. **4.46.** $c_1E_1 + c_2E_2$, $E_1 = (-3, 2, 1, 0, 0)^T$, $E_2 = (-5, 3, 0, 0, 1)^T$. **4.47.** The rows of matrix A do not form and those of matrix B do form a fundamental system of solutions for the given system of equations. *Hint.* If the rank of the matrix of the coefficients of the unknowns is equal to r, then it is necessary to check that (a) the rank of A (of B) is equal to $5 - r$; (b) the rows of matrix A (of B) are solutions of the original system. **4.48.** $a_1 = 2$, $X = c_1E_1$, $E_1 = (1, 0, -2)^T$; $a_2 = -4$, $X = c_1E_1$, $E_1 = (1, -24/5, -4/5)^T$. **4.49.** $a_1 = -1$, $X = c_1E_1$, $E_1 = (-5/3, 1/3, 1)^T$. **4.50.** $X_0 + c_1E_1 + c_2E_2 + c_3E_3$, $X_0 = (1/3, 1/3, 0, 0, 0)^T$, $E_1 = (0, 1, 1, 0, 0)^T$, $E_2 = (0, 1, 0, 1, 0)^T$, $E_3 = (1/3, -5/3, 0, 0, 1)^T$. **4.51.** $X_0 + c_1E_1 + c_2E_2 + c_3E_3$, $X_0 = (2/3, 1/6, 0, 0, 0)^T$, $E_1 = (0, 1/2, 1, 0, 0)^T$, $E_2 = (0, -1/2, 0, 1, 0)^T$, $E_3 = (1/3, 5/6, 0, 0, 1)^T$. **4.52.** $X_0 + c_1E_1 + c_2E_2 + c_3E_3 + c_4E_4$, $X_0 = (1/3, -1/3, 0, 0, 0, 0)^T$, $E_1 = (1, 1, 0, 0, 0, 0)^T$, $E_2 = (-1, 0, 1, 0, 0, 0)^T$, $E_3 = (0, 0, 0, 1, 1, 0)^T$, $E_4 = (0, 0, 0, -1, 0, 1)^T$. **4.53.** $X_0 + c_1E_1 + c_2E_2 + c_3E_3$, $X_0 = (1, -1/2, 0, 0, 0)$, $E_1 = (0, -3/2, 1, 0, 0)^T$, $E_2 = (0, -2, 0, 1, 0)^T$, $E_3 = (0, -5/2, 0, 0, 1)^T$. **4.54.** $(1, -1, -1, 1)^T$. **4.55.** $(6 - c, -5 + c, 3, -1 - c, c)^T$. **4.56.** The system is incompatible. **4.57.** $\left(\frac{5}{2} - \frac{3}{2}c_1, c_1, 0, 0, \frac{11}{5} - \frac{6}{5}c_2, c_2\right)^T$

4.58. $(-1 + c_1 + 2c_2, -3 + c_1 + 2c_2, c_1, c_2)^T$.

5.1.

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 11 \end{pmatrix}, \quad BA = \begin{pmatrix} 6 & -8 & -5 & -15 \\ 2 & 4 & 0 & 10 \\ -1 & 6 & 2 & 13 \\ -2 & 0 & 1 & 1 \end{pmatrix}, \quad AC = \begin{pmatrix} 0 & 0 & 22 \\ 0 & 11 & 15 \end{pmatrix}.$$

5.2.

$$AB = \begin{pmatrix} 5 & 17 & 14.25 & 5.75 \\ 9.35 & 25.3 & 22.275 & 9.625 \\ 10.4 & 15.6 & 8.775 & 6.5 \\ 4.9 & 1.4 & 3.15 & 3.5 \end{pmatrix},$$

$$BA=\begin{pmatrix}72.1 & -48.1 & 30.85 & -28.25\\ 52.85 & -34.85 & 28.75 & -25.7\\ 46.875 & -31.225 & 10.25 & -10.125\\ 12.275 & -8.225 & 4.75 & -4.925\end{pmatrix}, \quad A^{\top}B^{\top}=(BA)^{\top}.$$

5.3. $\begin{pmatrix}0.12 & -0.42 & -0.34 & -0.44\\ 0.42 & 0.12 & 0.44 & -0.34\\ 0.34 & -0.44 & 0.12 & 0.42\\ 0.44 & 0.34 & -0.42 & 0.12\end{pmatrix}$.

5.4. $\begin{pmatrix}2 & 1 & 1 & 1 & 1\\ 1 & 1.5 & 1 & 1 & 1\\ 1 & 1 & 1.25 & 1 & 1\\ 1 & 1 & 1 & 1.125 & 1\\ 1 & 1 & 1 & 1 & 1.0625\end{pmatrix}$.

5.5. $\begin{pmatrix}12.7273 & -7.2727 & -19.0909 & 10.909\\ -9.0909 & 5.4545 & 13.6364 & -8.1818\\ -31.8182 & 18.1818 & 50.9091 & 29.0909\\ 22.7272 & -13.6364 & -36.3636 & 21.8182\end{pmatrix}$.

5.6. $\begin{pmatrix}20 & 2.5 & -3.3333 & -6.6667\\ -9.5833 & -0.8333 & 1.6666 & 2.9167\\ 8.3333 & 0.8333 & -1.6666 & -2.5\\ -4.1667 & 0 & 0.8333 & 0.8333\end{pmatrix}$.

5.7. 207.36. **5.8.** 234.375. **5.9.** 2.03. **5.10.** 1.7368. **5.11.** 0.378.

5.12. $X=\begin{pmatrix}5\\ -4\\ 3\\ -2\end{pmatrix}$. **5.13.** $X=\begin{pmatrix}-0.46\\ 2.76\\ 1.802\\ 1.85\end{pmatrix}$.

5.14. $X=\begin{pmatrix}1.12\\ -2.4\\ 2.25\\ 0.53\\ 1.05\end{pmatrix}$. **5.15.** $X=\begin{pmatrix}5.2\\ -4.2\\ 3\\ -1.8\end{pmatrix}$. **5.16.** $X=\begin{pmatrix}1.5\\ -2.7\\ 2.5\\ 3.1\\ 4.3\end{pmatrix}$.

Chapter 4

ELEMENTS OF LINEAR ALGEBRA

SEC. 4.1.
LINEAR VECTOR SPACES AND SPACES WITH A SCALAR PRODUCT

1. Linear Vector Space. The set $\mathcal{L}$ is defined as a *linear vector space* if the following conditions are fulfilled:

(1) Introduced in $\mathcal{L}$ is the operation of addition of elements, i.e. $\forall x, y \in \mathcal{L}$ the mapping

$$\langle x, y \rangle \to z \in \mathcal{L}$$

(designation: $z = x + y$) is defined which possesses the following properties:

(1a) $x + y = y + x$;
(1b) $(x + y) + z = x + (y + z)$;
(1c) $\exists 0 \in \mathcal{L} \forall x \in \mathcal{L}$ $(x + 0 = x)$ (**0** is called a *zero* element);
(1d) $\forall x \in \mathcal{L}\ \exists(-x) \in \mathcal{L}$ $(x + (-x) = 0)$ (the element $-x$ is called *opposite* to the element x).

(2) Introduced in $\mathcal{L}$ is the operation of multiplication of elements by real (complex) numbers, i.e. $\forall \lambda \in \mathbb{R}$ $(\lambda \in \mathbb{C})$, $\forall x \in \mathcal{L}$ the mapping

$$\langle \lambda, x \rangle \to y \in \mathcal{L}$$

(designation: $y = \lambda x$) is defined which possesses the following properties:

(2a) $1 \cdot x = x$;
(2b) $\lambda (\lambda x) = (\lambda \mu) x$.

(3) The operations of addition of elements and their multiplication by numbers satisfy the distributive laws:

(3a) $\lambda (x + y) = \lambda x + \lambda y$;
(3b) $(\lambda + \mu) x = \lambda x + \mu x$.

The elements of a linear vector space are called *vectors*. The space $\mathcal{L}$ is said to be *real* if the operation of multiplication of vectors by a number is defined in it only for real numbers, and it is said to be *complex* if this operation is defined for complex numbers.

In Problems 1.1 to 1.5 check to see that the given set is a linear vector space.

1.1. The set $\mathscr{V}_3$ of all geometrical vectors (the operations on geometrical vectors are defined in Chapter 2, Sec. 2.1).

1.2. The set $\mathbb{R}^n$ of all arithmetical n-component vectors $x = (x_1, \ldots, x_n)$ (the operations on arithmetical vectors are defined in Chapter 3, Sec. 3.3).

1.3. The set $\mathscr{P}_n$ of all polynomials

$$p(t) = a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$$

of degree $\leqslant n - 1$ with naturally introduced operations of addition and multiplication of polynomials by numbers.

1.4. The set $C_{[a, b]}$ of all functions $f(t)$, continuous on the closed interval $[a, b]$, with naturally introduced operations of addition and multiplication of functions by numbers.

1.5. The set $\mathscr{M}_{m,n}$ of all matrices of dimension $m \times n$ (the operations on matrices are defined in Chapter 3, Sec. 3.2).

In Problems 1.6 to 1.12 find out whether the given set is a linear vector space.

1.6. The set $\mathscr{V}_1$ of all geometrical vectors collinear with a fixed straight line.

1.7. The set of all geometrical vectors emanating from the origin whose end points lie on a fixed straight line.

1.8. The set of all geometrical vectors satisfying the condition $|x| > a$, where $a > 0$ is a fixed number.

1.9. The set of all converging sequences.

1.10. The set of all diverging sequences.

1.11. The set of all functions integrable on the interval $[a, b]$.

1.12. The set of all transformations of a three-dimensional geometrical space by rotating it about a fixed axis.

A system of vectors $\{x_1, \ldots, x_s\} \subset \mathscr{L}$ is called *linearly dependent* if there can be found numbers $\lambda_1, \ldots, \lambda_s$, not all zero and such that $\lambda_1 x_1 + \ldots + \lambda_s x_s = 0$; otherwise this system is said to be *linearly independent.*

Let $Q \subset \mathscr{L}$ be an arbitrary set of vectors of a linear vector space. An ordered system of vectors $\mathfrak{B} = (e_1, \ldots, e_s)$ is called a *basis* in Q if:

(a) $e_k \in Q$, $k = 1, 2, \ldots, s$;

(b) the system $\mathfrak{B} = (e_1, \ldots, e_s)$ is linearly independent;

(c) for any $x \in Q$ there exist numbers $x_1, \ldots, x_s$ such that

$$x = \sum_{k=1}^{s} x_k e_k. \qquad (1)$$

Formula (1) is known as the *decomposition of the vector x with respect to the basis* $\mathfrak{B}$.

The coefficients $x_1, \ldots, x_s$ are determined identically by the vector x and are called the *coordinates* of this vector in the basis $\mathfrak{B}$.

If a set $Q \subset \mathcal{L}$ possesses bases, then all of them consist of the same number of vectors. This number is called the *rank of* Q and is denoted as rank Q. In particular, if the entire space $\mathcal{L}$ has a basis, then it is termed *finite-dimensional* and is symbolized $\mathcal{L}_n$, where $n = \dim \mathcal{L}$ is the number of vectors in any basis called the *dimension* of a space. Otherwise, the space $\mathcal{L}$ is *infinite-dimensional.*

Let $\mathcal{L}$ be an arbitrary n-dimensional space and let $\mathfrak{B} = (e_1, \ldots, e_n)$ be a fixed basis in it. Then to any vector $x \in \mathcal{L}$ there corresponds one-to-one the column of its coordinates in this basis:

$$x = x_1 e_1 + \ldots + x_n e_n \Leftrightarrow X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

And this is how the linear operations on vectors in coordinate form look like:

$$z = x + y \Leftrightarrow Z = X + Y.$$
$$y = \lambda x \Leftrightarrow Y = \lambda X.$$

Let $\mathfrak{B} = (e_1, \ldots, e_n)$ and $\mathfrak{B}' = (e_1', \ldots, e_n')$ be two distinct bases in $\mathcal{L}_n$. Let us now decompose each of the vectors of the basis $\mathfrak{B}'$ in the basis $\mathfrak{B}$

$$e_k' = t_{1k} e_1 + \ldots + t_{nk} e_n \Leftrightarrow E_k' = \begin{pmatrix} t_{1k} \\ \vdots \\ t_{nk} \end{pmatrix}, \quad k = 1, 2, \ldots, n.$$

The *transformation matrix* $T_{\mathfrak{B} \to \mathfrak{B}'}$ for passing from the basis $\mathfrak{B}$ to basis $\mathfrak{B}'$ is defined as the matrix

$$T_{\mathfrak{B} \to \mathfrak{B}'} = \begin{pmatrix} t_{11} & \ldots & t_{1n} \\ \cdot & \ldots & \cdot \\ t_{n1} & \ldots & t_{nn} \end{pmatrix},$$

whose kth column is the column E_k' of the coordinates of the vector e_k' in the basis $\mathfrak{B}$. If x is an arbitrary vector from $\mathcal{L}_n$, X and X' are the columns of its coordinates in the bases $\mathfrak{B}$ and $\mathfrak{B}'$ respectively, then the following equality takes place:

$$X' = (T_{\mathfrak{B} \to \mathfrak{B}'})^{-1} X \tag{2}$$

(*formula for transforming the coordinates* when the basis is transformed).

Example 1. Find the coordinates of the geometrical vector $x = -i + 2j + k$ in the basis $\mathfrak{B}'$ consisting of the vectors $e_1' = i + j$, $e_2' = j + k$, $e_3' = i + k$.

Let us write the coordinates of the vectors e_1', e_2', e_3' in the original basis $\mathfrak{B} = (i, j, k)$:

$$E_1' = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad E_2' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad E_3' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence the transformation matrix $T_{\mathfrak{B}\to\mathfrak{B}'}$ has the form

$$T_{\mathfrak{B}\to\mathfrak{B}'} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Transforming the matrix $T_{\mathfrak{B}\to\mathfrak{B}'}$ and using formula (2), we find

$$X' = (T_{\mathfrak{B}\to\mathfrak{B}'})^{-1} X = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},$$

i.e. $x = 2e_2' - e_3'$.

1.13. Let Q be an arbitrary system of vectors from $\mathcal{L}$. A subsystem $\{e_1, \ldots, e_s\} \subset Q$ is called a *maximal linearly independent subsystem* in Q if $\{e_1, \ldots, e_s\}$ is a linearly independent system and any extended system $e_1, \ldots, e_s, x$, where x is an arbitrary vector from Q, is linearly dependent. Prove that any basis in Q is a maximal linearly independent subsystem in Q, and vice versa.

1.14. Given k arbitrary vectors $x_1, \ldots, x_k$, we can make up not more than k linearly independent combinations, using these vectors. Using this result, prove that if $\mathfrak{B}$ and $\mathfrak{B}'$ are two distinct bases in system Q, then they consist of the same number of vectors (i.e. the notion of the rank of system Q holds).

1.15. Given in space $\mathcal{V}_3$ are three vectors:

$$e_1' = i + j, \quad e_2' = i - j, \quad e_3' = -i + 2j - k.$$

Prove that the system $\mathfrak{B}' = (e_1', e_2', e_3')$ is a basis in $\mathcal{V}_3$, and write the transformation matrix $T_{\mathfrak{B}\to\mathfrak{B}'}$, where $\mathfrak{B} = (e_1 = i, e_2 = j, e_3 = k)$. Find the coordinates of the vector $x = i - 2j + 2k$ in the basis $\mathfrak{B}'$.

Let $\mathfrak{B} = (i, j, k)$ and $\mathfrak{B}' = (i', j', k')$ be rectangular bases in $\mathcal{V}_3$. In Problems 1.16 to 1.18 find the transformation matrix $T_{\mathfrak{B}\to\mathfrak{B}'}$ and write the column of the coordinates of the vector $x = i - 2j + k$ in the basis $\mathfrak{B}'$.

1.16. Basis $\mathfrak{B}'$ is obtained by reversing the sense of all the three basis unit vectors of $\mathfrak{B}$.

1.17. Basis $\mathfrak{B}'$ is obtained by permutation of $\boldsymbol{i}' = \boldsymbol{j}$, $\boldsymbol{j}' = \boldsymbol{k}$, $\boldsymbol{k}' = \boldsymbol{i}$.

1.18. Basis $\mathfrak{B}'$ is obtained by rotating the basis $\mathfrak{B}$ about the unit vector $\boldsymbol{i}$ through an angle φ anticlockwise.

1.19. Find the rank and some basis of the system of geometrical vectors $\boldsymbol{x}_1 = -\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{x}_2 = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{x}_3 = -4\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{x}_4 = 3\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$.

1.20. Given in space $\mathbb{R}^4$ are the following vectors: $\boldsymbol{e}_1' = (1, 2, -1, -2)$, $\boldsymbol{e}_2' = (2, 3, 0, -1)$, $\boldsymbol{e}_3' = (1, 2, 1, 4)$, $\boldsymbol{e}_4' = (1, 3, -1, 0)$. Prove that $\mathfrak{B}' = (\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3', \boldsymbol{e}_4')$ is a basis in $\mathbb{R}^4$, and write the transformation matrix $T_{\mathfrak{B}\to\mathfrak{B}'}$, where $\mathfrak{B}$ is the canonical basis in $\mathbb{R}^4$ (see Chapter 3, Sec. 3.3). Find the coordinates of the vector $\boldsymbol{x} = (7, 14, -1, 2)$ in $\mathfrak{B}'$.

1.21. Prove that the system of arithmetical vectors $\boldsymbol{x}_1 = (1, 2, 0, 4)$, $\boldsymbol{x}_2 = (-1, 0, 5, 1)$, $\boldsymbol{x}_3 = (1, 6, 10, 14)$ is linearly dependent, and write a nontrivial relation of the form $\lambda_1\boldsymbol{x}_1 + \lambda_2\boldsymbol{x}_2 + \lambda_3\boldsymbol{x}_3 = \boldsymbol{0}$. Find the rank and all the bases of this system.

1.22. Prove that the system of matrices of the form

$$A_{\alpha\beta} = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & & \dots & \dots & \dots & & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & & \dots & \dots & \dots & & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \alpha \\ \\ \\ \\ \end{matrix}, \qquad \begin{matrix} \alpha = 1, \dots, m, \\ \beta = 1, \dots, n \end{matrix}$$

$$\begin{matrix} & & & \beta & & & \end{matrix}$$

forms a basis in the space $\mathcal{M}_{m\,n}$ of all matrices of dimension $m \times n$ and, consequently, $\dim \mathcal{M}_{m\,n} = mn$. What are the coordinates of an arbitrary matrix $A = (a_{ij}) \in \mathcal{M}_{m\,n}$ in this basis?

1.23. Prove that the system of polynomials $1, t, t^2, \dots, t^{n-1}$ forms a basis in the space $\mathcal{P}_n$ of all polynomials of degree $\leqslant n - 1$ and, consequently, $\dim \mathcal{P}_n = n$ (this basis is called *canonical*). Find the coordinates of the following polynomials:

(a) $-3t^2 + 1$ in the canonical basis of the space $\mathcal{P}_3$;
(b) $t^2 - 2t$ in the canonical basis of the space $\mathcal{P}_4$.

1.24. Prove that the system of polynomials $t^3 + t^2 + t + 1$, $t^2 + t + 1$, $t + 1$, 1 is linearly independent.

1.25. Prove that the system of polynomials $t^2 + 1$, $-t^2 + 2t$, $t^2 - t$ forms a basis in the space $\mathcal{P}_3$. Write the column of the coordinates of the polynomial $-2t^2 + t - 1$ in this basis.

1.26. Prove that for an arbitrary t_0 the system of polynomials 1, $t - t_0$, $(t - t_0)^2$, . . ., $(t - t_0)^{n-1}$ forms a basis in $\mathcal{P}_n$.

1.27. Find the transformation matrix for passing from the canonical basis 1, t, t^2, . . ., t^{n-1} to the basis 1, $t - t_0$, $(t - t_0)^2$, . . ., $(t - t_0)^{n-1}$ in $\mathcal{P}_n$.

1.28. Find the coordinates of the polynomial $t^2 - t + 2$ in the basis 1, $t - 1$, $(t - 1)^2$.

1.29. Prove that the space $\mathcal{P}$ of all polynomials is infinite-dimensional. Hence deduce that the space $C_{[a, b]}$ of function, $f(t)$, continuous on $[a, b]$ is also infinite-dimensional.

In Problems 1.30 to 1.34 the vectors e'_1, e'_2, . . ., e'_n and x are given in an arbitrary space $\mathcal{L}_n$ by their coordinates in some basis $\mathfrak{B}$. Prove that the system $\mathfrak{B}' = (e'_1, \ldots, e'_n)$ is a basis in $\mathcal{L}_n$, and find the column X' of the coordinates of the vector x in this basis.

1.30. $E'_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $E'_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $E'_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $X = \begin{pmatrix} 6 \\ 9 \\ 14 \end{pmatrix}$.

1.31. $E'_2 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, $E'_2 = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$, $E'_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,

$X = \begin{pmatrix} 6 \\ 2 \\ -7 \end{pmatrix}$.

1.32. $E'_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}$, $E'_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix}$, $E'_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}$,

$$E'_4 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} 7 \\ 14 \\ -1 \\ 2 \end{pmatrix}.$$

1.33. $E'_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, $E'_2 = \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$, $X = \begin{pmatrix} i \\ -2i \end{pmatrix}$.

1.34. $E'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $E'_2 = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$, $E'_3 = \begin{pmatrix} -1 \\ 1 & -i \\ 1 \end{pmatrix}$, $X = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

1.35. Prove the following statements:

(a) the transformation matrix $T_{\mathfrak{B}\to\mathfrak{B}'}$ is always nonsingular, and $T_{\mathfrak{B}'\to\mathfrak{B}} = (T_{\mathfrak{B}\to\mathfrak{B}'})^{-1}$;

(b) if

$$T = \begin{pmatrix} t_{11} & \dots & t_{1n} \\ & \dots & \\ t_{n1} & \dots & t_{nn} \end{pmatrix}$$

is a nonsingular matrix and $\mathfrak{B} = (e_1, \dots, e_n)$ is some basis in $\mathscr{L}_n$; then the system of vectors

$$e'_i = t_{1i}e_1 + \dots + t_{ni}e_n, \quad i = 1, 2, \dots, n,$$

also forms a basis in $\mathscr{L}_n$.

1.36. Prove that in $\mathfrak{B}$, $\mathfrak{B}'$ and $\mathfrak{B}''$ are bases in $\mathscr{L}_n$, the following matrix equality holds true:

$$T_{\mathfrak{B}\to\mathfrak{B}''} = T_{\mathfrak{B}\to\mathfrak{B}'} \cdot T_{\mathfrak{B}'\to\mathfrak{B}''}.$$

In Problems 1.37 and 1.38 the vectors $e_1, e_2, \dots, e_n$ and $e'_1, e'_2, \dots, e'_n$ are given in an arbitrary space $\mathscr{L}_n$ by their coordinates in some basis. Prove that the systems $\mathfrak{B} = (e_1, \dots, e_n)$ and $\mathfrak{B}' = (e'_1, \dots, e'_n)$ are bases in $\mathscr{L}_n$, and, using the results obtained in Problems 1.35 an 1.36, write the transformation matrix $T_{\mathfrak{B}\to\mathfrak{B}'}$.

1.37. $E_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$, $E_3 = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$;

$$E_1' = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad E_2' = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \quad E_3' = \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix}.$$

1.38. $E_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix},$

$$E_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \end{pmatrix};$$

$$E_1' = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \quad E_2' = \begin{pmatrix} -2 \\ -3 \\ -5 \\ -4 \end{pmatrix},$$

$$E_3' = \begin{pmatrix} 2 \\ 2 \\ 5 \\ 4 \end{pmatrix}, \quad E_4' = \begin{pmatrix} -2 \\ -3 \\ -4 \\ -4 \end{pmatrix}.$$

Let $\mathcal{L}$ and $\mathcal{L}'$ be two real (or complex) linear spaces. The mapping $\varphi: \mathcal{L} \to \mathcal{L}'$ of the space $\mathcal{L}$ onto the space $\mathcal{L}'$ is called isomorphism if:

(a) φ is one-to-one;

(b) $\varphi(\lambda x) = \lambda\varphi(x)$ and $\varphi(x + y) = \varphi(x) + \varphi(y)$ for any x, $y \in \mathcal{L}$ and for any number λ. If there exists isomorphism of $\mathcal{L}$ on $\mathcal{L}'$, then the spaces $\mathcal{L}$ and $\mathcal{L}'$ are called *isomorphic*: $\mathcal{L} \simeq \mathcal{L}'$.

In Problems 1.39 to 1.41* find out whether the given mapping $\mathcal{V}_3$ on $\mathbb{R}^3$ is an isomorphism.

1.39. $\varphi(xi + yj + zk) = (2x - y, z, x + y + z)$.

1.40. $\varphi(xi + yj + zk) = (x + y - 1, 2z, 3y)$.

1.41. $\varphi(xi + yj + zk) = (x + y, -y + 2z, x + 2y - 2z)$.

* Let us agree to use in the present chapter the lower-case letters x, y, z, but not the capital ones (as it was done in Chapter 2), for designating the coordinates of geometrical vectors in the rectangular basis (i, j, k), since here the capital letters will be used to denote column vectors.

1.42. The mapping $\varphi: \mathcal{L}_n \to \mathbb{R}^n$ of an arbitrary space $\mathcal{L}_n$ onto the space $\mathbb{R}^n$ of arithmetical vectors has the form

$$\varphi(x_1 e_1 + \ldots + x_n e_n) \quad (x_1, \ldots, x_n) \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix},$$

where $\mathfrak{B} = (e_1, \ldots, e_n)$ is some basis in $\mathcal{L}_n$, and $A = (a_{ij})$ is a nonsingular matrix of order n. Prove that this mapping is an isomorphism and, consequently, that $\mathcal{L}_n \simeq \mathbb{R}^n$.

1.43. Prove that the set of all complex numbers with ordinary addition and multiplication by real numbers forms a real space isomorphic with the space $\mathbb{R}^2$. Write the transformation matrix for passing from the basis $\mathfrak{B} = (1, i)$ to the basis $\mathfrak{B}' = (1+i, -i)$ in this space; write the decomposition of the number $-2 + 3i$ with respect to the basis $\mathfrak{B}'$.

2. Subspaces and Linear Manifolds. A *subspace* of a linear vector space $\mathcal{L}$ is a subset $\mathcal{L}' \subset \mathcal{L}$ which possesses the following properties:

(a) $x, y \in \mathcal{L}' \Rightarrow x + y \in \mathcal{L}'$;

(b) $x \in \mathcal{L}' \Rightarrow \lambda x \in \mathcal{L}'$ for any number λ.

If $\mathcal{L}'$ is some subspace in $\mathcal{L}$, then the set of vectors

$$\mathcal{L}' + x_0 = \{x \in \mathcal{L} \mid x = x' + x_0, \quad x' \in \mathcal{L}' \text{ for some } x_0 \in \mathcal{L}\}$$

is called a *linear manifold* obtained by a shear transformation of the subspace $\mathcal{L}'$ over the vector x_0.

1.44. Prove that any subspace $\mathcal{L}'$ of a linear vector space $\mathcal{L}$ is also a linear vector space, and $\dim \mathcal{L}' \leqslant \dim \mathcal{L}$.

In Problems 1.45 to 1.49 find out whether the given set is a subspace in the appropriate space. If the answer is in the affirmative, find its dimension.

1.45. The set of all geometrical vectors from $\mathcal{V}_3$:

(a) coplanar with a fixed plane;

(b) satisfying the condition $(x, a) = 0$, where a is a fixed vector;

(c) satisfying the condition $|x| = 1$.

1.46. The set of all vectors from $\mathbb{R}^n$ of the form:

(a) $x = (0, x_2, 0, x_4, x_5, \ldots, x_n)$;

(b) $x = (1, x_2, 1, x_4, x_5, \ldots, x_n)$.

1.47*. The set of all vectors of an arbitrary space $\mathcal{L}_n$ whose coordinates in a fixed basis satisfy the following conditions:

(a) $x_1 = x_n$; (b) $x_1 + x_2 + \ldots + x_n = 0$;

(c) $x_1 - x_2 = 1$;

(d) $a_{11}x_1 + \ldots + a_{1n}x_n = 0$,

$a_{m1}x_1 + \ldots + a_{mn}x_n = 0$,

or, in the matrix form, $AX = 0$, where A is a given matrix of dimension $m \times n$.

1.48. The set of all matrices A of order n satisfying the following conditions:

(a) $A^\top = A$ (symmetric matrices); (b) $\det A = 0$.

1.49. The set of all functions $f(t)$, continuous on the interval $[a, b]$ and satisfying the following conditions:

(a) $f(t_0) = 0$ for some $t_0 \in [a, b]$;

(b) $f(t_0) = 1$ for some $t_0 \in [a, b]$;

(c) $f(t) = a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$, i.e. $f(t)$ is a polynomial of degree not higher than $n - 1$.

Let Q be an arbitrary system of vectors from a linear vector space $\mathcal{L}$.

The linear span of the system Q is defined as the set of vectors

$$\mathcal{L}(Q) = \{x \mid x = \lambda_1 x_1 + \ldots + \lambda_s x_s, \quad x_1, \ldots, x_s \in Q\}.$$

1.50. Prove that:

(a) $\mathcal{L}(Q)$ is a subspace in $\mathcal{L}$;

(b) $\dim \mathcal{L}(Q) = \operatorname{rank} Q$, and any basis of system Q may be taken as the basis in $\mathcal{L}(Q)$.

1.51. Find the dimension of the linear span $\mathcal{L}(x_1, x_2)$ of the arithmetical vectors $x_1 = (1, 0, 2, -1)$, $x_2 = (0, -1, 2, 0)$. Show that the vector $x = (1, -1, 4, -1)$ belongs to $\mathcal{L}(x_1, x_2)$.

In Problems 1.52 and 1.53 find the dimension and some basis of the linear span of the given system of arithmetical vectors.

1.52. $x_1 = (1, 0, 0, -1)$, $x_2 = (2, 1, 1, 0)$, $x_3 = (1, 1, 1, 1)$, $x_4 = (1, 2, 3, 4)$, $x_5 = (0, 1, 2, 3)$.

1.53. $x_1 = (1, 1, 1, 1, 0)$, $x_2 = (1, 1, -1, -1, -1)$, $x_3 = (2, 2, 0, 0, -1)$, $x_4 = (1, 1, 5, 5, 2)$, $x_5 = (1, -1, -1, 0, 0)$.

1.54*. Show that the linear span of the system of polynomials $-3t^2 - 1$, $2t^2 + t$, $-t$ coincides with the space of all polynomials of degree $\leqslant 2$.

Let V be an arbitrary system of geometrical vectors. The *geometrical image* of the system V is defined as the set of points which are the terminal points of the vectors from V, provided that all the vectors emanate from the origin.

1.55. Write the equation of the geometrical image of the linear span $\mathcal{L}(\boldsymbol{a})$ and the manifold $\mathcal{L}(\boldsymbol{a}) + \boldsymbol{b}$ if $\boldsymbol{a} = -2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$.

1.56. Write the equation of the geometrical image of the linear span $\mathcal{L}(\boldsymbol{a}_1, \boldsymbol{a}_2)$ and the manifold $\mathcal{L}(\boldsymbol{a}_1, \boldsymbol{a}_2) + \boldsymbol{b}$ if $\boldsymbol{a}_1 = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{a}_2 = 2\boldsymbol{j} - \boldsymbol{k}$, and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{k}$.

1.57. Given a system of equations:

$$\begin{aligned} x_1 + x_2 - 3x_3 - x_4 + x_5 &= 1, \\ 3x_1 - x_2 + x_3 + 4x_4 + 3x_5 &= 4, \\ x_1 - 5x_2 - 9x_3 - 8x_4 + x_5 &= 0. \end{aligned}$$

(a) Prove that the set of solutions of this system is a linear manifold in $\mathbb{R}^5$.

(b) What space should be sheared to obtain this linear manifold? Find the rank and some basis of this subspace.

(c) Find some shearing vector.

3. Spaces with a Scalar Product. A real linear space $\mathcal{E}$ is called an *Euclidean space* if every pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ from $\mathcal{E}$ is associated with a real number denoted by $(\boldsymbol{x}, \boldsymbol{y})$ and called the *scalar product* of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, the following conditions being fulfilled:

(1) $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{y}, \boldsymbol{x})$;

(2) $(\boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{y}) = (\boldsymbol{x}_1, \boldsymbol{y}) + (\boldsymbol{x}_2, \boldsymbol{y})$;

(3) $(\lambda \boldsymbol{x}, \boldsymbol{y}) = \lambda (\boldsymbol{x}, \boldsymbol{y})$, $\lambda \in \mathbb{R}$;

(4) $(\boldsymbol{x}, \boldsymbol{x}) \geqslant 0$, where $(\boldsymbol{x}, \boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = 0$.

The *length* of the vector $\boldsymbol{x}$ is defined as the number

$$|\boldsymbol{x}| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})}.$$

A vector $\boldsymbol{x}$ whose length is equal to unity is said to be *normed*.

For any vectors $\boldsymbol{x}$, $\boldsymbol{y}$ of an Euclidean space the *Cauchy-Buniakovski inequality*

$$|(\boldsymbol{x}, \boldsymbol{y})|^2 \leqslant (\boldsymbol{x}, \boldsymbol{x})(\boldsymbol{y}, \boldsymbol{y}),$$

holds which enables us to determine the angle between two nonzero vectors in the following way:

$$\cos\varphi = \frac{(x, y)}{|x| \cdot |y|}.$$

The nonzero vectors $x, y \in \mathcal{E}$ are said to be *orthogonal* if $(x, y) = 0$.

The basis $\mathfrak{B} = (e_1, \ldots, e_n)$ of an n-dimensional Euclidean space $\mathcal{E}_n$ is called *orthonormal* if

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

If an arbitrary basis $(f_1, f_2, \ldots, f_n)$ is given in $\mathcal{E}_n$, then the vectors

$$e_1 = f_1, \quad e_k = f_k = \sum_{i=1}^{k-1} c_i e_i, \quad k = 2, 3, \ldots, n,$$

where $c_i = \dfrac{(f_k, e_i)}{(e_i, e_i)}$, form an orthogonal basis in this space (the *Gram-Schmidt process*).

A complex linear space $\mathcal{U}$ is called *unitary* if every pair of vectors x, y from $\mathcal{U}$ is associated with a complex number (symbolized (x, y)) called the *scalar product* of the vectors x and y, the following conditions being fulfilled:

(1) $(x, y) = \overline{(y, x)}$;
(2) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$;
(3) $(\lambda x, y) = \lambda (x, y)$, $\lambda \in \mathbb{C}$;
(4) $(x, x) \geqslant 0$, where $(x, x) = 0 \Leftrightarrow x = 0$.

In a unitary space the angle between vectors is not defined. But the rest of the definitions and results formulated above for an Euclidean space remain valid for a unitary space as well.

Euclidean and unitary spaces will be called *spaces with a scalar product*.

1.58. Prove the following properties of the scalar product of a unitary space:

(a) $(x_1, y_1 + y_2) = (x, y_1) + (x, y_2)$;
(b) $(x, \lambda y) = \bar{\lambda} (x, y)$;
(c) $(x_1 - x_2, y) = (x_1, y) - (x_2, y)$;
(d) $(x, 0) = 0$.

1.59. Prove that the basis $\mathfrak{B} = (e_1, \ldots, e_n)$ in a unitary space $\mathcal{U}_n$ is orthonormal if and only if any of the following conditions is fulfilled:

(a) if $x = x_1 e_1 + \ldots + x_n e_n$ and $y = y_1 e_1 + \ldots + y_n e_n$, then $(x, y) = x_1 \bar{y}_1 + \ldots + x_n \bar{y}_n$;

(b) if $x = x_1e_1 + \ldots + x_ne_n$, then $x_k = (x, e_k)$, $k = 1, \ldots, n$.

1.60. Prove that any system of pairwise orthogonal vectors is linearly independent.

1.61. Using the Cauchy-Buniakovski inequality, prove the following *triangle inequalities*:

(a) $|x + y| \leqslant |x| + |y|$;

(b) $||x| - |y|| \leqslant |x + y|$.

1.62. (a) Prove that in space $\mathbb{R}^n$ the formula

$$(x, y) = x_1y_1 + \ldots + x_ny_n,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. specify a scalar product (the Euclidean space of arithmetical vectors thus obtained will also be denoted by the symbol $\mathbb{R}^n$).

(b) Show that in the Euclidean space $\mathbb{R}^n$ the canonical basis (see Chapter 3, Sec. 3.3) is orthonormal.

(c) Write the Cauchy-Buniakovski inequality for the Euclidean space $\mathbb{R}^n$.

(d) Write the triangle inequalities in the Euclidean space $\mathbb{R}^n$.

1.63. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be arbitrary vectors of the arithmetical space $\mathbb{R}^2$. Show that the scalar product in $\mathbb{R}^2$ can be defined in the following ways:

(a) $(x, y) = 2x_1y_1 + 5x_2y_2$;

(b) $(x, y) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$.

Compute the scalar product of the vectors $x = (1, -2)$ and $y = (5, 1)$, using either of the indicated methods.

1.64. Prove that in the space $\mathcal{P}_n$ of polynomials of degree $\leqslant n-1$ the scalar product of the polynomials

$$p(t) = a_0 + a_1t + \ldots + a_{n-1}t^{n-1}$$

and

$$q(t) = b_0 + b_1t + \ldots + b_{n-1}t^{n-1}$$

can be defined by the following methods:

(a) $(p, q) = a_0b_0 + a_1b_1 + \ldots + a_{n-1}b_{n-1}$;

(b) $(p, q) = \sum_{k=1}^{n} p(t_k)\, q(t_k)$, $t_1, \ldots, t_n$ are arbitrary pairwise distinct real numbers.

Compute the scalar product of the polynomials $p(t) = 1 + t + t^2$ and $q(t) = t - 2t^2 + 3t^3$, using either of the indicated methods ($n = 4$) if in case (b) $t_1 = -2$, $t_2 = -1$, $t_3 = 1$, $t_4 = 2$.

1.65. (a) Prove that in space $C_{[a, b]}$ the relation

$$(f, g) = \int_a^b f(t)\, g(t)\, dt$$

defines a scalar product.

(b) Write the Cauchy-Buniakovski inequality for this space.

(c) Write the triangle inequalities for this space.

Apply the orthogonalization process to the following systems of vectors of Euclidean space $\mathbb{R}^n$ (see Problem 1.62).

1.66. $f_1 = (1, -2, 2)$, $f_2 = (-1, 0, -1)$, $f_3 = (5, -3, -7)$.

We set $e_1 = f_1 = (1, -2, 2)$ and find the vector e_2 in the form $e_2 = f_2 - c_1 e_1$. Since $(f_2, e_1) = -3$ and $(e_1, e_1) = 9$, we obtain $c_1 = \frac{(f_2, e_1)}{(e_1, e_1)} = -1/3$. Hence, $e_2 = (-2/3, -2/3, -1/3)$. Finally, we find the vector e_3 in the form of the following linear combination: $e_3 = f_3 - c_1 e_1 - c_2 e_2$. Computing the scalar products $(f_3, e_1) = -3$, $(f_3, e_2) = 1$, $(e_2, e_2) = 1$, we evaluate the coefficients: $c_1 = \frac{(f_3, e_1)}{(e_1, e_1)} = -1/3$, $c_2 = \frac{(f_3, e_2)}{(e_2, e_2)} = 1$. Consequently, $e_3 = (6, -3, -6)$.

1.67. $f_1 = (1, 1, 1, 1)$, $f_2 = (3, 3, -1, -1)$, $f_3 = (-2, 0, 6, 8)$.

1.68. $f_1 = (1, 2, 1, 3)$, $f_2 = (4, 1, 1, 1)$, $f_3 = (3, 1, 1, 0)$.

1.69. $f_1 = (1, 2, 2, -1)$, $f_2 = (1, 1, -5, 3)$, $f_3 = (3, 2, 8, -7)$.

1.70. $f_1 = (2, 1, 3, -1)$, $f_2 = (7, 4, 3, -3)$, $f_3 = (1, 1, -6, 0)$, $f_4 = (5, 7, 7, 8)$.

In Problems 1.71 to 1.74 check the orthogonality of the given system of vectors in the Euclidean space $\mathbb{R}^n$ and add it to obtain an orthogonal basis.

1.71*. $e_1 = (1, -2, 1, 3)$, $e_2 = (2, 1, -3, 1)$.

1.72. $e_1 = (2/3, 1/3, 2/3)$, $e_2 = (1/3, 2/3, -2/3)$.

1.73. $e_1 = (1, 1, 1, 2)$, $e_2 = (1, 2, 3, -3)$.

1.74. $e_1 = (1, 1, 1, 2, 1)$, $e_2 = (1, 0, 0, 1, -2)$, $e_3 = (2, 1, -1, 0, 2)$.

1.75. Let L be a linear subspace in $\mathscr{E}_n$. Prove that:

(a) any vector $x \in \mathscr{E}_n$ is uniquely representable in the form $x = y + z$, where $y \in L$ and z is orthogonal to L (y is called the *orthogonal projection* of the vector x on L, and z the *orthogonal component* of x with respect to L);

(b) if $\mathfrak{B} = (e_1, \ldots, e_k)$ is the basis of L, then $y = \sum_{i=1}^{n} c_i e_i$, where the coefficients c_i, $i = 1, 2, \ldots, k$, are found identically from the system of equations

$$\sum_{i=1}^{k} (e_j, e_i) c_i = (e_j, x), \quad j = 1, 2, \ldots, k,$$

and $z = x - y$.

In Problems 1.76 to 1.78, using the result obtained in Problem 1.75, find the orthogonal projection y and the orthogonal component z of the vector x on the linear subspace L of the Euclidean space $\mathbb{R}^n$;

1.76. $x = (4, -1, -3, 4)$, L is spanned on the vectors: $e_1 = (1, 1, 1, 1)$, $e_2 = (1, 2, 2, -1)$, $e_3 = (1, 0, 0, 3)$.

1.77. $x = (5, 2, -2, 2)$, L is spanned on the vectors: $e_1 = (2, 1, 1, -1)$, $e_2 = (1, 1, 3, 0)$, $e_3 = (1, 2, 8, 1)$.

1.78. $x = (-3, 5, 9, 3)$, L is spanned on the vectors: $e_1 = (1, 1, 1, 1)$, $e_2 = (2, -1, 1, 1)$, $e_3 = (2, -7, -1, -1)$.

1.79. Prove that in a real Euclidean space there hold true the Pythagoras' theorem and also its converse: two vectors x and y are orthogonal if and only if $|x - y|^2 = |x^2| + |y|^2$.

1.80*. Prove that Pythagoras' theorem holds in a unitary space as well: if two vectors are mutually orthogonal, then $|x - y|^2 = |x|^2 + |y|^2$. Show that in this case the converse is false.

SEC. 4.2.
LINEAR OPERATORS

1. Algebra of Linear Operators. A *linear operator* in linear vector space $\mathcal{L}$ is defined as any mapping A: $\mathcal{L} \to \mathcal{L}$ of the space $\mathcal{L}$ into itself possessing the following properties:

$$A(\lambda x) = \lambda Ax \text{ and } A(x + y) = Ax + Ay.$$

Let A be a linear operator in a finite-dimensional space $\mathcal{L}_n$ and let $\mathfrak{B} = (e_1, \ldots, e_n)$ be some fixed basis. Decompose the vectors Ae_k, $k = 1, \ldots, n$, in the basis $\mathfrak{B}$:

$$Ae_k = a_{1k}e_1 + \ldots + a_{nk}e_n, \quad k = 1, \ldots, n.$$

Then the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

is called the *matrix of operator A in basis* $\mathfrak{B}$. The matrix of operator A will sometimes be denoted by the symbol $[A]$ or $[A]_{\mathfrak{B}}$ if it is essential what basis is meant.

By giving a matrix, the operator is defined uniquely, namely: if $y = Ax$, then $Y = AX$, where X, Y are columns of the coordinates of the vectors x, y and A is the matrix of the operator A in the basis $\mathfrak{B}$.

Let A and A' be the matrices of the operator A in the bases $\mathfrak{B}$ and $\mathfrak{B}'$ and let $T = T_{\mathfrak{B} \to \mathfrak{B}'}$ be the transformation matrix for passing from the basis $\mathfrak{B}$ to basis $\mathfrak{B}'$. Then, when transforming the basis, the formula for transforming the matrix of the operator has the form

$$A' = T^{-1}AT. \tag{1}$$

Example 1. In the basis $\mathfrak{B} = (i, j, k)$ write the matrix of the operator P_α for projecting on the plane α: $x + y + z = 0$.

The operator for projecting on the plane α is defined by the equality $P_\alpha x = x_\alpha$, where x_α is the orthogonal projection of the vector x on the plane α. We have

$$P_\alpha x = x - x_n = x - \operatorname{pr}_n x \cdot \frac{n}{|n|} = x - \frac{(n, x)}{|n|^2} n,$$

where n is the normal vector of the plane α. In the case under consideration $n = i + j + k$ and, consequently,

$$P_\alpha i = i - \frac{1}{3} n = \frac{2}{3} i - \frac{1}{3} j - \frac{1}{3} k,$$

$$P_\alpha j = j - \frac{1}{3} n = -\frac{1}{3} i + \frac{2}{3} j - \frac{1}{3} k,$$

$$P_\alpha k = k - \frac{1}{3} n = -\frac{1}{3} i - \frac{1}{3} j + \frac{2}{3} k,$$

whence

$$P_\alpha = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}.$$

On linear operators functioning in a fixed space $\mathscr{L}$ the following operations are introduced:

(a) addition of operators: $(A + B)\,x = Ax + Bx$; where $[A + B] = A + B$;

(b) multiplication of operators by numbers: $(\lambda A)\,x = \lambda\,(Ax)$; and $[\lambda A] = \lambda A$;

(c) multiplication of operators: $(AB)\,x = A(Bx)$; where $[AB] = AB$.

The *inverse* of the operator A is defined as an operator A^{-1} such that $AA^{-1} = A^{-1}A = E$, where E is a *unit* operator realizing an identical mapping. The operator A has an inverse (and in this case it is called *nonsingular*) if and only if its matrix A is nonsingular (in any basis); in this case $[A^{-1}] = A^{-1}$.

In Problems 2.1 to 2.7 determine which of the given mappings of the space $\mathscr{V}_3$ into itself are linear operators; write their matrices in the rectangular basis $\mathfrak{B} = (i, j, k)$.

2.1. $Ax = \lambda x$, λ is a fixed number.

2.2. $Ax = \lambda x + y$, λ and a are fixed.

2.3. $Ax = (x, e)\,e$, where e is a given unit vector. Find out the geometrical sense of this mapping.

2.4. $Ax = [a, x]$, a is a fixed vector.

2.5. $Ax = (a, x)\,x$, a is a fixed vector.

2.6.** $U(e, \varphi)$ is a mapping consisting in rotation through an angle φ about an axis given by the unit vector e.

2.7. If $x = xi + yj + zk$, then

$$Ax = (y + z)\,i + (2x + z)\,j + (3x - y + z)\,k.$$

In Problems 2.8 to 2.10 determine which of the given mappings of the space of arithmetical vectors $\mathbb{R}^3$ into itself are linear operators; write their matrices in the canonical basis.

2.8. $Ax = (x_2 + x_3,\ 2x_1 + x_3,\ 3x_1 - x_2 + x_3)$.

2.9. $Ax = (x_1,\ x_2 + 1,\ x_3 + 2)$.

2.10. $Ax = (0,\ x_2 - x_3,\ 0)$.

2.11. Given in $\mathscr{L}_4$ is a linear operator A whose matrix in some basis $\mathscr{B} = (e_1, e_2, e_3, e_4)$ is equal to

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

Find the matrix of this operator in the following bases:

(a) $\mathscr{B}' = (e_1, e_3, e_2, e_4)$;

(b) $\mathscr{B}' = (e_1, e_1 + e_2, e_1 + e_2 + e_3, e_1 + e_2 + e_3 + e_4)$.

2.12. Given in $\mathscr{L}_3$ are two bases:

$$\mathscr{B}'\colon\ e_1' = 8e_1 - 6e_2 + 7e_3,\ e_2' = -16e_1 + 7e_2 - 13e_3,$$
$$e_3' = 9e_1 - 3e_2 + 7e_3,$$
$$\mathscr{B}''\colon\ e_1'' = e_1 - 2e_2 + e_3,\ e_2'' = 3e_1 - e_2 + 2e_3,$$
$$e_3'' = 2e_1 + e_2 + 2e_3.$$

Find the matrix of operator A in the basis $\mathscr{B}''$ if its matrix in the basis $\mathscr{B}'$ has the form

$$A' = \begin{pmatrix} 1 & -18 & 15 \\ -1 & -22 & 20 \\ 1 & -25 & 22 \end{pmatrix}.$$

2.13. In space $\mathscr{L}_2$ operator A has the matrix $\begin{pmatrix} 3 & 5 \\ 4 & 3 \end{pmatrix}$ in the basis $\mathscr{B}'$: $e_1' = e_1 + 2e_2$, $e_2' = 2e_1 + 3e_2$, and operator B has the matrix $\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$ in the basis $\mathscr{B}''$: $e_1'' = 3e_1 + e_2$, $e_2'' = 4e_1 + 2e_2$. Find the matrix of the operator $A + B$ in the basis $\mathscr{B}''$.

2.14. Let $p(t) = a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$ be a polynomial and let A be a linear operator. Consider the operator $p(A)$ defined by the equality

$$p(A) = a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 E.$$

Find the matrix of the operator $p(A)$ if $p(t) = 3t^2 - 2t + 5$, and the operator A is given by the matrix $A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$.

2.15. Given in space $\mathcal{P}_n$ is a linear operator of differentiation $D = \frac{d}{dt}$. Find the matrix of this operator in the basis:

(a) $1, t, t^2, \ldots, t^{n-1}$;

(b) $1, (t-t_0), \frac{(t-t_0)^2}{2!}, \ldots, \frac{(t-t_0)^{n-1}}{(n-1)!}$, $t_0 \in \mathbb{R}$.

Prove the operator equality $D^n = O$ (O is a *zero operator*: $Ox = 0$).

2.16. Given in space $\mathcal{P}_4$ is the mapping

$$Ap(t) = \int_0^1 K(t, \tau)\, p(\tau)\, d\tau,$$

where $K(t, \tau)$ is a polynomial in two variables whose degree with respect to t does not exceed 3. Prove that A is a linear operator in $\mathcal{P}_4$; find its matrix in the basis $1, t, t^2, t^3$ for the case when $K(t, \tau) = t + \tau$.

2.17. Given in space $\mathcal{P}_4$ is the mapping

$$A_h p(t) = p(t + h),$$

where h is some fixed number. Prove that A_h is a linear operator and find its matrix in the basis $1, t, t^2, t^3$.

2.18. Given in a space of functions differentiable throughout the entire number axis are an operator of differentiation $D = \frac{d}{dt}$ and an operator $A = e^{\lambda t}$ of multiplication by the function $e^{\lambda t}$. Verify the equality $DA - AD = \lambda A$.

In Problems 2.19 to 2.26 determine which of the linear operators given in $\mathcal{V}_3$ are nonsingular and find for them the explicit form of their inverses (e is a fixed vector of unit length and $x = xi + yj + zk$).

2.19. $Ax = \lambda x$, λ is a fixed number.

2.20. $Ax = (x, e)\, e$. **2.21.** $Ax = [e, x]$.

2.22. $Ax = A - (x, e)\, e$. **2.23*.** $Ax = x - 2(x, e)\, e$.

2.24. $Ax = (y + z)\, i + (2x + z)\, j + (3x - y + z)\, k$.

2.25. $Ax = 2zi + (x - z)\, j + (2x + 3z)\, k$.

2.26. $A = U(e, \varphi)$ is an operator of rotation through an angle φ about the axis given by the vector e.

In Problems 2.27 to 2.29 determine which of the given linear operators in $\mathbb{R}^3$ are nonsingular and find the explicit form of their inverses:

2.27. $Ax = (x_1 - x_2 + x_3, x_3, x_2)$.

2.28. $Ax = (x_2 + 2x_3, -x_2, 2x_2 - x_3)$.

2.29. $Ax = (x_1 + 2x_2 + 2x_3, 2x_1 + x_2 - 2x_3, 2x_1 - 2x_2 + x_3)$.

The set T_A of all vectors Ax, $x \in \mathcal{L}$, is called the *image* of operator A. The set N_A of all vectors $x \in \mathcal{L}$, for which $Ax = 0$ is termed the *kernel* of operator A. The image and kernel of a linear operator are subspaces in $\mathcal{L}$. Here the dimension of the image $r_A = \dim T_A$ is called the *rank* and the dimension of the kernel $n_A = \dim N_A$ is the *defect* of operator A. The following equality holds: $r_A + n_A = n$, where n is the dimension of the space $\mathcal{L}$.

2.30. Describe the image and kernel of the following linear operators operating in space $\mathcal{V}_3$:

(a) $Ax = (x, e)\, e$, $|e| = 1$;

(b) $Ax = [x, a]$, $a \neq 0$.

2.31. Describe the image and kernel of the operator of differentiation D operating in space $\mathcal{P}_n$.

In Problems 2.32 to 2.34 for the indicated linear operators operating in space $\mathbb{R}^3$ determine the rank and defect and find the bases of the image and kernel.

2.32. $Ax = (x_1 + 2x_2 + x_3, x_1 - x_3, x_1 + x_2)$.

For representing the arithmetical vectors and the given linear operator let us take advantage of the canonical basis in $\mathbb{R}^3$. In this basis the matrix of the operator has the form

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

By definition, $y \in T_A$ if and only if there can be found a vector $x \in \mathbb{R}^3$ such that $y = Ax$ or, in the coordinate notation:

$$Y = AX = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \quad (2)$$

Equality (2) means that the image T_A coincides with the linear span of the system of columns of matrix A. Consequently, the rank of operator A coincides with the rank of its matrix, i.e. equals 2, and any of the bases of the system of columns of matrix A can be chosen as the basis T_A, for instance,

$$E_1=\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad E_2=\begin{pmatrix}2\\0\\1\end{pmatrix}.$$

Analogously, $x \in N_A$ if and only if $Ax = 0$, or, in the coordinate notation,

$$AX=\begin{pmatrix}1 & 2 & 1\\1 & 0 & -1\\1 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}. \tag{3}$$

Hence it follows that the kernel N_A coincides with the subspace of the solutions of homogeneous system (3), i.e. the defect of operator A is equal to $n_A = n - r_A = 3 - 2 = 1$, and, as the basis in N_A, we can choose a fundamental system of solutions of system (3), say, $E = \begin{pmatrix}1\\-1\\1\end{pmatrix}$.

2.33. $Ax = (2x_1 - x_2 - x_3,\ x_1 - 2x_2 + x_3,\ x_1 + x_2 - 2x_3)$.

2.34. $Ax = (x_1 + x_2 + x_3,\ x_1 + x_2 + x_3,\ x_1 + x_2 + x_3)$.

2.35. Prove that operator A is nonsingular if and only if its defect is equal to zero, and, consequently, its rank coincides with the dimension of the space.

2. Eigenvalues and Eigenvectors of a Linear Operator. Let a number λ and a vector $x \in \mathscr{L}$, $x \neq 0$, be such that

$$Ax = \lambda x. \tag{4}$$

Then the number λ is called the *eigenvalue* of the linear operator A, and the vector x is termed the *eigenvector* of this operator corresponding to the eigenvalue λ.

In a finite-dimensional space $\mathscr{L}_n$ vector equality (4) is equivalent to the matrix equality

$$(A - \lambda E)\, X = O, \quad X \neq O. \tag{5}$$

Hence it follows that the number λ is the eigenvalue of operator A if and only if $\det(A - \lambda E) = 0$, i.e. λ is a root of the polynomial $p(\lambda) = \det(A - \lambda E)$ called the *characteristic polynomial of operator* A. The column of coordinates X of any eigenvector corresponding to the eigenvalue λ is some nontrivial solution of homogeneous system (5).

Example 2. Find the eigenvalues and the eigenvectors of the operator P_{Oxy} of projecting on the xy-plane in space $\mathcal{V}_3$.

(1) Geometrical solution. The equality $P_{Oxy}x = \lambda x$, $x \neq 0$, means that the orthogonal projection of the vector x on the xy-plane is collinear with the vector x itself. But this is possible only in two cases:

(a) the vector $x \neq 0$ is coplanar with the xy-plane. For all such vectors $P_{Oxy}x = x$, i.e. all of them are eigenvectors of the operator P_{Oxy} corresponding to the eigenvalue $\lambda_1 = 1$.

(b) the vector $x \neq 0$ is orthogonal to the xy-plane. For all such vectors $P_{Oxy}x = 0 = 0 \cdot x$, i.e. all of them are eigenvectors of the operator P_{Oxy} corresponding to the eigenvalue $\lambda_2 = 0$.

Finally, we conclude that the operator P_{Oxy} has two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 0$. The corresponding eigenvectors are:

$$\lambda_1 = 1: \quad x^{(\lambda_1)} = xi + yj, \quad x^{(\lambda_1)} \neq 0,$$
$$\lambda_2 = 0: \quad x^{(\lambda_2)} = zk, \quad x^{(\lambda_2)} \neq 0.$$

(2) Analytical solution. The matrix of the operator P_{Oxy} in a rectangular basis $\mathfrak{B} = (i, j, k)$ has the form

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic equation is

$$\det(P - \lambda E) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(1-\lambda)^2 = 0,$$

whence we obtain the eigenvalues of the operator: $\lambda_1 = 1$ and $\lambda_2 = 0$.

Let us now find the eigenvectors corresponding to the eigenvalue $\lambda_1 = 1$. For $\lambda = 1$ system (5) takes the form

$$(P - E)\, X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The fundamental system of solutions is:

$$E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

and the general solution is:

$$xE_1 + yE_2 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

Hence we conclude that the eigenvectors corresponding to the eigenvalue $\lambda_1 = 1$ have the form

$$x^{(\lambda_1)} = xi + yj,$$

where x and y are arbitrary numbers, not both equal to zero.

The case $\lambda_2 = 0$ is considered in a similar way and we obtain

$$x^{(\lambda_2)} = zk,$$

where z is an arbitrary nonzero number.

In Problems 2.36 to 2.40 find the eigenvalues and eigenvectors of the operators given in $\mathcal{V}_3$. Solve these problems geometrically, i.e. in an invariant form not connected with the choice of some basis in $\mathcal{V}_3$ (see Example 2, (1) Geometrical solution). Then in Problems 2.36 to 2.38 give analytical solution.

2.36. $Ax = ax$, a is a fixed number.

2.37. $Ax = (x, i)\, i$ is an operator of projecting on the x-axis.

2.38. $Ax = [i, x]$.

3.39. $A = U(e, \varphi)$ is an operator of rotation through an angle φ about the axis given by the vector e.

2.40. $Ax = x - 2(x, e)\, e$ is an operator of the mirror image in a plane with the normal vector e.

In Problems 2.41 to 2.46 find the eigenvalues and the eigenvectors of linear operators given by their matrices.

2.41. $A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$.

2.42. $A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$.

2.43. $A = \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$.

2.44. $A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$.

2.45. $A=\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$.

2.46. $A=\begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}$.

2.47. Given in the space $\mathscr{V}_2$ of geometrical vectors on the plane is an operator of rotation $U(\varphi)$ through an angle $0 \leqslant \varphi < 2\pi$ about the origin. Check (geometrically and analytically) to see that for $\varphi \neq 0, \pi$ this operator has no eigenvalues. This example shows that a linear operator in a real space may have no eigenvalues (and eigenvectors).

2.48. In a complex space $\mathscr{L}_2$ operator A is given by the matrix

$$A(\varphi)=\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad 0 \leqslant \varphi < 2\pi.$$

Find its eigenvalues and eigenvectors. Compare the results obtained with the results of Problem 2.47.

2.49*. Let operator A functioning in a complex space $\mathscr{L}_n$ be given in some basis by a matrix with real elements. Prove that:

(a) if λ is an eigenvalue, then $\overline{\lambda}$ is also an eigenvalue;

(b) if $X^{(\lambda)}$ is a column of the coordinates of the vector corresponding to the eigenvalue λ, then $\overline{X^{(\lambda)}}$ is a column of the coordinates of the eigenvector corresponding to the eigenvalue $\overline{\lambda}$.

2.50*. In a complex space $\mathscr{L}_3$ find the eigenvalues and eigenvectors of the linear operator given by the real matrix

$$A=\begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}.$$

2.51. Show that if x is the eigenvector of operator A corresponding to the eigenvalue λ, then it is the eigenvector of the operator $p(A) = a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 E$ corresponding to the eigenvalue $p(\lambda)$.

2.52. Prove that:

(a) operator A has an inverse if and only if it has no zero eigenvalues;

(b) if operator A has an inverse, then A and A^{-1} have the same eigenvectors. What is the relation between the eigenvalues of these operators?

3. Linear Operators in Spaces with a Scalar Product. Let A be a linear operator functioning in a space with a scalar product (x, y). A linear operator A^* is said to be *conjugate to operator* A if for any vectors x, y the following equality is fulfilled:

$$(Ax, y) = (x, A^*y).$$

For any operator A there exists a conjugate operator A^* which is unique.

If operator A has the matrix $A = (a_{ij})$ in an orthonormal basis then the conjugate operator A^* in the same basis has the matrix $A^* = (a_{ij}^*)$, where $a_{ij}^* = \overline{a_{ji}}$ (matrix A^* is called conjugate to the matrix A). In the particular case of an Euclidean space $A^* = A^T$.

Example 3. The linear operator A: $\mathscr{E}_3 \to \mathscr{E}_3$ in the basis $\mathfrak{B}' = (e_1', e_2', e_3')$ has the matrix

$$[A]_{\mathfrak{B}'} = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 5 & -1 \\ 2 & 7 & -3 \end{pmatrix}.$$

It is known that $e_1' = e_1 + 2e_2 + e_3$, $e_2' = e_1 + e_2 + 2e_3$, $e_3' = e_1 + e_2$ and the basis $\mathfrak{B} = (e_1, e_2, e_3)$ is orthonormalized. Find the matrix of the conjugate operator A^* in the basis $\mathfrak{B}'$.

Since the basis $\mathfrak{B}'$ is not orthonormalized (check this!), then to take advantage of the assertion on the relation between the matrices of the operators A and A^*, it is necessary to find the matrix $[A]_{\mathfrak{B}}$ (by hypothesis, the basis $\mathfrak{B}$ is orthonormalized). We have

$$T_{\mathfrak{B}\to\mathfrak{B}'} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad T_{\mathfrak{B}'\to\mathfrak{B}} = T^{-1}_{\mathfrak{B}\to\mathfrak{B}'} = \frac{1}{2}\begin{pmatrix} -2 & 2 & 0 \\ 1 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix},$$

consequently,

$$[A]_{\mathfrak{B}} = T^{-1}_{\mathfrak{B}'\to\mathfrak{B}} \cdot [A]_{\mathfrak{B}'} \cdot T_{\mathfrak{B}'\to\mathfrak{B}} = \begin{pmatrix} 2 & -3 & 7 \\ 6 & -4 & 6 \\ 6 & -5 & 5 \end{pmatrix},$$

$$[A^*]_{\mathfrak{B}} = \begin{pmatrix} 2 & 6 & 6 \\ -3 & -4 & -5 \\ 7 & 6 & 5 \end{pmatrix}.$$

Hence we finally obtain:

$$[A^*]_{\mathfrak{B}'} = T^{-1}_{\mathfrak{B}\to\mathfrak{B}'} \cdot [A^*]_{\mathfrak{B}} \cdot T_{\mathfrak{E}\to\mathfrak{B}'} = \begin{pmatrix} -36 & -37 & -15 \\ 30 & 30 & 14 \\ 26 & 27 & 9 \end{pmatrix}.$$

2.53. Prove that the operation * of passage from the operator A to the conjugate operator A^* possesses the following properties:

(a) $(A^*)^* = A$.

Let us write a chain of equalities holding for any vectors x and y:

$$(Ax, y) = (x, A^*y) = \overline{(A^*y, x)} = \overline{(y, (A^*)^* x)} = \overline{((A^*)^* x, y)} = ((A^*)^* x, y),$$

i.e.

$$(Ax, y) = ((A^*)^* x, y).$$

Hence, by virtue of the arbitrariness of vectors x, y, we obtain $A = (A^*)^*$ (show this in more detail!).

(b) $(A + B)^* = A^* + B^*$; (c) $(AB)^* = B^*A^*$;

(d) $(\alpha A)^* = \bar{\alpha} A^*$;

(e) $(A^{-1})^* = (A^*)^{-1}$ if A is nonsingular.

A linear operator A in the basis $\mathfrak{B}' = (e_1', \ldots, e_n')$ has a matrix A. Find the matrix of the conjugate operator A^* in the same basis $\mathfrak{B}'$, if the vectors $e_1', \ldots, e_n'$ are given by the columns of their coordinates in some orthonormalized basis $\mathfrak{B} = (e_1, \ldots, e_n)$:

2.54. $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, $E_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $E_2' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2.55. $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 5 & -1 \\ 2 & 7 & -3 \end{pmatrix}$, $E_1' = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$,

$E_2' = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $E_3' = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

2.56. $A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{pmatrix}$, $\varepsilon = e^{\frac{2\pi i}{3}}$,

$$E_1' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad E_2' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad E_3' = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Given in the space of polynomials $\mathcal{P}_3$ is a scalar product

$$(f, g) = a_0b_0 + a_1b_1 + a_2b_2, \tag{6}$$

where $f(t) = a_0 + a_1t + a_2t^2$, $g(t) = b_0 + b_1t + b_2t^2$. Find the matrices of the operator of differentiation $D = \frac{d}{dt}$ and the conjugate operator D^* in the basis $\mathfrak{B}$:

2.57. $\mathfrak{B} = \left(\frac{1}{2}t^2 - \frac{1}{2}t,\ t^2 - 1,\ \frac{1}{2}t^2 + \frac{1}{2}t\right)$.

2.58. $\mathfrak{B} = \left(1,\ t,\ \frac{3}{2}t^2 - \frac{1}{2}\right)$.

2.59. Find the conjugate operator for rotating an Euclidean space through an angle α about the origin counterclockwise.

2.60. Let Oxy be a rectangular Cartesian coordinate system on the plane and let A be an operator of projecting on the x-axis parallel to the straight line l: $ax + by = 0$ $(a \neq 0)$. Find the matrix of the conjugate operator A^*.

2.61. Let Oxy be a rectangular Cartesian coordinate system on the plane and let A be an operator of mapping the points on the plane about the straight line l: $ax + by = 0$. Find the matrix of the operator A^*.

The concept of a conjugate operator can be utilized in investigating a nonhomogeneous system of linear equations for compatibility. Let $AX = B$ be the matrix notation of such a system, and $m = n$. Then X and B are columns of the coordinates of the corresponding arithmetic vectors in the canonical basis of the Euclidean space $\mathbb{R}_n$ and to the square matrix A in the same basis there corresponds some linear operator A: $\mathbb{R}^n \to \mathbb{R}^n$. The system $A^*X = 0$, where A^* is the matrix of the conjugate operator A^* in the canonical basis is called a *conjugate* homogeneous system. The following theorem **(Fredholm's theorem)** is valid: *for a system $AX = B$ to be compatible, it is necessary and sufficient that the column vector B be orthogonal to all the solutions of a conjugate homogeneous system.*

2.62.** Prove Fredholm's theorem.

Using this theorem, investigate for compatibility the following systems of linear equations:

2.63. $\begin{aligned} 3x_1 + 2x_2 + x_3 &= -1, \\ 7x_1 + 6x_2 + 5x_3 &= 5, \\ 5x_1 + 4x_2 + 3x_3 &= 2. \end{aligned}$

2.64. $\begin{aligned} x_1 + x_2 + x_3 &= 0, \\ x_1 + x_2 + x_3 &= 1, \\ x_1 + x_2 + x_3 &= -1. \end{aligned}$

2.65. $\begin{aligned} 2x_1 + x_2 - 2x_3 &= 1, \\ x_1 - 2x_2 + x_3 &= 1, \\ -2x_1 + x_2 + x_3 &= 1. \end{aligned}$

2.66. $\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ x_1 + x_2 + x_3 &= 1, \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$

2.67*. Prove *Fredholm's alternative*: either the system $AX = B$ is compatible for any right-hand member B, or the conjugate homogeneous system $A^*X = 0$ has nonzero solutions.

2.68. Which of the systems of linear equations indicated in Problems 2.63 to 2.66 are compatible for any right-hand member?

A linear operator H in a space with a scalar product is said to be *self-conjugate* if $H = H^*$. A self-conjugate operator in a unitary (Euclidean) space is also called *Hermitian* (*symmetric*). For an operator A to be Hermitian (symmetric), it is necessary and sufficient that in any orthonormal basis its matrix $A = (a_{ij})$ satisfies the relation $a_{ij} = \bar{a}_{ji}$ ($a_{ij} = a_{ji}$). Such matrices are termed *Hermitian* (*symmetric*).

A linear operator U in a unitary (Euclidean) space is called *unitary* (*orthogonal*) if

$$UU^* = U^*U = E, \text{ i.e. } U^* = U^{-1}.$$

For an operator A to be unitary (orthogonal), it is necessary and sufficient that in any orthonormal basis its matrix $A = (a_{ij})$ satisfies the relation $A^{-1} = A^*$ ($A^{-1} = A^T$). Such matrices are called *unitary* (*orthogonal*).

2.69. Prove the following properties of a self-conjugate operator:

(a) the eigenvalues are real;

(b) the eigenvectors corresponding to the appropriate eigenvalues are orthogonal.

2.70. Prove the following properties of a unitary operator:

(a) the eigenvalues are equal (by modulus) to unity;

(b) for a linear operator to be unitary, it is necessary and sufficient that it transforms an orthonormal basis again into an orthonormal basis;

(c) a unitary operator retains a scalar product;
(d) a unitary operator retains the lengths of vectors.

2.71. Show that in the space $\mathscr{V}_3$ the following operators are symmetric:
(a) $\boldsymbol{Ax} = \lambda \boldsymbol{x}$, λ is a fixed number;
(b) $\boldsymbol{Ax} = (\boldsymbol{x}, \boldsymbol{e})\,\boldsymbol{e}$, $|\boldsymbol{e}| = 1$;
(c) $\boldsymbol{Ax} = \boldsymbol{x} - 2(\boldsymbol{x}, \boldsymbol{e})\,\boldsymbol{e}$, $|\boldsymbol{e}| = 1$.

2.72. Show that in the space of polynomials $\mathscr{P}_3$ with scalar product (6) the following operators are symmetric:

(a) $f(t) \to f(-t)$; (b) $f(t) = t^n f\left(\frac{1}{t}\right)$.

2.73. Show that in the space $\mathscr{V}_2$ the operator $\boldsymbol{U}(\boldsymbol{e}, \varphi)$ of rotation through an angle φ about the axis given by the unit vector $\boldsymbol{e}$ (see Problem 2.6) is orthogonal.

2.74. Show that the operators from Problem 2.71 are orthogonal.

4. Reducing the Matrix of a Linear Operator to the Diagonal Form. If an operator A functioning in space $\mathscr{L}_n$ has n linearly independent eigenvectors $e_1, e_2, \ldots, e_n$, corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then in the basis formed from these vectors the matrix of operator A has the diagonal form

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}. \tag{7}$$

Example 4. Reduce the matrix A of a linear operator to the diagonal form and find the corresponding basis:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}.$$

The characteristic equation

$$\det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 2-\lambda & 0 \\ -2 & -2 & -1-\lambda \end{vmatrix} = (\lambda - 2)(1 - \lambda^2) = 0$$

has the roots $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$. Consequently, the matrix can be reduced to the diagonal form. We find the corresponding eigenvectors. For $\lambda = 2$ system (5) takes the form

$$(A-2E)\,X=\begin{pmatrix}-1 & 2 & 0\\ 0 & 0 & 0\\ -2 & -2 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

or

$$-x_1 + 2x_2 = 0$$
$$-2x_1 - 2x_2 - 3x_3 = 0.$$

The fundamental system of solutions consists of one vector $E_1 = (2, 1, -2)^T$. Analogously, for $\lambda = 1$ system (5) takes the form:

$$(A-E)\,X=\begin{pmatrix}0 & 2 & 0\\ 0 & 1 & 0\\ -2 & -2 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

From this system we find the second eigenvector $E_2 = (1, 0, -1)^T$. Finally, for $\lambda = -1$ from the system

$$(A+E)\,X=\begin{pmatrix}2 & 2 & 0\\ 0 & 3 & 0\\ -2 & -2 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

we find the third eigenvector $E_3 = (0, 0, 1)^T$.

The found vectors E_1, E_2, E_3 form the desired basis in which the matrix of the linear operator has the following diagonal form

$$\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Example 5. Show that matrix A of a linear operator cannot be reduced to the diagonal form:

$$A=\begin{pmatrix}2 & -1 & 2\\ 5 & -3 & 3\\ -1 & 0 & -2\end{pmatrix}.$$

The characteristic equation

$$\det(A-\lambda E)=\begin{vmatrix}2-\lambda & -1 & 2\\ 5 & -3-\lambda & 3\\ -1 & 1 & -2-\lambda\end{vmatrix}=-(\lambda+1)^3=0$$

has one root $\lambda = -1$. The corresponding system (5) for determining eigenvectors takes the form:

$$(A+E)\,X=\begin{pmatrix} 3 & -1 & 2 \\ 5 & -2 & 3 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$3x_1 - x_2 + 2x_3 = 0,$$
$$5x_1 - 2x_2 + 3x_3 = 0.$$

Whence we find $E_1 = (1, 1, -1)^T$. Hence, there is only one linearly independent eigenvector. This means that matrix A of the given linear operator cannot be diagonalized.

In Problems 2.75 to 2.80 find out which of the given matrices of linear operators can be diagonalized by passing to a new basis. Find this basis and the corresponding diagonal form of the matrix.

2.75. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. **2.76.** $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$.

2.77. $\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$. **2.78.** $\begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$.

2.79. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

2.80. $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

2.81*. Compute A^m if

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$; (b) $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

Compute:

2.82. $\begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix}^5$. 2.83. $\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}^6$.

Matrix A of a self-conjugate operator is always reduced to the diagonal form. Using the notion of a unitary operator, it can be represented in the form

$$A = UDU^{-1},$$

where U is the matrix of the unitary operator performing the passage from the original basis to the basis made up from the eigenvectors of operator A, and D is the diagonal matrix of form (7).

In Problems 2.84 to 2.86 find the orthonormal basis formed by eigenvectors and the matrix in this basis for a linear operator given in some orthonormal basis by matrix A (the desired basis is defined nonuniquely).

2.84. $A = \begin{pmatrix} 11 & 2 & -8 \\ 2 & 2 & 10 \\ -8 & 10 & 5 \end{pmatrix}$.

2.85. $A = \begin{pmatrix} 17 & -8 & 4 \\ -8 & 17 & -4 \\ 4 & -4 & 11 \end{pmatrix}$.

2.86. $A = \begin{pmatrix} 3 & -i & 0 \\ i & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

In Problems 2.87 to 2.91 for the given matrix A find the diagonal matrix D and unitary (orthogonal) matrix U such that $A = UDU^{-1}$.

2.87. $A = \begin{pmatrix} 3 & 2+2i \\ 2-2i & 1 \end{pmatrix}$.

2.88. $A = \begin{pmatrix} 3 & 2-i \\ 2+i & 7 \end{pmatrix}$.

2.89. $A = \begin{pmatrix} 1 & 4i & 0 \\ -4i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2.90.
$$A = \begin{pmatrix} 3 & 2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

2.91.
$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}.$$

SEC. 4.3.
BILINEAR AND QUADRATIC FORMS

1. Linear Forms. A *linear form* is said to be given in a real linear vector space $\mathcal{L}$ if every vector $x \in \mathcal{L}$ is associated with the number (x), the following conditions being fulfilled:

$$f(x+y) = f(x) + f(y), \quad x, y \in \mathcal{L},$$
$$f(\lambda x) = \lambda f(x), \quad x \in \mathcal{L}, \lambda \in \mathbb{R}.$$

3.1. Prove that in space $\mathcal{L}$ the function $f(x)$, $x \in \mathcal{L}$, is a linear form:

(a) $f(x) = \int_a^b x(t)\,dt$, $\mathcal{L} = C_{[a, b]}$, $x = x(t)$;

(b) $f(x) = x(t_0)$, $\mathcal{L} = C_{[a,b]}$, $x = x(t)$, $t_0 \in [a, b]$;

(c) $f(x) = (x, a)$, $\mathcal{L} = \mathcal{V}_3$, $a \in \mathcal{V}_3$ is a fixed vector.

3.2. Let a basis $\mathfrak{B} = (e_1, \ldots, e_n)$ be fixed in space $\mathcal{L}$, and let $f(e_i) = a_i$, $i = 1, 2, \ldots, n$, where $f(x)$ is a linear form in $\mathcal{L}$.

(a) Prove that $f(x) = a_1x_1 + \ldots + a_nx_n$, where $x_1, \ldots, x_n$ are the coordinates of the vector x in the basis $\mathfrak{B}$.

(b) Let us denote by $\mathcal{L}^*$ the set of linear forms $f(x)$ in which the operations of addition and multiplication by a number are defined in the following way:

$$g = f_1 + f_2 \quad \text{if} \quad \forall x \in \mathcal{L}\ (g(x) = f_1(x) + f_2(x));$$
$$h = \lambda f \quad \text{if} \quad \forall x \in \mathcal{L}\ (h(x) = \lambda f(x)).$$

Prove that $\mathcal{L}^*$ is a linear vector space.

(c) Prove that $\dim \mathcal{L}^* = n$ (space $\mathcal{L}^*$ is called *conjugate* to space $\mathcal{L}$).

3.3. Prove that:

(a) if $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, then the formula $f(x) = x_1$ defines a linear form;

(b) any linear form $f(x)$, $x \in \mathbb{R}^n$, which is not equal identically to zero, by a properly chosen basis can be reduced to the form $f(x) = x_1$, where x_1 is the first coordinate of the vector x in this basis.

2. Bilinear Forms. A numerical function $A(x, y)$: $\mathcal{L} \times \mathcal{L} \to \mathbb{R}$, defined in a real linear vector space $\mathcal{L}$ is called a *bilinear form* if, with y fixed, it is a linear form in x, and for a fixed x it is linear in y. A bilinear form is termed symmetric if $A(x, y) = A(y, x)$, $x, y \in \mathcal{L}$. If in space $\mathcal{L}_n$ some basis $\mathfrak{B} = (e_1, \ldots, e_n)$ is fixed, then the matrix $A = (a_{ij})$, $a_{ij} = A(e_i, e_j)$, is called a *matrix of the bilinear form* $A(x, y)$ in the basis $\mathfrak{B}$.

3.4. Prove that in space $\mathcal{L}$ the function $A(x, y)$ is a bilinear form:

(a) $A(x, y) = f_1(x) f_2(y)$, where f_1, f_2 are linear forms in $\mathcal{L}$;

(b) $A(x, y) = \int_a^b \int_a^b K(s, t) x(s) y(t) \, ds \, dt$, where $\mathcal{L} = C_{[a, b]}$, $x = x(t) \in C_{[a, b]}$, $y = y(t) \in C_{[a, b]}$, $K(s, t)$ is a continuous function of two variables;

(c) $A(x, y) = \sum_{i, j=1}^{n} a_{ij} x_i y_j$, where $\mathcal{L} = \mathbb{R}^n$, $x, y \in \mathbb{R}^n$, $A = (a_{ij})$ is some matrix.

3.5. Let in space $\mathcal{L}_n$ there be fixed a basis $\mathfrak{B} = (e_1, \ldots, e_n)$, and let $A(x, y)$ be a bilinear form in $\mathcal{L}_n$, and $A(e_1, e_i) = a_{ij}$. Prove that:

(a) $A(x, y) = \sum_{i, j=1}^{n} a_{ij} x_i y_j$, where x_i, y_j, $i, j = 1, 2, \ldots, n$, are the coordinates of the vectors x and y in the basis $\mathfrak{B}$;

(b) if $A' = (a'_{ij})$ is the matrix of the bilinear form $A(x, y)$ in the basis $\mathfrak{B}' = (e'_1, \ldots, e'_n)$, then $A' = T^{\top} A T$, where $T = T_{\mathfrak{B} \to \mathfrak{B}'}$ is a transformation matrix for passing from the basis $\mathfrak{B}$ to basis $\mathfrak{B}'$.

3.6. Let in space $\mathbb{R}^3$ there be given a bilinear form $A(x, y)$. Find its matrix in the basis $\mathfrak{B} = (e_1, e_2, e_3)$ if:

(a) $A(x, y) = x_1y_1 + 2x_2y_2 + 3x_3y_3$, $e_1 = (1, 1, 1)$, $e_2 = (1, 1, -1)$, $e_3 = (1, -1, -1)$;

(b) $A(x, y) = x_1y_2 + x_2y_3 + x_3y_1$, $e_1 = (1, 0, 0)$, $e_2 = (1, 1, 0)$, $e_3 = (1, 1, 1)$.

3.7. Given in space $\mathbb{R}^n$ is a bilinear form $A(x, y)$ in the basis $\mathfrak{B}$. Find its matrix in the basis $\mathfrak{B}'$ if:

(a) $n = 4$, $A(x, y) = x_1y_2 + x_2y_3 + x_3y_4$,

$$T_{\mathfrak{B}\to\mathfrak{B}'} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix};$$

(b) $n = 2$, $A(x, y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$,

$$T_{\mathfrak{B}\to\mathfrak{B}'} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

3.8. Prove that the scalar product (x, y) in an Euclidean space $\mathcal{E}$ is a bilinear form.

3. Quadratic Forms. Let $A(x,y)$ be a symmetric bilinear form. The form $A(x, x)$, obtained from $A(x, y)$ by setting $y = x$, is called *quadratic*. In this case $A(x, y)$ is said to be a bilinear form *polar* to the quadratic form $A(x, x)$.

If some basis $\mathfrak{B} = (e_1, \ldots, e_n)$ is fixed in a real linear space $\mathcal{L}_n$, then the quadratic form $A(x, x)$ in this basis has the form

$$A(x, x) = \sum_{i,\, j=1}^{n} a_{ij}x_ix_j, \tag{1}$$

where $A = (a_{ij})$ is the matrix of the quadratic form and $x = x_1e_1 + \ldots + x_ne_n$.

Let in some basis expression (1) does not contain the products x_ix_j $(i \neq j)$, i.e.

$$A(x, x) = \sum_{i=1}^{n} \lambda_i x_i^2. \tag{2}$$

Then expression (2) is called the *canonical form* of a quadratic form. In particular, if $\lambda_i = \pm 1, 0$, $i = 1, 2, \ldots, n$, then we obtain the *normal form* of the quadratic form $A(x, x)$.

For any quadratic form there exists such a basis $\mathfrak{B}'$ in which it has a canonical (and even a normal) form.

Below, we consider a couple of methods for reducing a quadratic form to the canonical form.

(1) *The method of separating perfect squares (Lagrange's method).* Let a quadratic form $A(x, y)$ in basis $\mathfrak{B}$ have form (1). If all the

coefficients a_{ii} (of the squares x_i^2), $i = 1, 2, \ldots, n$, are zero and at the same time the form is not equal identically to zero, then at least one product, say, $2a_{12}x_1x_2$ is nonzero. Let us transform the basis so that the coordinates of vectors in the old and new bases are related by the formulas

$$x_1 = x_1' + x_2',$$
$$x_2 = x_1' - x_2',$$
$$x_i = x_i', \quad i = 3, \ldots, n.$$

Then $2a_{12}x_1x_2 = 2a_{12}(x_1'^2 - x_2'^2) = 2a_{12}x_1'^2 - 2a_{12}x_2'^2$, and since, by the supposition, $a_{11} = a_{22} = 0$, the coefficient of $x_1'^2$ is nonzero.

Thus, always there can be found such a basis $\mathfrak{B}$ in which in notation (1) at least one coefficient of the square is nonzero.

We then assume that $a_{11} \neq 0$. (If $a_{11} = 0$, then the coefficient of the square of some other coordinate is nonzero, and we can arrive at the case under consideration by numbering the vectors $e_1, e_2, \ldots, e_n$ which is also a certain transformation of the basis.)

Consider the portion of the quadratic form containing x_1, i.e.

$$\sigma_1 = a_{11}x_1^2 + 2a_{12}x_1x_2 + \ldots + 2a_{1n}x_n.$$

Let us complete this sum to get a perfect square:

$$\sigma_1 = \frac{1}{a_{11}}(a_{11}x_1 + \ldots + a_{1n}x_n)^2 - \gamma,$$

where γ is an algebraic sum of the terms independent of x_1. If now we make the substitution

$$x_1' = a_{11}x_1 + \ldots + a_{1n}x_n,$$
$$x_i' = x_i, \quad i = 2, \ldots, n,$$

then the quadratic form in the new basis will take the form

$$A(x, x) = \frac{1}{a_{11}}x_1'^2 + \sum_{i,j=2}^{n} a_{ij}'x_i'x_j' = \frac{1}{a_{11}}x_1'^2 + A_1.$$

In the obtained form the term $\frac{1}{a_{11}}x_1'^2$ is singled out, and the remaining part A_1 is a quadratic form in $\mathcal{L}_{n-1}$. Our reasoning is then repeated for the quadratic form $A_1(x, x)$, and so on.

Example 1. Using Lagrange's method, reduce to the canonical form the quadratic form

$$A(x, x) = 2x_1x_2 + 4x_1x_3 - x_2^2 - 8x_3^2.$$

First transformation $x_1 = x_2'$, $x_2 = x_1'$, $x_3 = x_3'$ yields

$$A = -x_1'^2 + 2x_1'x_2' + 4x_2'x_3' - 8x_3'^2.$$

Second transformation: $x_1'' = -x_1' + x_2'$, $x_2'' = x_2'$, $x_3'' = x_3'$. We obtain a new expression for the quadratic form:

$$A = -x_1''^2 + x_2''^2 + 4x_2''x_3'' - 8x_3''^2.$$

Third transformation: $x_1''' = x_1''$, $x_2''' = x_2'' + 2x_3''$, $x_3''' = x_3''$, and the form takes the canonical form:

$$A(x, x) = -x_1'''^2 + x_2'''^2 - 12x_3'''^2.$$

Here

$$\begin{aligned} x_1''' &= x_1 - x_2, \\ x_2''' &= x_1 + 2x_3, \\ x_3''' &= x_3. \end{aligned}$$

(2) *The method of eigenvectors.* We shall consider quadratic form (1) in Euclidean space $\mathbb{R}^n$. Since its matrix $A = (a_{ij})$ is symmetric, it can be represented in the form $A = U^T D U$, where D is the diagonal matrix whose diagonal is occupied by the eigenvalues of matrix A, and U is an orthogonal matrix (see the preceding section). The columns of matrix U are the coordinates of some orthonormal basis $\mathfrak{B}' = (e_1, \ldots, e_n)$, in which matrix A has the diagonal form D, and, hence, the quadratic form has the required canonical form. The corresponding transformation of the coordinates is defined by the relation

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = U \begin{pmatrix} x_2' \\ \cdot \\ \cdot \\ x_n' \end{pmatrix}.$$

Example 2. Find the orthogonal transformation reducing the quadratic form $A(x, x) = 6x_1^2 + 5x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_1x_3$, given in Euclidean space $\mathbb{R}^3$, to the canonical form. Write this canonical form.

The matrix of the quadratic form has the form

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}.$$

(Notice how the elements a_{ij} $(i \neq j)$ are obtained from the explicit form of the quadratic form!) The eigenvalues of this matrix are $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 9$. The corresponding orthonormal eigenvectors are

$$\begin{aligned} e_1' &= (2/3, 2/3, -1/3), \\ e_2' &= (-1/3, 2/3, 2/3), \\ e_3' &= (2/3, -1/3, 2/3), \end{aligned}$$

and, consequently,

$$U=\frac{1}{3}\begin{pmatrix}2&-1&2\\2&2&-1\\-1&2&2\end{pmatrix},\quad U^{\top}=\frac{1}{3}\begin{pmatrix}2&2&-1\\-1&2&2\\2&-1&2\end{pmatrix}.$$

In the basis $\mathfrak{B}'=(e_1', e_2', e_3')$, the given quadratic form has the form $A(x, x)=3x_1'^2+6x_2'^2+9x_3'^2$, and the corresponding transforming of coordinates:

$$x_1=\frac{1}{3}(2x_1'-x_2'+2x_3'),$$

$$x_2=\frac{1}{3}(2x_1'+2x_2'-x_3'),$$

$$x_3=\frac{1}{3}(-x_1'+2x_2'+2x_3').$$

3.9. Prove that any quadratic form $A(x, x)$ in Euclidean space $\mathcal{E}_n$ can be written in the form $A(x, x)=(Ax, x)$, where (x, y) is a scalar product in $\mathcal{E}_n$ and A is some linear operator.

3.10. Prove that the polar bilinear form $A(x, y)$ is defined uniquely by its quadratic form $A(x, x)$.

In Problems 3.11 to 3.13, using Lagrange's method, find the normal form and the nonsingular linear transformation leading to this form for the following quadratic forms:

3.11. $x_1^2+5x_2^2-4x_3^2+2x_1x_2-4x_1x_3$.

3.12. $x_1x_2+x_2x_3+x_3x_1$.

3.13. $4x_1^2+x_2^2+x_3^2-4x_1x_2+4x_1x_3-3x_2x_3$.

In Problems 3.14 to 3.17 find the orthogonal transformation reducing the given form to the canonical form, and write this canonical form.

3.14. $11x_1^2+5x_2^2+2x_3^2+16x_1x_2+4x_1x_3-20x_2x_3$.

3.15. $x_1^2+x_2^2+5x_3^2-6x_1x_2+6x_1x_3-6x_2x_3$.

3.16. $x_1^2+x_2^2+x_3^2+4x_1x_2+4x_1x_3+4x_2x_3$.

3.17. $17x_1^2+14x_2^2+14x_3^2-4x_1x_2-4x_1x_3-8x_2x_3$.

4. Second-order Curves and Surfaces. A *hypersurface of the second order* in Euclidean space $\mathbb{R}^n$ is defined as the set of points whose coordinates satisfy the equation

$$A(x, x)+2(b, x)+c=\sum_{i,j=1}^{n}a_{ij}x_ix_j+2\sum_{k=1}^{n}b_kx_k+c=0, \qquad (3)$$

where in the left-hand side stands a polynomial in n variables x_1, x_2 ... ..., x_n of degree 2.

The problem of classification of second-order hypersurfaces consists in finding such a basis in $\mathbb{R}^n$ in which the left-hand member of the equation in the new variables x_1', x_2', ..., x_n' has the simplest form. To this end, we first look for an orthogonal transformation such that in the new variables the quadratic form $A(\boldsymbol{x}, \boldsymbol{x}) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ has the canonical form. In the new basis equation (3) is written in the following way:

$$\sum_{k=1}^{n} \lambda_k x_k'^2 + 2\sum_{k=1}^{n} b_k' x_k' + c = 0,$$

where not all λ_k, $i = 1, 2, \ldots, n$, are zero. If $\lambda_k \neq 0$, then, by translating the origin, we can eliminate the linear term:

$$\lambda_k x_k'^2 + 2b_k x_k' = \lambda_k \left(x_k' + \frac{b_k}{\lambda_k}\right)^2 - \frac{b_k^2}{\lambda_k} = \lambda_k x_k'' - \frac{b_k^2}{\lambda_k}.$$

After these transformations we obtain (changing the enumeration of the variables if it is necessary)

$$\lambda_1 x_1''^2 + \ldots + \lambda_s x_s''^2 + b_{s+1}'' x_{s+1}'' + c'' = 0. \tag{4}$$

Equation (4) is called the *canonical equation* of a hypersurface of the second order.

The set of points of plane $\mathbb{R}_2$ satisfying equation (3) is called a *curve of the second order*. In this case canonical equation (4) can take one of the following forms (in the variables x, y):

(1) $\lambda_1 x^2 + \lambda_2 y^2 + c = 0$ $(\lambda_1\lambda_2 \neq 0)$;

(2) $\lambda_1 x^2 + by = 0$ $(\lambda_1 \neq 0)$;

(3) $\lambda_1 x^2 + c = 0$ $(\lambda_1 \neq 0)$.

Example 3. Write the canonical equation of the second-order curve

$$3x^2 + 10xy + 3y^2 - 2x - 14y - 13 = 0,$$

determine its type and find the canonical coordinate system.

The matrix of the quadratic part of the polynomial of the second degree is equal to $\begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}$. Its eigenvalues are: $\lambda_1 = 8$ and $\lambda_2 = -2$; and eigenvectors: $e_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $e_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. The transformation

$$x = \frac{1}{\sqrt{2}}(x' + y'), \quad y = \frac{1}{\sqrt{2}}(x' - y'),$$

yields

$$8x'^2-2y'^2-\frac{16}{\sqrt{2}}x'+\frac{12}{\sqrt{2}}y'-13=0.$$

Since λ_1 and λ_2 are different from zero, we can single out a perfect square with respect to each of the new variables x' and y':

$$8x'^2-\frac{16}{\sqrt{2}}x'=8\left(x-\frac{\sqrt{2}}{2}\right)^2-4,$$

$$-2y'+\frac{12}{\sqrt{2}}y'=-2\left(y'-\frac{3}{\sqrt{2}}\right)^2+9.$$

By the change of variables

$$x''=x'-\frac{\sqrt{2}}{2},\quad y''=y'-\frac{3}{\sqrt{2}},$$

corresponding to a shift along either of the coordinate axes, we get

$$8x''^2-2y''^2-8=0,\quad \text{or}\quad x''^2-\frac{1}{4}y''^2=1.$$

The last equation is a canonical equation of the hyperbola. The resulting transformation of the coordinates has the form

$$x=\frac{1}{\sqrt{2}}(x''+y'')+2,$$

$$y=\frac{1}{\sqrt{2}}(x''-y'')-1,$$

and the canonical coordinate system is (O', e_1, e_2), where $O'(2, -1)$,

$$e_1=\frac{1}{\sqrt{2}}i+\frac{1}{\sqrt{2}}j,\quad e_2=\frac{1}{\sqrt{2}}i-\frac{1}{\sqrt{2}}j.$$

In Problems 3.18 to 3.24 write the canonical equation of the given second-order curve, determine its type, and find the canonical coordinate system.

3.18. $9x^2-4xy+6y^2+16x-8y-2=0.$

3.19. $x^2-2xy+y^2-10x-6y+25=0.$

3.20. $5x^2+12xy-22x-12y-19=0.$

3.21. $4x^2-4xy+y^2-6x+3y-4=0.$

3.22. $2x^2+4xy+5y^2-6x-8y-1=0.$

3.23. $x^2-4xy+4y^2-4x-3y-7=0.$

3.24. A second-order curve is defined by the equation: (a) $x^2-2y+\lambda(y^2-2x)=0$; (b) $x^2+2\lambda xy+y^2-1=0$.

Determine its type as the parameter λ varies from $-\infty$ to $+\infty$.

The set of points of Euclidean space $\mathbb{R}^3$ satisfying equation (3) is called a *surface of the second order*. In this case canonical equation (4) takes one of the following forms (in the variables x, y, z):

(1) $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + c = 0$ $(\lambda_1\lambda_2\lambda_3 \neq 0)$,

(2) $\lambda_1 x^2 + \lambda_2 y^2 + bz = 0$ $(\lambda_1\lambda_2 \neq 0)$,

(3) $\lambda_1 x^2 + \lambda_2 y^2 + c = 0$ $(\lambda_1\lambda_2 \neq 0)$,

(4) $\lambda_1 x^2 + by = 0$ $(\lambda_1 \neq 0)$,

(5) $\lambda_1 x^2 + c = 0$ $(\lambda_1 \neq 0)$.

The surfaces of types (3)-(5) are cylinders (elliptic, hyperbolic, etc. depending on the type of the curve in the section by the plane $z = 0$).

Example 4. Write the canonical equation of the second-order surface

$$4x^2 + 4y^2 - 8z^2 - 10xy + 4yz + 4zx - 16x - 16y - 8z + 72 = 0,$$

determine its type, and find the canonical coordinate system.

The matrix of the quadratic part of the second-degree polynomial is equal to $\begin{pmatrix} 4 & -5 & 2 \\ -5 & 4 & 2 \\ 2 & 2 & -8 \end{pmatrix}$. Its eigenvalues are $\lambda_1 = 9$, $\lambda_2 = -9$, $\lambda_3 = 0$, and the eigenvectors:

$$e_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \quad e_2 = \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{4}{3\sqrt{2}}\right),$$

$$e_3 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

Carrying out the transformation

$$x = \frac{1}{3\sqrt{2}}(3x' + y' + 2\sqrt{2}z'),$$

$$y = \frac{1}{3\sqrt{2}}(-3x' + y' + 2\sqrt{2}z'),$$

$$z = \frac{1}{3\sqrt{2}}(-4y' + \sqrt{2}z'),$$

we obtain

$$9x'^2 - 9y'^2 - 72z' + 72 = 0.$$

Shearing transformation should be performed only with respect to the variable z':

$$-72z' + 72 = -72(z' - 1) = -72z''.$$

The second transformation of the coordinates has the form

$$x'' = x', \quad y'' = y', \quad y'' = z' - 1,$$

whence we finally obtain the canonical equation of the hyperbolic paraboloid

$$\frac{x''^2}{8} - \frac{y''^2}{8} = z''.$$

The resulting transformation of the coordinates is:

$$x = \frac{1}{3\sqrt{2}}(3x'' + y'' + 2\sqrt{2}z'') + \frac{2}{3},$$

$$y = \frac{1}{3\sqrt{2}}(-3x'' + y'' + 2\sqrt{2}z'') + \frac{2}{3},$$

$$z = \frac{1}{3\sqrt{2}}(-4y'' + \sqrt{2}z'') + \frac{1}{3},$$

and the canonical system of coordinates is (O', e_1, e_2, e_3), where

$$O'\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \quad e_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right),$$

$$e_2 = \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{4}{3\sqrt{2}}\right), \quad e_3 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

In Problems 3.25 to 3.32 write the canonical equation of the given second-order surface, determine its type, and find the canonical coordinate system.

3.25. $7x^2 + 6y^2 + 5z^2 - 4xy - 4yz - 6x - 24y + 18z + 30 = 0.$

3.26. $2x^2 - 7y^2 - 4z^2 + 4xy + 20yz - 16zx + 60x - 12y + 12z - 90 = 0.$

3.27. $2x^2 + 2y^2 - 5z^2 + 2xy - 2x - 4y - 4z + 2 = 0.$

3.28. $2x^2 + 2y^2 + 3z^2 + 4xy + 2yz + 2zx - 4x + 6y - 2z + 3 = 0.$

3.29. $4x^2 + y^2 + 4z^2 - 4xy + 4yz - 8zx - 28x + 2y + 16z + 45 = 0.$

3.30. $2x^2 + 5y^2 + 2z^2 - 2xy - 4xz + 2yz + 2x - 10y - 2z - 1 = 0.$

3.31. $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx - 2x + 6y + 2z = 0.$

3.32. $x^2 - 2y^2 + z^2 + 4xy + 4yz - 10zx + 2x + 4y - 10z - 1 = 0.$

Answers

1.6. Yes. 1.7. Yes, if the straight line passes through the origin. 1.8. No. 1.9. Yes. 1.10. No. 1.11. Yes. 1.12. No. 1.15. $T_{\mathfrak{B}\to\mathfrak{B}'}=$

$$=\begin{pmatrix}1 & 1 & -1\\ 1 & -1 & 2\\ 0 & 0 & -1\end{pmatrix},\quad X'=\begin{pmatrix}1/2\\ -3/2\\ -2\end{pmatrix}.$$ 1.16. $T_{\mathfrak{B}\to\mathfrak{B}'}=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$,

$$X'=\begin{pmatrix}-1\\ 2\\ -1\end{pmatrix}.$$ 1.17. $$T_{\mathfrak{B}\to\mathfrak{B}'}=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad X'=\begin{pmatrix}-2\\ 1\\ 1\end{pmatrix}.$$

1.18. $$T_{\mathfrak{B}\to\mathfrak{B}'}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\varphi & \sin\varphi\\ 0 & \sin\varphi & -\cos\varphi\end{pmatrix},\quad X'=\begin{pmatrix}1\\ -2\cos\varphi+\sin\varphi\\ 2\sin\varphi+\cos\varphi\end{pmatrix}.$$

1.19. $r=2$; the basis is, for instance, the system (x_1, x_2).

1.20. $$T_{\mathfrak{B}\to\mathfrak{B}'}=\begin{pmatrix}1 & 2 & 1 & 1\\ 2 & 3 & 2 & 3\\ -1 & 0 & 1 & -1\\ -2 & -1 & 4 & 0\end{pmatrix},\quad X'=\begin{pmatrix}0\\ 2\\ 1\\ 2\end{pmatrix}.$$ 1.21. $3x_1+$

$+2x_2-x_3=0$, $r=2$, any pair of vectors form the basis of this system. 1.22. The coordinates of the matrix in this basis coincide with its elements. 1.23. (a) $\begin{pmatrix}1\\ 0\\ -3\end{pmatrix}$; (b) $\begin{pmatrix}0\\ -2\\ 1\\ 0\end{pmatrix}$. 1.25. $\begin{pmatrix}-1\\ 0\\ -1\end{pmatrix}$.

1.27. $$\begin{pmatrix}1 & -t_0 & t_0^2 & -t_0^3 & \dots & (-1)^{n-1}t_0^{n-1}\\ 0 & 1 & -2t_0 & 3t_0^2 & \dots & (-1)^{n-2}(n-1)t_0^{n-2}\\ & & & \dots & & \dots\\ 0 & 0 & 0 & 0 & & 1\end{pmatrix}.$$ 1.28. $\begin{pmatrix}2\\ 1\\ 1\end{pmatrix}$.

1.30. $\begin{pmatrix}1\\2\\3\end{pmatrix}$. **1.31.** $\begin{pmatrix}1\\1\\1\end{pmatrix}$. **1.32.** $\begin{pmatrix}0\\2\\1\\2\end{pmatrix}$. **1.33.** $\frac{1}{2(1+i)}\begin{pmatrix}1+3i\\-1-2i\end{pmatrix}$.

1.34. $\begin{pmatrix}2\\1\\1\end{pmatrix}$. **1.37.** $T_{\mathfrak{L}\to\mathfrak{B}'}=\begin{pmatrix}-42 & -71 & -41\\ 12 & 20 & 9\\ 7 & 12 & 8\end{pmatrix}$. **1.38.** $T_{\mathfrak{B}\to\mathfrak{B}'}=$

$$=\begin{pmatrix}2 & 0 & 1 & -1\\ -3 & 1 & -2 & 1\\ 1 & -2 & 2 & -1\\ 1 & -1 & 1 & -1\end{pmatrix}.$$ **1.39.** It is. **1.40.** It is not, since the condition of linearity of the mapping is violated. **1.41.** No, it is not, since the condition of one-to-one correspondence of the mapping is violated. **1.43.** $T_{\mathfrak{B}\to\mathfrak{B}}=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$, $-2+3i=-2(1+i)-5(-i)$. **1.45.** (a), (b) Subspace of dimension 2, the basis is any pair of noncollinear vectors from the given set; (c) it is not a subspace. **1.46.** (a) Subspace of dimension $n-2$; (b) it is not a subspace. **1.47.** The sets indicated in (a), (b), and (d) are subspaces, and the set from (c) is not a subspace. *Hint.* The condition satisfied by the coordinates in any of the problems belonging to this series can be written in the form $AX=O$, where A is some matrix having n columns, and X is a column of the coordinates in a fixed basis. Therefore the dimension of the corresponding subspace is equal to $n-\operatorname{rank}A$, and any fundamental system of solutions of the system of equations $AX=0$ may be taken as the basis. **1.48.** (a) Subspace of dimension $n^2-C_n^2=\frac{n(n+1)}{1}$; (b) it is not a subspace. **1.49.** (a) Infinite-dimensional subspace; (b) it is not a subspace; (c) subspace of dimension n. **1.51.** 2. **1.52.** 3; one of the basis is, for instance, $\mathfrak{B}=(x_1, x_2, x_4)$. **1.53.** 3; one of the bases is, for instance, $\mathfrak{B}$ (x_1, x_2, x_5). **1.54.** *Hint.* The given system of polynomials is linearly independent. **1.55.** $\mathfrak{L}(a)$ is a straight line $\frac{x}{-2}=\frac{y}{1}=\frac{z}{-1}$, $\mathfrak{L}(a)+b$ is a straight line $\frac{x-2}{-2}=\frac{y+1}{1}=\frac{z}{-1}$. **1.56.** $\mathfrak{L}(a_1, a_2)$ is a plane $-3x-y-2z=0$, $\mathfrak{L}(a_1a_2)+b$ is a plane $-3x-y-2z+5=0$. **1.57.** The set of solutions of a nonhomogeneous system is a linear manifold obtained from the subspace of dimension $n-\operatorname{rank}A=3$ of the solutions of the corresponding homogeneous system by a shift over an arbitrary

particular solution of the nonhomogeneous system. **1.62.** (c) $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leqslant \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$; (d) $\left|\sqrt{\sum_{i=1}^{n} x_i^2} - \sqrt{\sum_{i=1}^{n} y_i^2}\right| \leqslant \sqrt{\sum_{i=1}^{n} (x_i + y_i)^2} \leqslant \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}$. **1.63.** (a) 0; (b) $= -6$. **1.64.** (a) -1; (b) 24. **1.65.** (b) $\left(\int_a^b f(t)\, g(t)\, dt\right)^2 \leqslant \left(\int_a^b f^2(t)\, dt\right)\left(\int_a^b g^2(t)\, dt\right)$; (c) $\left|\left(\int_a^b f^2(t)\, dt\right)^{1/2} - \left(\int_a^b g^2(t)\, dt\right)^{1/2}\right| \leqslant \left(\int_a^b (f(t)+g(t))^2\, dt\right)^{1/2} \leqslant \left(\int_a^b f^2(t)\, dt\right)^{1/2} + \left(\int_a^b g^2(t)\, dt\right)^{1/2}$. **1.67.** $e_1 = f_1 = (1, 1, 1, 1)$, $e_2 = (2, 2, -2, -2)$, $e_3 = (-1, 1, -1, 1)$. **1.68.** $\mathfrak{B} = (e_1, e_2, e_3)$, $e_1 = (1, 2, 1, 3)$, $e_2 = (10, -1, 1, -3)$, $e_3 = (19, -87, -61, 72)$. **1.69.** $\mathfrak{B} = (e_1, e_2, e_3)$, $e_1 = (1, 2, 2, -1)$, $e_2 = (2, 3, -3, 2)$, $e_3 = (2, -1, -1, -2)$. **1.70.** $\mathfrak{B} = (e_1, e_2, e_3)$, $e_1 = (2, 1, 3, 1)$, $e_2 = (3, 2, -3, -1)$, $e_3 = (1, 5, 1, 10)$. **1.71.** $e_3 = (-4, 2, -1, 3)$, $e_4 = (2, 4, 3, 1)$. *Hint.* To determine the vector $e_3 = (x_1, x_2, x_3, x_4)$, it suffices to find some solution of the system with respect to the unknowns of the two linear equations $(e_3, e_1) = 0$, $(e_3, e_2) = 0$. The vector e_4 is determined from an analogous system consisting of three equations. **1.72.** $e_3 = (2/3, -2/3, -1/3)$. **1.73.** $e_3 = (1, -2, 1, 0)$, $e_4 = (25, 4, -17, -6)$. **1.74.** $e_4 = (0, 7, 3, -4, -2)$, $e_5 = (39, -37, 51, -29, 5)$. **1.76.** $y = (1, -1, -1, 5)$, $z = (3, 0, -2, -1)$. **1.77.** $y = (3, 1, -1, -2)$, $z = (2, 1, -1, 4)$. **1.78.** $y = (3, 1, -1, -2)$, $z = (1, 7, 3, 3)$. **1.80.** *Hint.* It follows from the equality $|x - y|^2 = |x|^2 + |y|^2$ that $(x, y) + (y, x) = (x, y) + \overline{(x, y)} = 0$, i.e. (x, y) is an imaginary number, not necessarily equal to zero.

2.1. It is; $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$. **2.2.** It is not. **2.3.** It is the operator of projecting on the axis given by the vector e. If $e = \cos\alpha \cdot i + \cos\beta \cdot j + \cos\gamma \cdot k$, then $A = \begin{pmatrix} \cos^2\alpha\cos\beta & \cos\alpha & \cos\gamma\cos\alpha \\ \cos\alpha\cos\beta & \cos^2\beta & \cos\gamma\cos\beta \\ \cos\alpha\cos\gamma & \cos\beta & \cos\gamma\cos^2\gamma \end{pmatrix}$.

2.4. It is; if $a = a_1 i + a_2 j + a_3 k$, then $A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & 0 & a_1 \end{pmatrix}$

2.5. It is not. **2.6.** It is. It is clear that.

$$y = U(e, \varphi)\, x = y_e + y_\alpha, \tag{1}$$

where y_e is the component of the vector y along e, and y_α is the component of the vector y coplanar with the plane α. The component

$$y_e = x_e = (x, e)\, e. \tag{2}$$

The component y_α is obtained from the vector x_α by rotating the latter in the plane α through an angle φ. To find y_α, let us introduce an auxiliary vector $[e, x_\alpha]$, lying in the plane α perpendicular to the vector x_α, the triple x_α, $[e, x_\alpha]$, e being right-handed. Let us decompose the vector y_α along x_α and $[e, x_\alpha]$:

$$y_\alpha = |x_\alpha| \cos\varphi \frac{x_\alpha}{|x_\alpha|} + |x_\alpha| \sin\varphi \frac{[e, x_\alpha]}{|[e, x_\alpha]|} = \cos\varphi \cdot x_\alpha + \sin\varphi \cdot [e, x_\alpha]. \tag{3}$$

Finally,

$$x_\alpha = x - x_e = x - (x, e)\, e. \tag{4}$$

Substituting (2), (3) and (4) into (1), we obtain

$$y = (x, e)\, e + \cos\varphi\,(x - (x, e)\, e) + \sin\varphi\,[e, x - (x, e)\, e] = \cos\varphi \cdot x + (1 - \cos\varphi)\,(x, e)\, e + \sin\varphi\,[e, x]. \tag{5}$$

From (5) it follows that the operator $U(e, \varphi)$ represents the sum of the operators from Problems 2.1, 2.3, and 2.4, whose matrices are known.

2.7. It is; $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}$. **2.8.** It is; $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}$. **2.9.** It is not. **2.10.** It is; $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. **2.11.** (a) $A' = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 3 & 5 & 1 \\ 3 & -1 & 0 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}$;

(b) $A' = \begin{pmatrix} -2 & 0 & 1 & 0 \\ 1 & -4 & -8 & -7 \\ 1 & 4 & 6 & 4 \\ 1 & 3 & 4 & 7 \end{pmatrix}$. **2.12.** $A'' = \begin{pmatrix} 1 & 2 & 2 \\ 3 & -1 & -2 \\ 2 & -3 & 1 \end{pmatrix}$.

2.13. $[A+B]_{\mathscr{B}''} = \begin{pmatrix} 44 & 44 \\ -29.5 & -25 \end{pmatrix}$. **2.14.** $[p(A)] = 3A^2 - 2A + 5E = \begin{pmatrix} -6 & 22 \\ -22 & 49 \end{pmatrix}$. **2.15.** (a) $\begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & 0 \\ & & 0 & 3 & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & 0 & n-1 \\ & & & & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}$.

2.16. $\begin{pmatrix} 1/2 & 1/3 & 1/4 & 1/5 \\ 1 & 1/2 & 1/3 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. **2.17.** $\begin{pmatrix} 1 & h & h^2 & h^3 \\ 0 & 1 & 2h & 3h^2 \\ 0 & 0 & 1 & 3h \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

2.19. The operator is invertible if and only if $\lambda \neq 0$; $A^{-1}x = \frac{1}{\lambda}x$.

2.20. The operator of projecting on the axis given by the vector e has no inverse. **2.21.** The operator has no inverse. **2.22.** The operator of projecting on the plane perpendicular to the vector e has no inverse. **2.23.** The operator of mirror reflection in the plane perpendicular to the vector e. It is invertible, and $A^{-1} = A$. *Hint.* The latter follows from the equality $A^2 = E$, which is geometrically obvious, but it can also be verified in the following way: $A^2x = A(x - 2(x, e)e) = x - 2(x, e)e - 2(x, e)Ae = x - 2(x, e)e - 2(x, e)(e - 2e) = x = Ex$, $x \in \mathcal{V}_3$. **2.24.** $A^{-1}x = (-x + 2y - z)i + (-x - 3y - 2z)j + (2x - 3y + 2z)k$. **2.25.** The operator has no inverse. **2.26.** $U^{-1}(e, \varphi) = U(e, -\varphi) = U(-e, \varphi)$. **2.27.** $A^{-1} = A$. **2.28.** The operator has no inverse. **2.29.** $A^{-1}x = \frac{1}{9}(x_1 + 2x_2 + 2x_3, 2x_1 + x_2 - 2x_3, 2x_1 - 2x_2 + x_3)$. **2.30.** (a) N_A is a two-dimensional subspace of all the vectors orthogonal to the vector e, T_A is a one-dimensional subspace of all the vectors collinear with the vector e; (b) N_A is a one-dimensional subspace of all the vectors collinear with a, T_A is a two-dimensional subspace of all the vectors orthogonal to a. **2.31.** $N_D = \mathcal{P}_0 = \mathbb{R}$, $T_D = \mathcal{P}_{n-1}$. **2.33.** $r_A = 2$, basis T_A: $e_1 = (2, 1, 1)$, $e_2 = (-1, -2, 1)$; $n_A = 1$, basis N_A: $e = (1, 1, 1)$. **2.34.** $r_A = 1$, basis T_A: $e = (1, 1, 1)$; $n_A = 2$, basis N_A: $e_1 = (1, -1, 0)$, $e_2 = (1, 0, -1)$. **2.36.** $\lambda = a$, $x^{(\lambda)}$ is any nonzero vector. **2.37.** $\lambda_1 = 1$, $x^{(\lambda_1)} = xi$, $x \neq 0$, $\lambda_2 = 0$, $x^{(\lambda_2)} = yj + zk$, $x^{(\lambda_2)} \neq 0$. **2.38.** $\lambda = 0$, $x^{(\lambda)} = xi$, $x \neq 0$. **2.39.** For $\varphi = 2\pi l$, $l = 0, \pm 1, \ldots$, the operator $U(e, \varphi)$ coincides with the unit operator. Therefore in this case $\lambda = 1$, and $x^{(\lambda)}$ is any nonzero vector. For $\varphi = (2l + 1)\pi$, $l = 0, \pm 1, \ldots$, $\lambda_1 = 1$, $x^{(\lambda_1)} = \alpha e$, $\alpha \neq 0$, $\lambda_2 = -1$, $x^{(\lambda_2)}$ is any nonzero vector perpendicular to the vector e. For $\varphi \neq \pi l$, $l = 0, \pm 1, \ldots$, the operator has the unique eigenvalue $\lambda = 1$, and $x^{(\lambda)} = \alpha e$, $\alpha \neq 0$. **2.40** $\lambda_1 = 1$, $x^{(\lambda_1)}$ is any nonzero vector coplanar with the reflection plane, $\lambda_2 = -1$, $x^{(\lambda_2)} = \alpha e$, $\alpha \neq 0$. **2.41.** $\lambda = -1$, $X^{(\lambda)} = \alpha \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\alpha \neq 0$. **2.42.** $\lambda = 2$, $X^{(\lambda)} = \alpha_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, α_1 and α_2 are not both zero. **2.43.** $\lambda_1 = 1$, $X^{(\lambda_1)} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\lambda_2 = 0$, $X^{(\lambda_2)} = \alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$; $\alpha \neq 0$. **2.44.** $\lambda = 1$,

$X^{(\lambda)} = \alpha \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$, $\alpha \neq 0$. **2.45.** $\lambda_1 = 3$, $X^{(\lambda_1)} = \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$; $\lambda_2 = -1$, $X^{(\lambda_2)} = \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; $\alpha \neq 0$. **2.46.** $\lambda_1 = 1$, $X^{(\lambda_1)} = \alpha_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, α_1, α_2 are not both zero; $\lambda_2 = -1$, $X^{(\lambda_2)} = \alpha \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$, $\alpha \neq 0$. **2.48.** For any $\varphi \neq 0$, π the operator A has two eigenvalues $\lambda_1(\varphi) = \cos\varphi + i \sin\varphi = e^{i\varphi}$, $\lambda_2(\varphi) = \cos\varphi - i \sin\varphi = e^{-i\varphi}$. The corresponding eigenvectors are $X^{(\lambda_1)}(\varphi) = \alpha \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $X^{(\lambda_2)}(\varphi) = \beta \begin{pmatrix} 1 \\ i \end{pmatrix}$, where α and β are arbitrary nonzero complex numbers. For $\varphi = 0$ and $\varphi = \pi$ the operator A has a single eigenvalue for either case: $\lambda(\varphi = 0) = 1$, $\lambda(\varphi = \pi) = -1$. In both cases the eigenvector is any nonzero vector from the complex space $\mathscr{L}_2$. **2.49.** *Hint.* To the equality $(A - \lambda E) X^{(\lambda)} = O$ apply the operation of complex conjugation.

2.50. $\lambda_1 = 1$, $X^{(\lambda_1)} = \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; $\lambda_2 = 2 + 3i$, $X^{(\lambda_2)} = \alpha \begin{pmatrix} 3-3i \\ 5-3i \\ 4 \end{pmatrix}$; $\lambda_3 = 2 - 3i$, $X^{(\lambda_3)} = \alpha \begin{pmatrix} 3+3i \\ 5+3i \\ 4 \end{pmatrix}$; $\alpha \in \mathbb{C}$, $\alpha \neq 0$. *Hint.* Take advantage of the result obtained in Problem 2.49.

2.52. (b) $\lambda_{A^{-1}} = 1/\lambda_A$. **2.54.** $\begin{pmatrix} 3 & 6 \\ -1 & -3 \end{pmatrix}$.

2.55. $\begin{pmatrix} -83 & -59 & -45 \\ 107 & 83 & 67 \\ 14 & 10 & 3 \end{pmatrix}$. **2.56.** $\begin{pmatrix} 17 & 13 & 13 \\ -11 & \varepsilon^2-8 & \varepsilon-8 \\ -11 & \varepsilon-8 & \varepsilon^2-8 \end{pmatrix}$.

2.57. $D = \begin{pmatrix} -3/2 & -2 & -1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & 2 & 3/2 \end{pmatrix}$, $D^* = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}$.

2.58. $D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, $D^* = \begin{pmatrix} 0 & 2/3 & 0 \\ 1 & 0 & -1/2 \\ 0 & 4/3 & 0 \end{pmatrix}$. **2.59.** Rotation through an angle α clockwise about the origin. **2.60.** $\begin{pmatrix} 1 & 0 \\ b/a & 0 \end{pmatrix}$.

2.61. $\frac{1}{a^2+b^2} \begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix}$, if $ab \neq 0$; $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $a=0$; $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for $b=0$. **2.62.** Let $a_1, \ldots, a_n \in \mathbb{R}^n$ correspond to the

for $b = 0$. **2.62.** Let $a_1, \ldots, a_n \in \mathbb{R}^n$ correspond to the column vectors of matrix A. By the Kronecker-Capelli theorem, a system is compatible if and only if rank A = rank $\overline{A}$, i.e. the vector b corresponding to the column vector B belongs to the linear span of the vectors $a_1, \ldots, a_n$, which also correspond to the row vectors of the conjugate matrix A^* (under consideration here is the real case). By definition, the arithmetical vector x is a solution of a conjugate system if and only if $(a_i, x) = 0$, $i = 1, 2, \ldots, n$, and, hence, $(b, x) = 0$. The theorem has been proved. In the complex case, the rows of matrix A^* are not vectors $a_1, \ldots, a_n$, but their complex conjugate vectors, but the proof is performed in a similar way. **2.63.** Compatible; the general solution of the conjugate system is ce, $e = (-1, -1, 2)$. **2.64.** Incompatible; the general solution of the conjugate system is $c_1e_1 + c_2e_2$, $e_1 = (-1, 1, 0)$, $e_2 = (-1, 0, 1)$. **2.65.** Compatible; the conjugate system has only a trivial solution. **2.66.** Compatible; the general solution of the conjugate system as in Problem 2.64. **2.67.** *Hint.* Take advantage of Fredholm's theorem.

2.68. Only the system from Problem 2.65. **2.75.** $E_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. **2.76.** The given matrix cannot be diagonalized. **2.77.** $E_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. **2.78.** $E_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. **2.79.** $E_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $E_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. **2.80.** $E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

2.81. (a) $\begin{pmatrix} 1 & 2^m - 1 \\ 0 & 2^m \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for an even m, $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ for an odd m. *Hint.* Use the formula $A^m = T^{-1}D^mT$.

2.82. $\begin{pmatrix} 3197 & -1266 \\ 7385 & -2922 \end{pmatrix}$. **2.83.** $\begin{pmatrix} 190 & 189 & -189 \\ 126 & 127 & -126 \\ 252 & 252 & -251 \end{pmatrix}$. **2.84.** $E_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$,

$$E_2 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix}, \quad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 18 \end{pmatrix}.$$

2.85. $E_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ -4/\sqrt{18} \end{pmatrix}$, $E_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$,

$D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 27 \end{pmatrix}$. **2.86.** $E_1 = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{pmatrix}$,

$E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. **2.87.** $U = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1+i}{\sqrt{6}} \\ \frac{1-i}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}$,

$D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$. **2.88.** $U = \frac{1}{\sqrt{6}} \begin{pmatrix} 2-i & -1 \\ 1 & 2+i \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$.

2.89. $U = \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2.90. $U = \begin{pmatrix} 2/3 & -2/3 & -1/3 \\ 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$.

2.91. $U = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{1}{3} \\ -\frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & -\frac{2}{3} \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}$.

3.6. (a) $A = \begin{pmatrix} 6 & 0 & -4 \\ 0 & 6 & 2 \\ -4 & 2 & 2 \end{pmatrix}$; (b) $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

3.7. (a) $A'=\begin{pmatrix} 3 & -1 & -1 & -1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -3 \\ -1 & -1 & 3 & -1 \end{pmatrix}$; (b) $A'=\begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$.

3.11. $x_1'^2+x_2'^2-x_3'^2$,

$$\begin{cases} x_1=x_1'-\frac{1}{2}x_2'+\frac{5}{6}x_3', \\ x_2=\frac{1}{2}x_2'-\frac{1}{6}x_3', \\ x_3=\frac{1}{3}x_3'. \end{cases}$$

3.12. $x_1'^2-x_2'^2-x_3'^2$,

$$\begin{cases} x_1=x_1'-x_2'-x_3', \\ x_2=x_1'+x_2'-x_3', \\ x_3=x_3'. \end{cases}$$

3.13. $x_1'^2+x_2'^2-x_3'^2$,

$$\begin{cases} x_1=\frac{1}{2}x_1'+x_2', \\ x_2=x_2'+x_3', \\ x_3=-x_2'+x_3'. \end{cases}$$

3.14. $9x_1'^2+18x_2'^2-9x_3'^2$,

$$\begin{cases} x_1=\frac{2}{3}x_1'+\frac{2}{3}x_2'-\frac{1}{3}x_3' \\ x_2=-\frac{1}{3}x_1'+\frac{2}{3}x_2'+\frac{2}{3}x_3', \\ x_3=\frac{2}{3}x_1'-\frac{1}{3}x_2'+\frac{2}{3}x_3'. \end{cases}$$

3.15. $3x_1'^2+6x_2'^2-2x_3'^2$,

$$\begin{cases} x_1=\frac{1}{\sqrt{3}}x_1'+\frac{1}{\sqrt{6}}x_2'+\frac{1}{\sqrt{2}}x_3', \\ x_2=-\frac{1}{\sqrt{3}}x_1'-\frac{1}{\sqrt{6}}x_2'+\frac{1}{\sqrt{2}}x_3', \\ x_3=\frac{1}{\sqrt{3}}x_1'-\frac{2}{\sqrt{6}}x_2'. \end{cases}$$

3.16. $5x_1'^2-x_2'^2-x_3'^2$,

$$\begin{cases} x_1=\frac{1}{\sqrt{3}}x_1'+\frac{1}{\sqrt{6}}x_2'+\frac{1}{\sqrt{2}}x_3', \\ x_2=\frac{1}{\sqrt{3}}x_1'+\frac{1}{\sqrt{6}}x_2'-\frac{1}{\sqrt{2}}x_3', \\ x_3=\frac{1}{\sqrt{3}}x_1'-\frac{2}{\sqrt{6}}x_2'. \end{cases}$$

3.17. $9x_1'^2+18x_2'^2+18x_3'^2$,

$$\begin{cases} x_1=\frac{1}{3}x_1'-\frac{2}{3}x_2'+\frac{2}{3}x_3', \\ x_2=\frac{2}{3}x_1'-\frac{1}{3}x_2'-\frac{2}{3}x_3', \\ x_3=\frac{2}{3}x_1'+\frac{2}{3}x_2'+\frac{1}{3}x_3'. \end{cases}$$

3.18. Ellipse $\frac{x'^2}{2}+y'^2=1$, $O'\times(-4/5, 2/5)$, $e_1=(1/\sqrt{5}, 2/\sqrt{5})$, $e_2=(-2/\sqrt{5}, 1/\sqrt{5})$. 3.19. Parabola $y'^2=4\sqrt{2}x'$, $O'(2, 1)$, $e_1=(1/\sqrt{2}, 1/\sqrt{2})$, $e_2=(-1/\sqrt{2}, 1/\sqrt{2})$. 3.20. Hyperbola $\frac{x'^2}{4}-\frac{y'^2}{9}=1$, $O'(1, 1)$,

$e_1 = (3/\sqrt{13},\ 2/\sqrt{13})$, $e_2 = (-2/\sqrt{13},\ 3/\sqrt{13})$. **3.21.** Parallel straight lines $x' = \pm\sqrt{5}/2$, $O'(-3/5, -3/10)$, $e_1 = (-2/\sqrt{5},\ 1/\sqrt{5})$, $e_2 = (1/\sqrt{5},\ 2/\sqrt{5})$ or, in the old variables, $2x - y + 1 = 0$, $2x - y - 4 = 0$. **3.22.** Ellipse $\frac{x'^2}{35/6} + \frac{y'^2}{35/36} = 1$, O' (7/6, 1/3), $e_1 = (2/\sqrt{5},\ -1/\sqrt{5})$, $e_2 = (1/\sqrt{5},\ 2/\sqrt{5})$. **3.23.** Parabola $y'^2 = \frac{1}{\sqrt{5}}x'$, O' (3, 2), $e_1 = \left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$, $e_2 = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$. **3.24:** (a) For $\lambda \in (-\infty, -1)$ hyperbola $(x-\lambda)^2 + \lambda\left(y - \frac{1}{\lambda}\right)^2 = \frac{\lambda^3+1}{\lambda}$, for $\lambda = -1$ two intersecting straight lines $x - y = 0$, $x + y + 2 = 0$, for $\lambda \in (-1, 0)$—hyperbola $(x-\lambda)^2 + \lambda\left(y - \frac{1}{\lambda}\right)^2 = \frac{\lambda^3+1}{\lambda}$, for $\lambda = 0$—parabola $x^2 = 2y$, for $\lambda \in (0, +\infty)$—ellipse $(x-\lambda)^2 + \lambda\left(y - \frac{1}{\lambda}\right)^2 = \frac{\lambda^3+1}{\lambda}$; (b) for $\lambda \in (-\infty, -1)$—hyperbola $(1-\lambda)x'^2 + (1+\lambda)y'^2 = 1$, O' (0, 0), $e_1 = (-1/\sqrt{2},\ 1/\sqrt{2})$, $e_2 = (-1/\sqrt{2},\ -1/\sqrt{2})$, for $\lambda = -1$ two parallel lines $x - y \pm 1 = 0$, for $\lambda \in (-1, 0)$—ellipse $(1-\lambda)x'^2 + (1+\lambda)y'^2 = 1$, O' (0, 0), $e_1 = (-1/\sqrt{2},\ 1/\sqrt{2})$, $e_2 = (-1/\sqrt{2},\ -1/\sqrt{2})$, for $\lambda = 0$—circle $x^2 + y^2 = 1$, for $\lambda \in (0, 1)$—ellipse $(1-\lambda)x'^2 + (1+\lambda)y'^2 = 1$, $O'(0, 0)$, $e_1 = (-1/\sqrt{2}, 1/\sqrt{2})$, $e_2 = (-1/\sqrt{2}, -1/\sqrt{2})$, for $\lambda = 1$—two parallel straight lines $x + y \pm 1 = 0$, for $\lambda \in (1, +\infty)$—hyperbola $(1-\lambda)x'^2 + (1+\lambda)y'^2 = 1$, O' (0, 0), $e_1 = (-1/\sqrt{2},\ 1/\sqrt{2})$, $e_2 = (-1/\sqrt{2},\ -1/\sqrt{2})$. **3.25.** Ellipsoid $\frac{x'^2}{2} + \frac{y'^2}{1} + \frac{z'^2}{2/3} = 1$, O' (1, 2, −1), $e_1 = (1/3, 2/3, 2/3)$, $e_2 = (2/3, 1/3, -2/3)$, $e_3 = (2/3, -2/3, 1/3)$. **3.26.** Hyperbolic paraboloid $\frac{x'^2}{2} - \frac{y'^2}{1} = -2z'$, O' (1, 2, 3), $e_1 = (-2/3, 1/3, 2/3)$, $e_2 = (1/3, -2/3, 2/3)$, $e_3 = (2/3, 2/3, 1/3)$. **3.27.** Hyperboloid of two sheets $\frac{x'^2}{4/5} + \frac{y'^2}{4/15} - \frac{z'^2}{4/25} = -1$, O' (0, 1, −2/5), $e_1 = (1/\sqrt{2},\ -1/\sqrt{2},\ 0)$, $e_2 = (1/\sqrt{2},\ 1/\sqrt{2},\ 0)$, $e_3 = (0, 0, 1)$. **3.28.** Elliptic paraboloid $\frac{x'^2}{5\sqrt{2}/4} + \frac{y'^2}{\sqrt{2}/2} = 2z'$, O' (−1/40, −19/40, 1/2), $e_1 = (1/\sqrt{6},\ 1/\sqrt{6},\ -2/\sqrt{6})$, $e_2 = (1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$, $e_3 = (1/\sqrt{2},\ -1/\sqrt{2},\ 0)$. **3.29.** Parabolic cylinder $y'^2 = \frac{4}{3}x'$, O' (2, 1, −1), $e_1 = (2/3, 2/3, 1/3)$, $e_2 = (2/3, -1/3, -2/3)$,

$e_3 = (1/3, -2/3, 2/3)$. **3.30.** Elliptic cylinder $\frac{x'^2}{2} + \frac{y'^2}{1} = 1$, $O'(0, 1, 0)$, $e_1 = (1/\sqrt{3}, 1/\sqrt{3}, -1/3)$, $e_2 = (1/\sqrt{6}, -2/\sqrt{6}, -1/\sqrt{6})$, $e_3 = (1/\sqrt{2}, 0, 1/\sqrt{2})$. **3.31.** Hyperboloid of one sheet $\frac{x'^2}{1/3} + \frac{y'^2}{1/6} - \frac{z'^2}{1/2} = 1$, $O'(-1/3, -2/3, 2/3)$, $e_1 = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, $e_2 = (1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6})$, $e_3 = (1/\sqrt{2}, 0, -1/\sqrt{2})$. **3.32.** Hyperbolic cylinder $\frac{x'^2}{1/3} - \frac{y'^2}{1/3} = 1$, $O'(1/6, 1/3, -5/6)$, $e_1 = (1/\sqrt{2}, 0, -1/\sqrt{2})$, $e_2 = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, $e_3 = (1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6})$.

Chapter 5

DIFFERENTIAL CALCULUS: FUNCTIONS OF ONE VARIABLE

SEC. 5.1. DERIVATIVES

1. The Derivative Defined. Differentiation of Explicitly Represented Functions. Let $\Delta f(x_0, \Delta x) = f(x_0 + \Delta x) - f(x_0)$ be the *increment of the function* $y = f(x)$ *at the point* x_0 corresponding to the *increment of the argument* Δx. The *derivative* of the first order (or the *first derivative*) of a function $y = f(x)$ at a given point x_0 is defined as the limit

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f(x_0, \Delta x)}{\Delta x}. \tag{1}$$

The numbers

$$f'_-(x_0) = \lim_{\Delta x \to -0} \frac{\Delta f(x_0, \Delta x)}{\Delta x}$$

and

$$f'_+(x_0) = \lim_{\Delta x \to +0} \frac{\Delta f(x_0, \Delta x)}{\Delta x}$$

are called, respectively, the *left-hand* and *right-hand derivatives* of the function $y = f(x)$ at the point x_0. For the derivative $f'(x_0)$ of the function $f(x)$ to exist at the point x_0, it is necessary and sufficient that its left-hand and right-hand derivatives exist and coincide at this point, i.e.

$$f'_-(x_0) = f'_+(x_0).$$

Example 1. Find $f'_-(0)$ and $f'_+(0)$ for the function $f(x) = |x|$.

By definition, we have

$$f'_-(0) = \lim_{\Delta x \to -0} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to -0} \frac{-\Delta x}{\Delta x} = -1$$

and

$$f'_+(0) = \lim_{\Delta x \to +0} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to +0} \frac{\Delta x}{\Delta x} = 1.$$

Note that the function $f(x) = |x|$ has no derivative at the point $x_0 = 0$, since $f'_-(0) \neq f'_+(0)$.

The derivative of the function $f(x)$, considered on the set of those points where it exists, is a function itself. The process of finding the derivative is also called *differentiation.*

Table of derivatives of basic elementary functions:

1. $(x^a)' = ax^{a-1}$, $a \neq 0$.
2. $(a^x)' = a^x \ln a$, $a > 0$; $(e^x)' = e^x$.
3. $(\log_a x)' = \log_a e \cdot \frac{1}{x}$, $a > 0$, $a \neq 1$; $(\ln x)' = \frac{1}{x}$.
4. $(\sin x)' = \cos x$.
5. $(\cos x)' = -\sin x$.
6. $(\tan x)' = \frac{1}{\cos^2 x}$.
7. $(\cot x)' = -\frac{1}{\sin^2 x}$.
8. $(\arcsin x)' = -(\arccos x)' = \frac{1}{\sqrt{1-x^2}}$.
9. $(\arctan x)' = -(\operatorname{arc\,cot} x)' = \frac{1}{1+x^2}$.

Rules for differentiation of functions:

I. Let C be a constant and let $f(x)$, $g(x)$ be differentiable functions. Then:

1. $(C)' = 0$.
2. $(f+g)' = f' + g'$.
3. $(Cf)' = Cf'$.
4. $(fg)' = f'g + fg'$.
5. $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$, $g \neq 0$.

II. Let the function $y = f(x)$ have a derivative at the point x_0 and let the function $z = g(y)$ have a derivative at the point $y_0 = f(x_0)$. Then the composite function $z = g(f(x))$ has a derivative at the point x_0 equal to

$$z'(x_0) = g'(y_0) f'(x_0) \tag{2}$$

(*the rule for differentiation of a composite function*).

Example 2. Find the derivative of the function $z = \log_3 (\arcsin x)$.

Setting $z = \log_3 y$ and $y = \arcsin x$, we have

$$z'(y) = \log_3 e \cdot \frac{1}{y} \text{ and } y'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Hence, according to (2), we obtain

$$z'(x) = \frac{\log_3 e}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}.$$

In Problems 1.1 to 1.3 find $\Delta f(x_0, \Delta x)$ if

1.1. $f(x) = x^3$, $x_0 = 1$, $\Delta x = 0.1$.

1.2. $f(x) = \sqrt{x}$, $x_0 = 1$, $\Delta x = 0.25$.

1.3. $f(x) = \log x$, $x_0 = 100$, $\Delta x = -90$.

In Problems 1.4 to 1.7 find $\Delta f(x_0, \Delta x)$ as a function of Δx if

1.4. $f(x) = \sin x$, $x_0 = \frac{\pi}{2}$.

We have

$$\Delta f\left(\frac{\pi}{2}, \Delta x\right) = \sin\left(\frac{\pi}{2} + \Delta x\right) - \sin\frac{\pi}{2}$$

$$= 2\sin\frac{\Delta x}{2}\cos\left(\frac{\pi}{2} + \frac{\Delta x}{2}\right) = -2\sin^2\frac{\Delta x}{2}.$$

1.5. $f(x) = x^2$, $x_0 = -1$. **1.6.** $f(x) = e^x$, $x_0 = 1$.

1.7. $f(x) = \log_2 x$, $x_0 = 1$.

In Problems 1.8 to 1.12, using only the definition of the derivative, find $f'(x)$.

1.8. $f(x) = \cot x$.

We have:

$$(\cot x)' = \lim_{\Delta x \to 0} \frac{\cot(x+\Delta x) - \cot x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(-\Delta x)}{\Delta x \sin x \sin(x+\Delta x)}$$

$$= -\lim_{\Delta x \to 0} \frac{1}{\sin x \sin(x+\Delta x)} = -\frac{1}{\sin^2 x}$$

1.9. $f(x) = \frac{1}{x^2}$. **1.10.** $f(x) = \sqrt{x}$.

1.11. $f(x) = 2^x$. **1.12.** $f(x) = \log_2 x$.

1.13. It is known that $f(0) = 0$ and there exists the limit $\lim\limits_{x \to 0} \frac{f(x)}{x}$. Prove that this limit is equal to $f'(0)$.

1.14*. Prove that if $f(x)$ has a derivative at the point x_0, then

$$\lim_{x \to x_0} \frac{xf(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0).$$

In Problems 1.15 to 1.19 find $f'_-(x_0)$ and $f'_+(x_0)$ for the given $f(x)$.

1.15. $f(x) = |x-1| + |x+1|$, $x_0 = \pm 1$.

1.16. $f(x) = \begin{cases} x, & x \leqslant 1, \\ -x^2+2x, & x>1, \end{cases} \quad x_0 = 1.$

We have

$$f'_-(1) = \lim_{x \to 1-0} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1-0} \frac{x-1}{x-1} = 1$$

and

$$f'_+(1) = \lim_{x \to 1+0} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x \to 1+0} \frac{-x^2+2x-1}{x-1} = -\lim_{x \to 1+0} (x-1) = 0.$$

1.17. $f(x) = \begin{cases} 0, & x \leqslant 0 \\ x^2 \ln x, & x>0, \end{cases} \quad x_0 = 0.$

1.18. $f(x) = \sqrt{1-e^{-x^2}}$, $x_0 = 0$.

1.19. $f(x) = \begin{cases} 0, & x = 0, \\ \dfrac{x}{1+e^{1/x}}, & x \neq 0, \end{cases} \quad x_0 = 0.$

1.20*. Show that the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is continuous for $x = 0$, but has neither a left-hand, nor a right-hand derivative at this point.

In Problems 1.21 to 1.76 find the derivatives of the given functions:

1.21. $y = 3 - 2x + \frac{2}{3} x^4$. **1.22.** $y = -\frac{5x^5}{a^2}$.

1.23. $y = -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3}$. **1.24.** $y = \frac{x-1}{x+1}$.

1.25. $y = \frac{x^2+1}{x^3-x}$. **1.26.** $y = (x^2-1)(x^2-4)(x^2+9)$.

1.27. $y = \frac{1+3x^2}{\sqrt{2\pi}}$. **1.28.** $y = \frac{1}{x^3+3x-1}$.

1.29. $y=\frac{a}{\sqrt[5]{x^3}}+\frac{\sqrt[3]{x^2}}{b}$. **1.30.** $y=\frac{a+bx}{c+dx}$.

1.31. $y=\frac{2}{2x-1}-\frac{1}{x}$. **1.32.** $y=\frac{2+\sqrt{x}}{2-\sqrt{x}}$.

1.33. $y=(\sqrt{x}-1)\left(\frac{1}{\sqrt{x}}+1\right)$. **1.34.** $y=3\sqrt[3]{x^2}-2\sqrt[4]{x^3}$.

1.35. $y=(3\sqrt[3]{x^2}+6\sqrt[3]{x})\sqrt[3]{x^4}$.

1.36. $y=\frac{4}{\sqrt[4]{x^3}}-\frac{3}{\sqrt[3]{x^2}}$. **1.37.** $y=x^3\cot x$.

1.38. $y=\frac{\tan x}{\sqrt[3]{x^2}}$. **1.39.** $y=\frac{\cos x}{1+\sin x}$.

1.40. $y=\sqrt{x}\sin x$. **1.41.** $y=x\sqrt[3]{x^2}(2\ln x-3^x)$.

1.42. $y=3x^3\log_2 x+\frac{x^2}{e^x}$. **1.43.** $y=2\sin x-3\tan x$.

1.44. $y=\frac{\sin x-\cos x}{\sin x+\cos x}$. **1.45.** $y=x^{\frac{3}{2}}\sqrt[3]{x^5+a}$.

1.46. $y=\sqrt{\frac{1-x^2}{1+x^2}}$. **1.47.** $y=\sin\frac{3x}{2}$.

1.48. $y=6\cos\frac{2x}{3}$. **1.49.** $y=(1+4x^2)^3$.

1.50. $y\sqrt[4]{(1+3x^2)^3}$. **1.51.** $y=\sin^2\frac{x}{2}$.

1.52. $y=\sqrt{1+\sin 4x}-\sqrt{1-\sin 4x}$.

1.53. $y=x\arcsin\ln x$. **1.54.** $y=\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

1.55. $y=\sqrt[4]{(1+\sin^2 x)^3}$. **1.56.** $y=x^2e^{-2x}$.

1.57. $y=e^{\frac{x}{3}}\cos^2\frac{x}{3}$.

1.58. $y=\frac{x}{2}\sqrt{x^2+a}+\frac{a}{2}\ln\left(x+\sqrt{x^2+a}\right)$.

1.59. $y=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$. **1.60.** $y=\ln\sqrt{\frac{1+x^2}{1-x^2}}$.

1.61. $y=\sqrt{\operatorname{arc\,cot}\frac{x}{2}}$. **1.62.** $y=\sqrt[3]{1+\tan\left(x+\frac{1}{x}\right)}$.

1.63. $y=\cos^2\left(\sin\frac{x}{3}\right)$. **1.64.** $y=\sqrt{\sin\sqrt{x}}$.

1.65. $y=\arctan\left(x-\sqrt{1+x^2}\right)$.

1.66. $y=\arccos\frac{b+a\cos x}{a+b\cos x}$.

1.67. $y=\sqrt{x}e^{\frac{x}{2}}$. **1.68.** $y=\frac{e^{-x^2}}{2x}$.

1.69. $y=2^{\frac{x}{\ln x}}$. **1.70.** $y=2^{\sqrt{\sin^2 x}}$.

1.71. $y=3^{2^x}$. **1.72.** $y=\ln x\cdot\log x-\ln a\cdot\log_a x$.

1.73. $y=\log_2\ln 2x$. **1.74.** $y=e^{\sqrt{\ln(ax^2+bx+c)}}$.

1.75. $y=\ln\arctan\sqrt{1+x^2}$. **1.76.** $y=\ln\left(x+\sqrt{a^2+x^2}\right)$.

In Problems 1.77 to 1.80 find the derivatives of the given *hyperbolic functions*:

1.77. $\sinh x=\frac{e^x-e^{-x}}{2}$ (hyperbolic sine),

1.78. $\cosh x=\frac{e^x+e^{-x}}{2}$ (hyperbolic cosine),

1.79. $\tanh x=\frac{\sinh x}{\cosh x}$ (hyperbolic tangent),

1.80. $\coth x=\frac{\cosh x}{\sinh x}$ (hyperbolic cotangent).

The *logarithmic derivative* of a function $y=f(x)$ is defined as the derivative of the logarithm of this function, i.e.

$$(\ln y)'=\frac{y'}{y}.$$

Taking the logarithm prior to finding the derivative often simplifies the computation of the derivative.

Example 3. Find the derivative of the function $y=\sqrt{\frac{x(x-1)}{x-2}}$.

We have

$$\ln y=\frac{1}{2}(\ln x+\ln(x-1)-\ln(x-2)),$$

$$(\ln y)'=\frac{y'}{y}=\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}-\frac{1}{x-2}\right),$$

whence

$$y' = (\ln y)' \cdot y = \frac{x^2 - 4x + 2}{2\sqrt{x(x-1)(x-2)^3}}.$$

Example 4. Find the derivative of an *exponential composite* function: $y = \left(1 + \frac{1}{x}\right)^x$.

Taking the logarithm, we obtain

$$\ln y = x \ln \left(1 + \frac{1}{x}\right).$$

Then we find the derivatives of the left-hand and right-hand members

$$(\ln y') = \frac{y'}{y} = \ln \left(1 + \frac{1}{x}\right) - \frac{1}{1+x}.$$

Consequently,

$$y' = (\ln y)' \cdot y = \left(1 + \frac{1}{x}\right)^x \left(\ln \left(1 + \frac{1}{x}\right) - \frac{1}{1-x}\right).$$

In Problems 1.81 to 1.92, taking first the logarithm of the given function, find then its derivative.

1.81. $y = \dfrac{(x-3)^2 (2x-1)}{(x+1)^3}$. **1.82.** $y = \sqrt[3]{\dfrac{(x+2)(x-1)^2}{x^5}}$.

1.83. $y = \dfrac{\sqrt{x+2}}{\sqrt[3]{(x-1)^2 (2x+1)}}$.

1.84. $y = x^3 \sqrt{\dfrac{x-1}{(x+2)\sqrt{x-2}}}$.

1.85. $y = x^x$. **1.86.** $y = x^{2^x}$.

1.87. $y = \sqrt{x \sqrt[3]{x}}$. **1.88.** $y = (\ln x)^{1/x}$.

1.89. $y = (\sin x)^{\arcsin x}$. **1.90.** $y = x^{x^x}$.

1.91. $y = \dfrac{(\ln x)^x}{x^{\ln x}}$. **1.92*.** $y = x^{x^2} + x^{2^x} + 2^{x^x}$.

In Problems 1.93 to 1.97, introducing intermediate variables, compute the derivatives of the given functions:

1.93*. $y = \ln \left(\cos^2 x + \sqrt{1 + \cos^2 x}\right)$.

1.94. $y = (\arccos x)^2 \ln (\arccos x)$.

1.95. $y = \dfrac{e^{-x^2} \arcsin (e^{-x^2})}{\sqrt{1 - e^{-2x^2}}}$.

1.96. $y=\frac{1-a^{2x}}{1+a^{2x}}\operatorname{arc\,tan} a^{-x}$.

1.97*. Let

$$f(x)=\begin{cases} x^2+2x, & x\leqslant 0, \\ ax+b, & x>0. \end{cases}$$

Find the coefficients a and b such that the function $f(x)$ is continuous and differentiable at the point $x_0=0$.

1.98. Let

$$f(x)=\begin{cases} \frac{1}{|x|}, & |x|\geqslant 1, \\ ax^2+b, & |x|<1. \end{cases}$$

Find the coefficients a and b such that the function $f(x)$ is continuous and differentiable at any point.

In Problems 1.99 to 1.132 find the derivatives of the given functions.

1.99. $y=\sqrt{\frac{1+x^3}{1-x^3}}$. **1.100.** $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$.

1.101. $y=\sqrt[m+n]{(1-x)^m(1+x)^n}$.

1.102. $y=\sin(\cos^2 x)\cos(\sin^2 x)$. **1.103.** $y=\frac{1}{\cos^n mx}$.

1.104. $y=\left(\frac{a}{b}\right)^x\left(\frac{b}{x}\right)^a\left(\frac{x}{a}\right)^b$, $a, b>0$.

1.105. $y=\ln(\ln^n mx)$.

1.106. $y=\frac{1}{2\sqrt{6}}\ln\frac{x\sqrt{3}-\sqrt{2}}{x\sqrt{3}+\sqrt{2}}$.

1.107. $y=\log_2\sin\left(2\pi x+\frac{\pi}{2}\right)$.

1.108. $y=\operatorname{arc\,tan}(\tan^2 x)$.

1.109. $y=\log_x e$. **1.110.** $y=(\sin x)^{\cos x}$.

1.111. $y=\sqrt{x}^{\sin^2 x}$. **1.112.** $y=\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}$.

1.113. $y=\ln(\sinh x)+\frac{1}{2\sinh x}$.

1.114. $y=\operatorname{arc\,tan}(\tanh x)$.

1.115. $y = e^{-x} \sinh ax$. **1.116.** $y = \arccos\left(\frac{1}{\cosh x}\right)$.

1.117. $y = \ln |x|$.

The function $y = \ln |x|$ is defined $\forall x \in \mathbb{R}$, $x \neq 0$, and

$$\ln |x| = \begin{cases} \ln x, & x > 0, \\ \ln(-x), & x < 0. \end{cases}$$

Hence,

$$(\ln |x|)' = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{x}, & x < 0 \end{cases} = \frac{1}{x}, \quad x \neq 0.$$

1.118. $y = \arcsin \frac{1}{|x|}$. **1.119.** $y = |\sin x|$. **1.120.** $y = |\arctan x|$.

1.121. $y = [x]\, x$, where $[x]$ is the integral part of the number x.

The function $y = [x]\, x$ is defined $\forall x \in \mathbb{R}$. If $k \in \mathbb{Z}$, then $y = kx$ for $x \in [k, k+1)$. Therefore

$$y' = k, \quad x \in (k, k+1),$$

and at the points $x = k$, $k \in \mathbb{Z}$:

$$f'_-(k) = k - 1, \quad f'_+(k) = k.$$

1.122. $y = \begin{cases} 1 - x, & x \leqslant 0, \\ e^{-x}, & x > 0. \end{cases}$

1.123. $y = \begin{cases} x, & x < 0, \\ \ln(1+x), & x \geqslant 0. \end{cases}$

1.124. $y = \begin{cases} x^2 e^{-x^2}, & |x| \leqslant 1, \\ 1/e, & |x| > 1. \end{cases}$

1.125. $y = \frac{x^2}{1-x} \sqrt[3]{\frac{3-x}{(3+x)^2}}$.

1.126. $y = (x - a_1)^{\alpha_1} (x - a_2)^{\alpha_2} \ldots (x - a_n)^{\alpha_n}$.

1.127. $y = a^{x^a}$. **1.128.** $y = (\log_x a)^x$.

1.129. $y = \sin(\sin(\sin x))$. **1.130.** $y = (1/x)^{1/x}$.

1.131. $y = \frac{\ln 3 \cdot \sin x + \cos x}{3^x}$.

1.132. $y = 3^{\frac{\sin ax}{\cos bx}} + \frac{1}{3}\,\frac{\sin^3 ax}{\cos^3 bx}$.

1.133. Prove that the derivative of an even function is an odd function, and the derivative of an odd function is an even function.

1.134. Prove that the derivative of a periodic function is again a periodic function.

1.135*. Find $f'(x_0)$ if $f(x) = (x - x_0)\,\varphi(x)$, where the function $\varphi(x)$ is continuous at the point x_0.

Let $\varphi(x)$ and $\psi(x)$ be differentiable functions. Find the derivatives of the following composite functions (Problems 1.136 to 1.139):

1.136. $y = \sqrt{\varphi^2(x) + \psi^2(x)}$. **1.137.** $y = \arctan \frac{\varphi(x)}{\psi(x)}$.

1.138. $y = \psi(x)^{\varphi(x)}$, $\psi(x) > 0$.

1.139. $y = \log_{\varphi(x)} \psi(x)$, $\varphi(x) > 0$, $\psi(x) > 0$, $\varphi(x) \neq 1$.

Let us pass to natural logarithms:

$$y = \log_{\varphi(x)} \psi(x) = \frac{\ln \psi(x)}{\ln \varphi(x)}.$$

Hence we find

$$y' = \frac{\frac{\psi'(x)}{\psi(x)} \ln \varphi(x) - \frac{\varphi'(x)}{\varphi(x)} \ln \psi(x)}{\ln^2 \varphi(x)} = \frac{1}{\ln \varphi(x)} \left(\frac{\psi'(x)}{\psi(x)} - y \frac{\varphi'(x)}{\varphi(x)} \right).$$

Let $f(x)$ be an arbitrary differentiable function. Find y' (Problems 1.140 to 1.143):

1.140. $y = f(\ln x)$. **1.141.** $y = \ln(f(x))$.

1.142. $y = f(e^x)\, e^{f(x)}$.

We have $y' = f'(e^x)\, e^x e^{f(x)} + f(e^x)\, e^{f(x)} f'(x) = e^{f(x)} (e^x f'(e^x) + f'(x) f(e^x))$.

1.143. $y = f(f(x))$.

2. Differentiation of Functions Represented Implicitly or Parametrically. The function $y = f(x)$, $x \in (a, b)$ is said to be *represented implicitly* by the equation $F(x, y) = 0$ if for all $x \in (a, b)$

$$F(x, f(x)) = 0. \tag{3}$$

To compute the derivative of the function $y = f(x)$, one should differentiate identity (3) with respect to x (regarding the left-hand member as a composite function of x) and then solve the obtained equation with respect to $f'(x)$.

Example 5. The equation $x^2 + y^2 = 1$ defines implicitly on the interval $(-1, 1)$ two functions:

$$y_1(x) = \sqrt{1 - x^2}, \qquad (4)$$
$$y_2(x) = -\sqrt{1 - x^2}.$$

Find their derivatives without using explicit expression (4).

Let $y(x)$ be any of these functions. Then, differentiating the identity $x^2 + y^2(x) = 1$ with respect to x, we obtain

$$2x + 2y(x)\, y'(x) = 0.$$

Hence

$$y'(x) = -\frac{x}{y(x)},$$

i.e.

$$y_1'(x) = -\frac{x}{y_1(x)} = -\frac{x}{\sqrt{1-x^2}}$$

and

$$y_2'(x) = -\frac{x}{y_2(x)} = \frac{x}{\sqrt{1-x^2}}.$$

Example 6. Deduce the rule for differentiation of an inverse function.

If $x = f^{-1}(y)$, $y \in E$, is the inverse of the function $y = f(x)$, $x \in D$, then for all $y \in E$ the following equality is fulfilled:

$$f(f^{-1}(y)) - y = 0.$$

In other words, the inverse function $x = f^{-1}(y)$ is a function represented implicitly by the equation

$$f(x) - y = 0. \qquad (5)$$

To compute the derivative of the function $x = f^{-1}(y)$, we differentiate (5) with respect to y:

$$f'(x(y))\, x'(y) - 1 = 0,$$

whence

$$x'(y) = \frac{1}{f'(x(y))}.$$

For implicitly represented functions as also for composite functions we shall also use symbols of type y_x' to denote their derivatives whenever it is necessary to indicate with respect to which variable differentiation is carried out.

1.144. Compute the derivative y'_x at the point $x = 1$ if

$$x^3 - 2x^2y^2 + 5x + y - 5 = 0,\ y(1) = 1.$$

1.145. Find y'_x at the point $(0, 1)$ if $e^y + xy = e$.

In Problems 1.146 to 1.157 find y'_x for the indicated implicitly represented functions.

1.146. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **1.147.** $x^4 + y^4 = x^2y^2$.

1.148. $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $a > 0$. **1.149.** $2y \ln y = x$.

1.150. $e^x \sin y - e^y \cos x = 0$. **1.151.** $\sin(xy) + \cos(xy) = 0$.

1.152. $2^x + 2^y = 2^{x+y}$. **1.153.** $x - y = \arcsin x - \arcsin y$.

1.154. $\arctan \frac{y}{x} = \ln \sqrt{x^2 + y^2}$.

1.155. $xy = \arctan \frac{x}{y}$.

1.156. $x^y = y^x$. **1.157.** $a^{\frac{x}{y}} = \left(\frac{x}{y}\right)^a$.

1.158. Prove that the function y defined by the equation $xy - \ln y = 1$ satisfies the equation $y^2 + (xy - 1)y' = 0$ as well.

In Problems 1.159 to 1.162 find the derivative of the inverse of the given function.

1.159. $y = \sinh x$.

By definition, we have $\sinh x = \frac{e^x - e^{-x}}{2}$ since $(\sinh x)' = \frac{e^x + e^{-x}}{2} > 0$ for all $x \in \mathbb{R}$, the function $\sinh x$ increases monotonically throughout the entire real axis, and, consequently, has an inverse denoted as $\sinh^{-1} x$. According to the differentiation rule for an inverse function, we obtain

$$x'_y = (\sinh^{-1} y)' = \frac{1}{y'_x} = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x} = \frac{1}{\sqrt{1 + \sinh^2 x}} = \frac{1}{\sqrt{1 + y^2}}.$$

Consequently, passing to ordinary notation, we have

$$(\sinh^{-1} x)' = \frac{1}{\sqrt{1 + x^2}}.$$

1.160*. $y = \cosh x$. **1.161.** $y = \arcsin 2^x$.

1.162. $y = 2x^2 - x$, $x \in (1/2, +\infty)$.

Let $y = \alpha(x)$ be the inverse of the given function $y = f(x)$. Express $\alpha'(x)$ in terms of x and $\alpha(x)$ if (Problems 1.163 to 1.167):

1.163. $y = x^x$.

Taking into consideration that

$$(x^x)' = x^x (\ln x + 1),$$

we obtain:

$$x'_y = \frac{1}{y'_x} = \frac{1}{x^x (\ln x + 1)} = \frac{1}{y (\ln \alpha(y) + 1)},$$

since $x = \alpha(y)$. In ordinary notation:

$$\alpha'(x) = \frac{1}{x (\ln \alpha(x) + 1)}.$$

1.164. $y = x + e^x$. **1.165.** $y = (1/2)x + x^3$.

1.166. $y = x + \log_2 x$. **1.167.** $y = x \ln x$.

Let there be given two functions:

$$x = \varphi(t), \quad y = \psi(t), \quad t \in (\alpha, \beta). \tag{6}$$

If the function $x = \varphi(t)$ has an inverse $t = \varphi^{-1}(x)$ on the interval (α, β) then a new function is defined:

$$y(x) = \psi(\varphi^{-1}(x)), \tag{7}$$

which is said to be *reprsented parametrically* by relations (6). Differentiating (7) with respect to x and applying the rule for differentiation of an inverse function (Example 6), we obtain

$$y'_x = \psi'_t \cdot t'_x = \frac{\psi'_t}{\varphi'_t} = \frac{y'_t}{x'_t}. \tag{8}$$

Example 7. Find y'_x if

$$x = \cos^2 t, \quad y = \sin t, \quad t \in (0, \pi/2).$$

Since $\psi'_t = \cos t$, $\varphi'_t = -2 \cos t \sin t$, by formula (8), we find

$$y'_x = \frac{1}{2 \sin t}.$$

In Problems 1.168 to 1.179 find y'_x for the functions represented parametrically:

1.168. $x = 2t$, $y = 3t^2 - 5t$, $t \in (-\infty, +\infty)$.

1.169. $x = t^3 + 2$, $y = 0.5t^2$, $t \in (-\infty, +\infty)$.

1.170. $x = \frac{1}{t+1}$, $y = \left(\frac{t}{t+1}\right)^2$, $t \neq -1$.

1.171. $x = 2^{-t}$, $y = 2^{2t}$, $t \in (-\infty, +\infty)$.

1.172. $x = a \cos \varphi$, $y = b \sin \varphi$, $\varphi \in (0, \pi)$.

1.173. $x = \tan t$, $y = \sin 2t + 2 \cos 2t$, $t \in (-\pi/2, \pi/2)$.

1.174. $x = \arccos \frac{1}{\sqrt{1+t^2}}$, $y = \arcsin \frac{t}{\sqrt{1+t^2}}$, $t \in (0, +\infty)$.

1.175. $x = \ln (1 + t^2)$, $y = t - \arctan t$, $t \in (0, +\infty)$.

1.176. $x = 3 \log_2 \cot t$, $y = \tan t + \cot t$, $t \in (0, \pi/2)$.

1.177. $x = \arcsin (t^2 - 1)$, $y = \arccos 2t$, $t \in (0, \sqrt{2})$.

1.178. $x = \sqrt[3]{1 - \sqrt{t}}$, $y = \sqrt{1 - \sqrt[3]{t}}$, $t \in (1, +\infty)$.

1.179. $x = a \sinh t$, $y = b \cosh t$, $t \in (0, +\infty)$.

In Problems 1.180 to 1.183 find y'_x at the indicated points.

1.180. $x = t \ln t$, $y = \frac{\ln t}{t}$, $t = 1$.

1.181. $x = t (t \cos t - 2 \sin t)$, $y = t (t \sin t + 2 \cos t)$, $t = \pi/4$.

1.182. $x = e^t \cos t$, $y = e^t \sin t$, $t = \pi/6$.

1.183. $x = \frac{3at}{1+t^2}$, $y = \frac{3at^2}{1+t^2}$, $t = 2$.

3. Derivatives of Higher Orders. The *derivative of the second order* of the function $y = f(x)$ is defined as the derivative of its first derivative, i.e.

$$y''(x) = (y'(x))'.$$

In general, the *derivative of order* n (or *the nth derivative*) is defined as the derivative of the derivative of order $(n - 1)$, i.e.

$$y^{(n)}(x) = (y^{(n-1)}(x))', \quad n = 2, 3, \ldots$$

For the derivative of order n the notation $\frac{d^n y}{dx^n}$ is also used.

Example 8. Find y'' if $y = \ln (x + \sqrt{1 + x^2})$. We have $y' = \frac{1}{\sqrt{1 + x^2}}$. Consequently,

$$y'' = \left(\frac{1}{\sqrt{1+x^2}}\right)' = -\frac{x}{(1+x^2)^{3/2}}.$$

In Problems 1.184 to 1.189 find the derivatives of the second order of the given functions.

1.184. $y = \cos^2 x$. **1.185.** $y = \operatorname{arc\,tan} x^2$.

1.186. $y = \log_2 \sqrt[3]{1-x^2}$. **1.187.** $y = e^{-x^2}$.

1.188. $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$. **1.189*.** $y = x^{\sqrt{x}}$.

1.190. Find $y'(0)$, $y''(0)$, $y'''(0)$ if $y(x) = e^{2x} \sin 3x$.

1.191. Find $y'''(2)$ if $y = \ln(x-1)$.

1.192. Find $y^{IV}(1)$ if $y = x^3 \ln x$.

1.193. Find $y(0)$, $y'(0)$, $y''(0)$ if $y = 2^{\sin x} \cos(\sin x)$.

Let $f(u)$ be a twice differentiable function. Find y' and y'' if (Problems 1.194 and 1.195):

1.194. $y = f\left(\dfrac{1}{x^2}\right)$. **1.195.** $y = \ln f(e^x)$.

Let $u(x)$ and $v(x)$ be twice differentiable functions. Find y' and y'' if (Problems 1.196 to 1.198):

1.196. $y = y^v$ $(u > 0)$.

We have $\ln y = v \ln u$. Hence we find

$$\frac{y'}{y} = v' \ln u + \frac{v'}{u} u',$$

i.e.

$$y' = y\left(v' \ln u + \frac{v}{u} u'\right) = u^v \left(v' \ln u + \frac{v}{u} u'\right),$$

$$y'' = y'\left(v' \ln u + \frac{v}{u} u'\right) + y\left(v'' \ln u + \frac{v'}{u} u' + \frac{v'u'u + vu''u - vu'^2}{u^2}\right)$$

$$u^v \left(\left(v \frac{u'}{u} + u' \ln u\right)^2 + v \frac{uu'' - u'^2}{u^2} + \frac{2u'v'}{u} + v'' \ln u\right).$$

1.197. $y = \sqrt{u^2 + v^3}$. **1.198.** $y = \ln \dfrac{u}{v}$.

In Problems 1.199 to 1.204 find the formula for the nth derivative of the given function.

1.199. $y = x^m$, $m \in \mathbb{N}$. **1.200.** $y = a^{kx}$, $k \in \mathbb{R}$.

1.201*. $y = \sin x$. **1.202.** $y = \ln x$.

1.203*. $y = \cos^2 x$. **1.204.** $y = \dfrac{1+x}{1-x}$.

In Problems 1.205 to 1.207 find the indicated derivatives of the given functions, by expanding them into a linear combination of simpler functions.

1.205. $y=\frac{2x}{x^2-1}$, find $y^{(n)}$.

Let us transform the given expression to the form

$$y=\frac{2x}{x^2-1}=\frac{1}{x+1}+\frac{1}{x-1}.$$

Since

$$\left(\frac{1}{x\pm 1}\right)^{(n)}=(-1)^n\, n!\,\frac{1}{(x\pm 1)^{n+1}},$$

we have

$$y^{(n)}=(-1)^n\, n!\left(\frac{1}{(x+1)^{n+1}}+\frac{1}{(x-1)^{n+1}}\right).$$

1.206. $y=\frac{1}{x^2-3x+2}$, find $y^{(50)}$.

1.207*. $y=\frac{1+x}{\sqrt{1-x}}$, find $y^{(20)}$.

Let $u(x)$ and $v(x)$ have derivatives up to order n inclusively. Then for the nth derivative of their product $u(x)\,v(x)$ *Leibniz' formula* is valid:

$$(uv)^{(n)}=u^{(n)}v+nu^{(n-1)}v'+\frac{n(n-1)}{1\cdot 2}u^{(n-2)}v''+\ldots+uv^{(n)}$$

$$=\sum_{k=0}^{n}C_n^k u^{(n-k)}v^{(k)},$$

where

$u^{(0)}=u$, $v^{(0)}=v$ and $C_n^k=\frac{n(n-1)\ldots(n-k+1)}{1\cdot 2\cdot\ldots\cdot k}=\frac{n!}{k!\,(n-k)!}$

are binomial coefficients.

In Problems 1.208 to 1.212 find the derivative of the indicated order of the given function, applying Leibniz' formula.

1.208. $y=(x^2+x+1)\sin x$, find $y^{(15)}$.

1.209. $y=(x^2-x)\,e^x$, find $y^{(20)}$.

1.210. $y=\sin x\cdot e^{-x}$, find $y^{(5)}$.

1.211. $y=x\log_2 x$, find $y^{(10)}$.

1.212. $y = x \sinh x$, find $y^{(100)}$.

1.213*. Show that

$$(e^{ax} \cos bx)^{(n)} = r^n e^{ax} \cos (bx + n\varphi),$$

where $r = \sqrt{a^2 + b^2}$, $\tan \varphi = \frac{b}{a}$.

1.214. Prove that $(x^{n-1} e^{1/x})^{(n)} = (-1)^n \frac{1/x}{x^{n+1}}$.

1.215. Compute the nth derivative of the function $y = \frac{3x+2}{x^2-2x+5}$ at the point $x = 0$.

By hypothesis, we have

$$y(x)(x^2 - 2x + 5) = 3x + 2.$$

Let us differentiate this identity n times, using Leibniz' formula. Then (for $n \geqslant 2$) we obtain

$$y^{(n)}(x)(x^2-2x+5) + ny^{(n-1)}(2x-2) + \frac{n(n-1)}{2} y^{(n-2)}(x) \cdot 2 = 0,$$

whence for $x = 0$

$$5y^{(n)}(0) - 2ny^{(n-1)}(0) + n(n-1) y^{(n-2)}(0) = 0,$$

or

$$y^{(n)}(0) = \frac{2}{5} ny^{(n-1)}(0) - \frac{n(n-1)}{5} y^{(n-2)}(0).$$

We have obtained a recurrence formula for determining the nth derivative at the point $x = 0$ ($n \geqslant 2$); $y(0)$ and $y'(0)$ are found directly:

$$y(0) = \frac{2}{5}, \quad y'(0) = \frac{-3x^2-4x+19}{(x^2-2x+5)^2}\Big|_{x=0} = \frac{19}{25}.$$

Then, setting in turn $n = 2, 3, 4, \ldots$, with the aid of the recurrence formula, we compute the derivatives of higher orders.

For instance,

$$y''(0) = \frac{2}{5} \cdot 2 \cdot \frac{19}{25} - \frac{2 \cdot 1}{5} \cdot \frac{2}{5} = \frac{56}{125},$$

$$y'''(0) = \frac{2}{5} \cdot 3 \cdot \frac{56}{125} - \frac{3 \cdot 2}{5} \cdot \frac{19}{25} = -\frac{235}{625}.$$

In Problems 1.216 and 1.217, applying the method described in Problem 1.215, find the derivative of the fourth order of the given function at the point $x = 0$.

1.216. $y=\frac{ax+b}{cx+d}$, $c\neq 0$. **1.217.** $y=\frac{x^2+x+1}{x^2-x+1}$.

1.218. Show that the function $y = \arcsin x$ satisfies the differential equation $(1-x^2)\,y''=xy'$.

1.219. Show that the function $y=C_1e^{-2x}+C_2e^{2x}+e^x$ satisfies the differential equation $y''-4y'+4y=e^x$.

1.220. Show that the function $y=e^{-x}\cos x$ satisfies the differential equation $y^{(IV)}+4y=0$.

1.221. Show that the function $y=x^n(\cos(\ln x)+\sin(\ln x))$ satisfies the differential equation $x^2y''+(1-2n)xy'+(1+n^2)y=0$.

In Problems 1.222 to 1.226 find the second derivative of the function represented implicitly.

1.222. $\sqrt{x^2+y^2}=ae^{\operatorname{arc\,tan}\frac{y}{x}}$, $a>0$.

Differentiating the equation defining the function $y(x)$, we obtain

$$\frac{x+yy'}{\sqrt{x^2+y^2}}=ae^{\operatorname{arc\,tan}\frac{y}{x}}\cdot\frac{y'x-y}{x^2+y^2}=\frac{y'x-y}{\sqrt{x^2+y^2}}.$$

Hence,

$$x+yy'=xy'-y \tag{9}$$

and, consequently,

$$y'=\frac{x+y}{x-y}. \tag{10}$$

Differentiating (9) and making use of the expression (10), found for y', we get

$$y''=-\frac{2(x^2+y^2)}{(x-y)^3}.$$

1.223. $y^2=2px$. **1.224.** $y=1+xe^y$.

1.225. $y=\tan(x+y)$. **1.226.** $e^{x-y}=xy$.

1.227. Deduce the formula for the second derivative of the inverse of the given function $y=f(x)$.

1.228. Prove that if $(a+bx)\,e^{y/x}=x$, then $x^3y''=(xy'-y)^2$.

In Problems 1.229 to 1.233 find the second derivative of the function represented parametrically.

1.229. $x=\ln t$, $y=t^2$, $t\in(0,+\infty)$.

We have

$$y'_x=\frac{y'_t}{x'_t}=3t^3$$

and

$$y''_{xx} = (y'_x)'_x = (y'_x)'_t \cdot t'_x = \frac{(y'_x)'_t}{x'_t} = \frac{9t^2}{1/t} = 9t^3.$$

Note that in the present case the parameter t is readily eliminated from the given equations by putting $t = e^x$. Consequently, the expression for y''_{xx} as a function of x has the form $y''_{xx} = 9e^{3x}$.

In the general case, if $x = \varphi(t)$, $y = \psi(t)$, then y''_{xx} is computed by the formula

$$y''_{xx} = \frac{\psi''(t)\varphi'(t) - \varphi''(t)\psi'(t)}{(\varphi'(t))^3} = \frac{\begin{vmatrix} \varphi'(t) & \psi'(t) \\ \varphi''(t) & \psi''(t) \end{vmatrix}}{(\varphi'(t))^3}.$$

1.230. $x = \sec t$, $y = \tan t$, $t \in (0, \pi/2)$.

1.231. $x = \arcsin t$, $y = \ln(1 - t^2)$, $t \in (-1, 1)$.

1.232. $x = \arctan t$, $y = \ln(1 + t^2)$, $t \in (-\infty, +\infty)$.

1.233. $x = a\cos^3 t$, $y = a\sin^3 t$, $t \in (0, \pi/2)$.

1.234. Show that the function $y(x)$ represented parametrically by the equations $x = \sin t$, $y = ae^{t\sqrt{2}} + be^{t\sqrt{2}}$, $t \in (-\pi/2, \pi/2)$ satisfies the differential equation $(1 - x^2) \times y''_{xx} - xy'_x = 2y$ for any constants a and b.

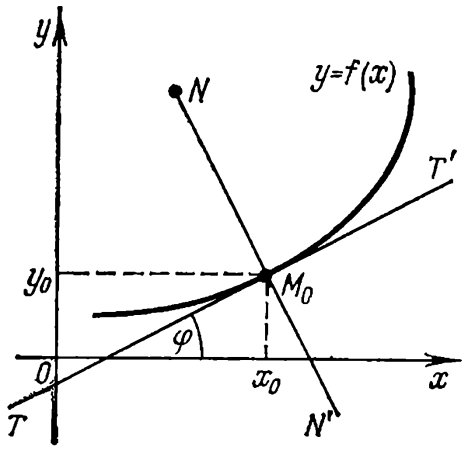

Fig. 46.

4. Geometrical and Mechanical Applications of the Derivative. The magnitude of the derivative $f'(x_0)$ of the function $y = f(x)$ at the point x_0 is equal to the slope $k = \tan\varphi$ of the tangent line TT' to the graph of this function drawn through the point $M_0(x_0, y_0)$, where $y_0 = f(x_0)$ (see Fig. 46) (*geometrical meaning of the derivative*).

The equation of the *tangent line* TT' to the graph of the function $y = f(x)$ at its point $M_0(x_0, y_0)$ has the form

$$y - y_0 = f'(x_0)(x - x_0).$$

The straight line NN' passing through the point of tangency M_0 perpendicular to the tangent (line) is called the *normal* to the graph of the function $y = f(x)$ at this point. The equation of the normal is

$$(x - x_0) + f'(x_0)(y - y_0) = 0.$$

In Problems 1.235 to 1.240 write the equations of the tangent and the normal to the graph of the given function $y = f(x)$ at the indicated point if:

1.235. $y = x^2 - 5x + 4$, $x_0 = -1$.

1.236. $y = x^3 + 2x^2 - 4x - 3$, $x_0 = -2$.

1.237. $y = \sqrt{x}$, $x_0 = 4$.

1.238. $y = \tan 2x$, $x_0 = 0$.

1.239. $y = \ln x$, $x_0 = 1$.

1.240. $y = e^{1-x^2}$, $x_0 = -1$.

1.241. Write the equations of the tangent and the normal to the curve $x = \frac{1+t}{t^3}$, $y = \frac{3}{2t^2} + \frac{1}{2t}$, $t \neq 0$, at its point M_0 (2, 2).

1.242. Write the equations of the tangent lines to the curve

$$x = t \cos t, \quad y = t \sin t, \quad t \in (-\infty, +\infty)$$

at the origin and at the point $t = \pi/4$.

1.243. Write the equations of the tangent and the normal to the curve $x^3 + y^2 + 2x - 6 = 0$ at the point with ordinate $y_0 = 3$.

1.244. Write the equation of the tangent line to the curve $x^5 + y^5 - 2xy = 0$ at the point M_0 (1, 1).

1.245. At what angle does the graph of the function $y = e^{x/2}$ intersect the straight line $x = 2$?

1.246. At what point M_0 of the curve $y^2 = 2x^3$ is the tangent line perpendicular to the straight line $4x - 3y + 2 = 0$?

1.247. Find the coefficients b and c in the equation of the parabola $y = x^2 + bx + c$ which touches the straight line $y = x$ at the point M_0 (1, 1).

1.248. Show that the tangents to the hyperbola $y = \frac{x-4}{x-2}$ at the points of its intersection with the coordinate axes are parallel to each other.

1.249. Write the equation of the normal to the graph of the function $y = -\sqrt{x} + 2$ at the point of intersection with the bisector of the first quadrant.

1.250. Write the equation of such a normal to the parabola $y = x^2 - 6x + 6$ which is perpendicular to the straight

line connecting the origin with the vertex of the parabola.

1.251. At the points of intersection of the straight line $x - y + 1 = 0$ and the parabola $y = x^2 - 4x + 5$ normals are drawn to the parabola. Find the area of the triangle formed by the normals and the chord subtending the indicated points of intersection.

1.252. Show that the normals to the involute of a circle $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ are tangent lines to the circle $x^2 + y^2 = a^2$.

The *angle* ω *between two curves* $y = f_1(x)$ and $y = f_2(x)$ at their common point $M_0(x_0, y_0)$ is defined as the angle between the tangent lines to these curves at the point M_0.

1.253. Prove that

$$\tan \omega = \frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) f_2'(x_0)}.$$

In Problems 1.254 to 1.257 find the angles at which the given curves intersect.

1.254 $y = x^2$ and $y = x^3$.

1.255. $y = (x - 2)^2$ and $y = 4x - x^2 + 4$.

1.256. $y = \sin x$ and $y = \cos x$, $x \in [0, 2\pi]$.

1.257. $x^2 + y^2 = 8ax$ and $y^2 = \frac{x}{2a - x}$.

1.258. Prove that the sum of the segments intercepted by the tangent line to the curve $x^{1/2} + y^{1/2} = a^{1/2}$ on the coordinate axes is equal to a for all of its points.

1.259. Show that the segment of the tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ contained between the coordinate axes has a constant length equal to a.

1.260. Find the distance from the origin to the normal to the line $y = e^{2x} + x^2$ drawn at the point with abscissa $x = 0$.

1.261. Prove that the segment of the tangent line to the tractrix

$$y \quad \frac{a}{2} \ln \frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 + x^2}} - \sqrt{a^2 - x^2},$$

contained between the axis of ordinates and the point of tangency has a constant length.

If a curve is given in polar coordinates by the equation $r = r(\varphi)$, then the angle θ formed by the tangent line TT' and the radius vector $\overline{OM}$ of the point of tangency M (Fig. 47) is determined by the relation

$$\tan\theta = \frac{r(\varphi)}{r'_\varphi}. \tag{12}$$

1.262.** Derive formula (12).

1.263. Find the angle θ between the tangent line and the radius vector of the point of tangency for the logarithmic spiral $r = ae^{k\varphi}$.

1.264. Find the angle θ between the tangent line and the radius vector of the point of tangency for the lemniscate $r^2 = a^2 \cos 2\varphi$.

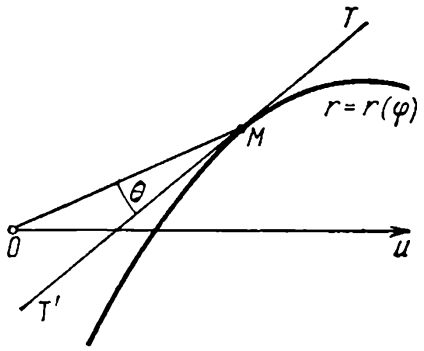

Fig. 47.

If $x = x(t)$ is a function describing the law of motion of a material point, then the first derivative $\frac{dx}{dt} = \dot{x}$ is the velocity, and the second derivative $\frac{d^2x}{dt^2} = \ddot{x}$ is the acceleration of this point at instant t (*mechanical interpretation of the first and second derivatives*).

1.265. The law of motion of a material point in a straight line has the form $x = (1/4)t^4 - 4t^3 + 16t^2$.

(a) At what instants of time is the point found at the origin?

(b) At what instants of time does the direction of its motion coincide with the positive direction of the x-axis?

(c) At what instants of time does its acceleration equal zero?

1.266. Find the velocity of a harmonic oscillation with amplitude a, frequency ω, and initial phase $\varphi = 0$.

1.267. A body of mass 4 performs a rectilinear motion according to the law $x = t^2 + t + 1$. Determine the kinetic energy of the body at the moment of time $t = 5$.

1.268. At what instant $t \in [0, 2\pi]$ the forces must cease to act so that a point performing harmonic oscillation $x = \cos 3t$ will continue to move uniformly with velocity $v = 3/2$.

1.269. A point moves along a logarithmic spiral $r = e^{a\varphi}$. Find the rate of change of the polar radius if it is known that it rotates with a constant angular velocity ω.

1.270. A point moves in a circle $r = 2a \cos \varphi$. Find the rates of change of the abscissa and ordinate of the point if the polar radius rotates with angular velocity ω.

1.271. At what point of the ellipse $16x^2 + 9y^2 = 400$ does the ordinate decrease at the same rate with which the abscissa increases?

1.272. The radius of a sphere changes at a rate v. At what rate do the volume and surface of the sphere change?

1.273. A wheel revolves so that the angle of rotation is proportional to the square of time. The first revolution was completed by the wheel during the time $T = 8$ s. Find the angular velocity ω at the instant of time $t = 32$ s after the beginning of motion.

SEC. 5.2.
THE DIFFERENTIAL

1. Differential of First Order. A function $y = f(x)$ is called *differentiable* at a point x_0 if its increment $\Delta y\ (x_0,\ \Delta x)$ can be represented in the form

$$\Delta y\ (x_0,\ \Delta x) = A\,\Delta x + o\ (\Delta x). \tag{1}$$

The principal linear part $A\,\Delta x$ of the increment Δy is called the *differential* of this function at the point x_0, corresponding to the increment Δx, and is denoted by the symbol $dy\ (x_0,\ \Delta x)$.

For a function $y = f(x)$ to be differentiable at a point x_0, it is necessary and sufficient that the derivative $f'(x_0)$ exists; the equality $A = f'(x_0)$ being valid.

This assertion allows any function having a derivative to be called differentiable. It is in this sense that we have used this expression in the preceding section.

The expression for the differential has the form

$$dy\ (x_0,\ dx) = f'(x_0)\,dx,$$

where $dx = \Delta x$. It follows from formula (1) that if $f'(x_0) \neq 0$, then, as $\Delta x \to 0$, the increment of the function and its differential dy at a fixed point are equivalent infinitesimals which allows us to write the following approximate equality:

$$\Delta y \approx dy \text{ for } |\Delta x| \ll 1. \tag{2}$$

Example 1. Find approximately the volume V of the sphere of radius $r = 1.02$ m.

Since $V(r) = \frac{4}{3}\pi r^3$, setting $r_0 = 1$, $\Delta r = 0.02$ and using formula (2), we obtain:

$$V(1.02) = V(1) + \Delta V(1, 0.02) \approx V(1) + V'(1) \cdot 0.02$$
$$= \frac{4}{3}\pi + 4\pi \cdot 0.02 \approx 4.43 \text{ m}^3.$$

Geometrical meaning of the differential. The differential dy $(x_0, \Delta x)$ is equal to the increment of the ordinate of the tangent line TT' to the graph of the function $y = f(x)$ at the point $M_0(x_0, y_0)$ with the increment of the argument equal to Δx (Fig. 48).

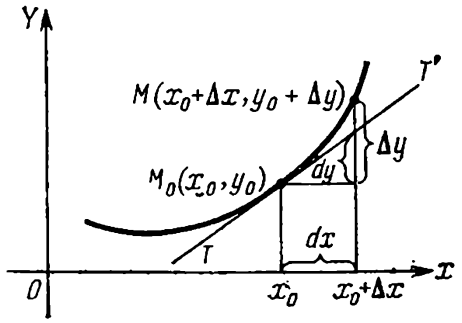

Fig. 48.

2.1. Prove that for the linear function $y = ax + b$ the increment Δy and differential dy coincide.

2.2. Find the increment Δy and differential dy of the function $y = x^3$, corresponding to the value of the argument $x_0 = 2$ and to two distinct increments of the argument: $(\Delta x)_1 = 0.1$ and $(\Delta x)_2 = 0.01$.

2.3. Find the increment ΔS and differential dS of the area S of a square corresponding to the increment Δx of its side x. With the aid of a drawing, give a geometrical interpretation of ΔS, dS and of the difference $\Delta S - dS$.

2.4. A material point M performs rectilinear motion according to the law $s = f(t)$, where t is an instant of time and s the path covered during the time interval from 0 to t. Give a mechanical interpretation of the differential of path ds corresponding to the time interval $\Delta t = t_2 - t_1$.

2.5. Using the result derived in the preceding problem and formula (2), find approximately the path Δs covered by the point M during the time interval from $t_1 = 3$ to $t_2 = 4$ if the law of motion of the point M is given by the formula $s = 1 + \arctan t$. Compare the answer with the exact value of Δs.

2.6. For the functions (a) $f(x) = x^n$ and (b) $\varphi(x) = \sin x$ find the values of the argument x for which the differentials of these functions are not equivalent to their increments as $\Delta x \to 0$.

2.7. Given: the interval $[x_0, x_0 + \Delta x]$ of variation of the argument x of the function $y = f(x)$; the corresponding increment and differential of this function: Δy and dy, respectively. Are the following equalities possible: (a) $dy = \frac{3}{2}\Delta y$, (b) $dy = \Delta y$, (c) $dy = \frac{1}{2}\Delta y$?

2.8. The edges of a cube are extended by 1 cm. In this case the differential dV of the volume V of the cube turns out to be equal to 12 cm^3. Find the original length of the edges.

2.9. The radius of a circle is extended by 1 cm. In this case the differential of the area of the circle turns out to be equal to 6π cm^2. Find the original length of the radius.

In Problems 2.10 to 2.17 find the differentials of the indicated functions for arbitrary values of the argument x and its arbitrary increment $\Delta x = dx$.

2.10. $x\sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a} - 5$.

2.11. $\sin x - x \cos x + 4$.

2.12. $x \arctan x - \ln \sqrt{1 + x^2}$. **2.13.** $x \ln x - x + 1$.

2.14. $x \arcsin x + \sqrt{1 - x^2} - 3$. **2.15.** $y^5 + y - x^2 - 1$.

2.16. $\arctan \frac{y}{x} - \ln \sqrt{x^2 + y^2}$. **2.17.** $e^y = x + y$.

In Problems 2.18 to 2.22 carry out the indicated approximate computations by replacing the increment Δy of a suitable function $y = f(x)$ by the differential dy of this function for a small absolute value of the increment Δx of the argument x.

2.18. Compute approximately:

(a) arc sin 0.05;

(b) arc tan 1.04;

(c) ln 1.2.

2.19. Justify the approximate formula

$$\sqrt[3]{x + \Delta x} \approx \sqrt[3]{x} + \frac{\Delta x}{3\sqrt[3]{x^2}}$$

and compute $\sqrt[3]{25}$ using this formula.

2.20. Find the approximate value of the function $f(x) = e^{x^2 - x}$ for $x = 1.2$.

2.21*. Find the approximate expression for the increment ΔV of the volume V of a right circular cylinder with altitude h for a change in the radius r of the base by the quantity Δr.

2.22*. According to the Clapeyron equation of state for gases, the volume V occupied by a gas, its pressure p, and the absolute temperature T are related by the equation $pV = RT$, where R is the gas constant. Find the approximate expression for the increment ΔV of the volume V with p changing by a quantity Δp and the temperature T unchanged.

2. Differentials of Higher Orders. Consider the differential $dy\ (x, \Delta_1 x) = f'(x)\ \Delta_1 x$ as a function x with a fixed $\Delta x = \Delta_1 x$. Assuming that the function $y = f(x)$ is twice differentiable at the point x, we find the differential of $dy\ (x, \Delta_1 x)$ for $\Delta x = \Delta_2 x$:

$$d\ (dy\ (x,\ \Delta_1 x))\ |_{x,\Delta x=\Delta_2 x} = f''(x)\ \Delta_1 x \Delta_2 x.$$

The value of the obtained expression for $\Delta_1 x = \Delta_2 x = dx$ is known as the *second differential* or the *differential of the second order* of the function $y = f(x)$ and is denoted by the symbol $d^2y\ (x, dx)$.

Hence,

$$d^2y = f''(x)\ dx^2.$$

Analogously,

$$d^3y = d\ (d^2y) = f'''(x)\ dx^3,$$

. . . . . .

$$d^ny = d\ (d^{n-1}y) = f^{(n)}(x)\ dx^n.$$

In Problems 2.23 to 2.30 find the second differentials of the indicated functions y of the argument x.

2.23. $y = a \sin (bx + c)$. **2.24.** $y = 3^{-x}$.

2.25. $y = \dfrac{\sin x}{x}$. **2.26.** $y = ax^2 + bx + c$.

2.27. $xy + y^2 = 1$. **2.28.** $(x - a)^2 + (y - b)^2 = R^2$.

2.29. $x^3 + y^3 = y$. **2.30.** $x = y - a \sin y$.

SEC. 5.3.
THEOREMS ON DIFFERENTIABLE FUNCTIONS. TAYLOR'S FORMULA

1. Mean-value Theorems.

Rolle's theorem. *If a function $f(x)$ is continuous on the closed interval $[a, b]$ differentiable for $x \in (a, b)$ and $f(a) = f(b)$, then there exists at least one point $\zeta \in (a, b)$ such that $f'(\zeta) = 0$.*

Points at which $f'(x) = 0$ are called the *stationary points* of the function $f(x)$.

Lagrange's theorem. *If a function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable for $x \in (a, b)$, then there exists at least one point $\xi \in (a, b)$ such that*

$$f(b) - f(a) = f'(\xi) \cdot (b - a) \quad (\textit{Lagrange's formula}).$$

Cauchy's theorem. *If two functions $f(x)$ and $g(x)$ are continuous on the closed interval $[a, b]$, differentiable for $x \in (a, b)$, and $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists at least one point $\xi \in (a, b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)} \quad (\textit{Cauchy's formula}).$$

3.1. The function $f(x) = \frac{5-x^2}{x^4}$ has equal values at the end points of the closed interval $[-1, 1]$ (check this!). Its derivative $f'(x)$ is equal to zero only at two points $x = \pm\sqrt{10}$ (check!) situated beyond the limits of this interval. Why is the conclusion of Rolle's theorem violated in this case?

3.2. Show that the function $f(x) = x^2 - 1$ satisfies the conditions of Rolle's theorem on the closed interval $[-1, 1]$. Find all the stationary points of this function.

3.3. Let $f(x) = x(x-1)(x-2)(x-3)$. Prove that all the three roots of the equation $f'(x) = 0$ are real.

3.4*. Prove that the equation $16x^4 - 64x + 31 = 0$ cannot have two distinct real roots on the interval $(0, 1)$.

3.5*. Prove that the equation $e^{x-1} + x - 2 = 0$, having the root $x = 1$ (check this!), has no other real roots.

3.6*. Prove that if a function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the interval (a, b), the function $F(x) = (f(x) - f(a))(b-a) - (f(b) - f(a))(x-a)$ has at least one stationary point on the interval (a, b).

3.7. Write Lagrange's formula for the function $f(x) = \sqrt{3x^3 + 3x}$ on the closed interval $[0, 1]$, and find the corresponding value of ξ on the open interval $(0, 1)$.

3.8. Prove that if the derivative $f'(x)$ is identically equal to zero on the interval (a, b), then the function $f(x)$ is constant on this interval.

3.9. Prove that if $f'(x) > 0$ ($f'(x) < 0$) on the interval (a, b) then the function $f(x)$ increases (decreases) monotonically on this interval.

A function $f(x)$ satisfies a *Lipschitz condition* on the interval (a, b) if there exists a K such that

$$|f(x_2) - f(x_1)| \leqslant K \cdot |x_2 - x_1|$$

for any $x_1, x_2 \in (a, b)$.

3.10. Prove that if $\sup\limits_{a<x<b} f'(x) = M$, then the function $f(x)$ on the interval (a, b) satisfies a Lipschitz condition with constant K equal to M.

3.11*. Let $f(x)$ and $\varphi(x)$ be twice differentiable functions on the interval (a, b). Prove that if $f''(x) = \varphi''(x)$ on (a, b), then $f(x)$ and $\varphi(x)$ differ by a linear term.

3.12. Prove that if a function $f(x)$ satisfies the conditions of Lagrange's theorem on $[a, b]$ then $|f(b) - f(a)| \geqslant m \cdot (b - a)$, where $m = \inf\limits_{a \leqslant x \leqslant b} f'(x)$.

3.13. Write Cauchy's formula for the functions $f(x) = 2x^3 + 5x + 1$ and $g(x) = x^2 + 4$ on the interval $[0, 2]$ and find the values of ξ.

2. L'Hospital-Bernoulli Rule. *Evaluation of indeterminate forms* $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$ Let, as $x \to a$, the functions $f(x)$ and $\varphi(x)$ be both infinitely small or infinitely large. Then their ratio is not defined at the point $x = a$, and it is said to represent an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, respectively. But this ratio may have a limit at the point x a, finite or infinite. Finding this limit is known as evaluation of an indeterminate form. The L'Hospital-Bernoulli rule is one of the methods for evaluation of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. It is based on the following L'Hospital-Bernoulli theorem.

Theorem. *Let in some neighbourhood U of the point x — a the functions $f(x)$ and $\varphi(x)$ be differentiable everywhere except possibly at x — a, and let $\varphi'(x) \neq 0$ in U. If the functions $f(x)$ and $\varphi(x)$ are both either infinitesimals, or infinitely large as $x \to a$ and $\frac{f'(x)}{\varphi'(x)}$ approaches a limit as x approaches a, then $\frac{f(x)}{\varphi(x)}$ approaches the same limit, i.e.*

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \lim_{x \to a} \frac{f'(x)}{\varphi'(x)}. \tag{1}$$

The rule is also applicable when $a = \infty$.

Example 1. Find $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\operatorname{arc\,tan} 5x}$ (i.e. evaluate the indeterminate form $\dfrac{0}{0}$).

Using formula (1), we obtain

$$\lim_{x\to 0}\frac{e^{2x}-1}{\operatorname{arc\,tan} 5x}=\lim_{x\to 0}\frac{2e^{2x}}{\dfrac{1}{1+25x^2}\cdot 5}=\frac{2}{5},$$

since $e^{2x}\to 1$ and $\dfrac{1}{1+25x^2}\to 1$ as $x\to 0$.

In some cases evaluation of the indeterminate forms $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ may require repeated application of the L'Hospital-Bernoulli rule.

Example 2. Find $\lim\limits_{x\to+\infty}\dfrac{\ln^2 x}{x^3}$ (i.e. evaluate the indeterminate form $\dfrac{\infty}{\infty}$).

Applying formula (1) twice, we obtain:

$$\lim_{x\to+\infty}\frac{\ln^2 x}{x^3}=\lim_{x\to+\infty}\frac{\dfrac{2\ln x}{x}}{3x^2}=\frac{2}{3}\lim_{x\to+\infty}\frac{\ln x}{x^3}=\frac{2}{3}\lim_{x\to+\infty}\frac{\dfrac{1}{x}}{3x^2}=0.$$

At each step of applying the L'Hospital-Bernoulli rule it is advisable to use various identical transformations to simplify the ratio, and also to combine this rule with any other methods for evaluation of limits.

Example 3. Find $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$ (i.e. evaluate the indeterminate form $\dfrac{0}{0}$).

Using formula (1), we obtain

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\dfrac{1}{\cos^2 x}-\cos x}{3x^2}=\frac{1}{3}\lim_{x\to 0}\frac{1-\cos^3 x}{x^2\cos^2 x}$$

We then eliminate the factor $\cos^2 x$ from the denominator, since it has the limit 1 as $x\to 0$. Expanding the difference of cubes standing in the numerator, we eliminate then the factor $(1+\cos x+\cos^2 x)$ having the limit 3 as $x\to 0$. After these simplifications we get

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{x^2}.$$

Applying (1) once again, we obtain:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}.$$

Using the first remarkable limit, we obtain the final answer 1/2 without resorting to the L'Hospital-Bernoulli rule once again.

In Problems 3.14 to 3.33 evaluate the given indeterminate forms $\left(\frac{0}{0}\text{ or }\frac{\infty}{\infty}\right)$.

3.14. $\lim\limits_{x\to 0}\dfrac{\ln\cos 2x}{\sin 2x}$. **3.15.** $\lim\limits_{x\to 0}\dfrac{x-\operatorname{arc}\tan x}{x^3}$.

3.16. $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}$, $m\neq n$, $a\neq 0$.

3.17. $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{c^x-d^x}$, $a\neq b$, $c\neq d$.

3.18. $\lim\limits_{x\to 0}\dfrac{\ln\sin ax}{\ln\sin bx}$. **3.19.** $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\operatorname{arc}\sin 3x}$.

3.20. $\lim\limits_{x\to 5}\dfrac{\sqrt[3]{x}-\sqrt[3]{5}}{\sqrt{x}-\sqrt{5}}$. **3.21.** $\lim\limits_{x\to 0}\dfrac{\ln\cos ax}{\ln\cos bx}$.

3.22. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$. **3.23.** $\lim\limits_{x\to+\infty}\dfrac{\pi-2\operatorname{arc}\tan x}{e^{3/x}-1}$.

3.24. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x-\tan x}$. **3.25.** $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\ln(1+x)}$.

3.26. $\lim\limits_{x\to 0}\dfrac{e^{3x}-3x-1}{\sin^2 5x}$.

3.27. $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\cot x-1}{\sin 4x}$. **3.28.** $\lim\limits_{x\to 1}\dfrac{x^3-4x^2+5x-2}{x^3-5x^2+7x-3}$.

3.29. $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x^m}$, $m>0$. **3.30.** $\lim\limits_{x\to 0}\dfrac{\ln x}{1+2\ln\sin x}$.

3.31. $\lim\limits_{x\to 1}\dfrac{\tan\frac{\pi x}{2}}{\ln(1-x)}$.

3.32. $\lim\limits_{x\to 3}\dfrac{\cos x\cdot\ln(x-3)}{\ln(e^x-e^3)}$.

3.33. $\lim\limits_{x\to 1}\dfrac{\ln(1-x)+\tan\frac{\pi x}{2}}{\cot\pi x}$.

Evaluation of the indeterminate forms $0\cdot\infty$ and $\infty-\infty$. To evaluate $\lim\limits_{x\to a} f(x)\,\varphi(x)$, where $f(x)$ is an infinitesimal, and $\varphi(x)$ is an infinite as $x\to a$ (evaluation of the indeterminate form $0\cdot\infty$), it is necessary to transform the product to the form $\dfrac{f(x)}{1/\varphi(x)}$ (the indeterminate form $\dfrac{0}{0}$) or to the form $\dfrac{\varphi(x)}{1/f(x)}$ (the indeterminate form $\dfrac{\infty}{\infty}$) and then apply the L'Hospital-Bernoulli rule.

Example 4. Find $\lim\limits_{x\to 1}\sin(x-1)\cdot\tan\dfrac{\pi x}{2}$ (i.e. evaluate the indeterminate form $0\cdot\infty$).

We have: $\lim\limits_{x\to 1}\sin(x-1)\cdot\tan\dfrac{\pi x}{x}=\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{\cot\frac{\pi x}{2}}=\lim\limits_{x\to 1}\times$

$$\times\frac{\cos(x-1)}{-\frac{\pi}{2}\frac{1}{\sin^2\frac{\pi x}{2}}}=-\frac{2}{\pi}\lim_{x\to 1}\cos(x-1)\sin^2\frac{\pi x}{2}=-\frac{2}{\pi}.$$

To evaluate $\lim\limits_{x\to a}(f(x)-\varphi(x))$, where $f(x)$ and $\varphi(x)$ are infinitely large quantities as $x\to a$ (evaluation of the indeterminate form $\infty-\infty$), it is necessary to transform the difference to the form $f(x)\left(1-\dfrac{\varphi(x)}{f(x)}\right)$ and then evaluate the indeterminate form $\dfrac{\varphi(x)}{f(x)}$ $\left(\text{i.e. }\dfrac{\infty}{\infty}\right)$. If $\lim\limits_{x\to a}\dfrac{\varphi(x)}{f(x)}\neq 1$, then $\lim\limits_{x\to a}(f(x)-\varphi(x))=\infty$. And if $\lim\limits_{x\to a}\dfrac{\varphi(x)}{f(x)}=1$, we obtain the above considered indeterminate form $\infty\cdot 0$.

Example 5. Find $\lim\limits_{x\to+\infty}(x-\ln^3 x)$ (i.e. evaluate the indeterminate form $\infty-\infty$).

We have

$$\lim_{x\to+\infty}(x-\ln^3 x)=\lim_{x\to+\infty}x\left(1-\frac{\ln^3 x}{x}\right).$$

Since

$$\lim_{x\to+\infty}\frac{\ln^3 x}{x}=\lim_{x\to+\infty}\frac{3\ln^2 x\cdot\frac{1}{x}}{1}=3\lim_{x\to+\infty}\frac{\ln^2 x}{x}$$

$$=3\lim_{x\to+\infty}\frac{2\ln x\cdot\frac{1}{x}}{1}=6\lim_{x\to+\infty}\frac{\ln x}{x}=6\lim_{x\to+\infty}\frac{\frac{1}{x}}{1}=6\lim_{x\to+\infty}\frac{1}{x}=0,$$

we have

$$\lim_{x\to+\infty}(x-\ln^3 x)=+\infty.$$

In Problems 3.34 to 3.48 evaluate the indicated indeterminate forms $0\cdot\infty$ or $\infty-\infty$.

3.34. $\lim\limits_{x\to\infty} x\,(e^{1/x}-1)$. **3.35.** $\lim\limits_{x\to 0}\left(\cot x-\frac{1}{x}\right)$.

3.36. $\lim\limits_{x\to\infty} x^n e^{-x}$. **3.37.** $\lim\limits_{x\to 0} x\ln^3 x$.

3.38. $\lim\limits_{x\to\pi}(\pi-x)\tan\frac{x}{2}$. **3.39.** $\lim\limits_{x\to 0}(e^x-e^{-x}-2)\cot x$.

3.40. $\lim\limits_{x\to 0} x^2 e^{1/x^2}$. **3.41.** $\lim\limits_{x\to 1}(x-1)\cot\pi(x-1)$.

3.42. $\lim\limits_{x\to\infty} x\sin\frac{a}{x}$.

3.43. $\lim\limits_{x\to 1}\ln x\cdot\ln(x-1)$. **3.44.** $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$.

3.45. $\lim\limits_{x\to 0}\left(\frac{1}{\operatorname{arc\,tan} x}-\frac{1}{x}\right)$.

3.46. $\lim\limits_{x\to 1}\left(\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}\right)$.

3.47. $\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{x}{\cot x}-\frac{\pi}{2\cos x}\right)$. **3.48.** $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$.

Evaluation of the forms 0^0, ∞^0, 1^∞. In all the three cases we have to evaluate the limit of the expression $(f(x))^{\varphi(x)}$, where $f(x)$ is an infinitesimal in the first case, an infinite in the second, and a function having the limit equal to unity in the third. As to the function $\varphi(x)$, it is an infinitesimal in the first two cases and an infinite in the third.

We proceed as follows: taking first the logarithm of $y=(f(x))^{\varphi(x)}$, we obtain the equality

$$\ln y=\varphi(x)\ln f(x) \tag{2}$$

and find the limit of $\ln y$, whereupon the limit of y is found. In all the three cases, by virtue of (2), $\ln y$ is the determinate form $0\cdot\infty$ (check this!) which is evaluated by the above discussed method.

Example 6. Find $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^{2x}$ (in other words, evaluate the indeterminate form 1^∞).

Let us put $y=\left(1+\frac{1}{x}\right)^{2x}$. Then $\ln y=2x\ln\left(1+\frac{1}{x}\right)$ is the indeterminate form $\infty\cdot 0$. Transforming the expression $\ln y$ to the

form $\ln y = 2 \dfrac{\ln\left(1+\frac{1}{x}\right)}{1/x}$ and applying the L'Hospital-Bernoulli rule, we find

$$\lim_{x\to\infty} \ln y = 2 \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = 2 \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 2.$$

Consequently,

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{2x} = e^2.$$

In Problems 3.49 to 3.60 evaluate the indeterminate forms 0^0, ∞^0, 1^∞.

3.49. $\lim\limits_{x\to 0} x^{\sin x}$.

3.50. $\lim\limits_{x\to 0} (\arcsin x)^{\tan x}$.

3.51. $\lim\limits_{x\to \frac{\pi}{2}} (\pi - 2x)^{\cos x}$. **3.52.** $\lim\limits_{x\to 0} \dfrac{1}{x^{\ln(e^x-1)}}$

3.53. $\lim\limits_{x\to+\infty} x^{1/x}$. **3.54.** $\lim\limits_{x\to+\infty} (x+2^x)^{1/x}$.

3.55. $\lim\limits_{x\to 0} (\cot x)^{1/\ln x}$. **3.56.** $\lim\limits_{x\to \frac{\pi}{2}} (\tan x)^{2x-\pi}$.

3.57. $\lim\limits_{x\to 1} x^{1/(1-x)}$. **3.58.** $\lim\limits_{x\to+\infty} \left(1+\frac{1}{x^2}\right)^x$.

3.59. $\lim\limits_{x\to 0} (\cos 2x)^{3/x^2}$. **3.60.** $\lim\limits_{x\to 0} (e^x + x)^{1/x}$.

3. Taylor's Formula. If a function $y = f(x)$ has the derivatives up to the $(n+1)$th order (inclusively) in some neighbourhood $U_\delta(a) = \{x \mid |x-a| < \delta\}$ of the point a, then for any $x \in U_\delta(a)$ Taylor's formula (of order n) holds true:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

$$\ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x).$$

where

$$R_{n+1}(x) = \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1},\ 0<\theta<1$$

(the remainder in *Lagrange's form*). Thus, Taylor's formula of order n enables us to represent the function $y = f(x)$ in the form of the sum of a polynomial of the nth degree and a remainder.

In particular, for $a = 0$ we have

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

$$\ldots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1},\quad 0<\theta<1$$

(*Maclaurin's formula*).

3.61. Expand the polynomial $2x^3 - 3x^2 + 5x + 1$ in powers of the binomial $x + 1$.

3.62. Write Taylor's formula of the second order at the point $a = 1$ for the polynomial $x^4 + 4x^2 - x + 3$. Write the remainder in Lagrange's form and find the value of θ corresponding to the following values of the argument: (a) $x = 0$; (b) $x = 1$; (c) $x = 2$.

3.63. Let $P(x)$ be a polynomial of the 4th degree, $P(2) = -1$, $P'(2) = 0$, $P''(2) = 2$, $P'''(2) = -12$, $P^{(IV)}(2) = 24$. Compute $P(-1)$, $P'(0)$, and $P''(1)$.

In Problems 3.64 to 3.69 write Maclaurin's formula of the nth order for the given functions.

3.64. $y = e^x$. **3.65.** $y = \sin x$. **3.66.** $y = \cos x$.

3.67. $y = \ln(1 + x)$. **3.68*.** $y = \arctan x$. **3.69.** $y = (1 + x)^\alpha$.

In Problems 3.70 to 3.74 using Maclaurin's formula obtained in Problems 3.64 to 3.69 write the first n terms of Maclaurin's formula (without the remainder) for the given functions.

3.70*. $y = e^{-x^2/2}$. **3.71*.** $y = \sin^2 x$.

3.72. $y = \sin\frac{5x}{2}$. **3.73*.** $y = \ln(4 + x^2)$.

3.74. $y = \sqrt[3]{8 + x^2}$.

3.75. Write Taylor's formula of the 3rd order for the function $y = \frac{x}{x-1}$ at the point $a = 2$. Construct the

graphs of the given function and its Taylor polynomial of the 3rd degree.

3.76. Write Taylor's formula of the 2nd order for the function $y = \tan x$ at the point $a = 0$. Construct the graphs of the given function and its Taylor polynomial of the 2nd degree.

3.77. Write Taylor's formula of the 3rd order for the function $y = \arcsin x$ at the point $a = 0$. Construct the graph of the given function and its Taylor polynomial of the 3rd degree.

3.78. Write Taylor's formula of the 3rd order for the function $y = \frac{1}{\sqrt{x}}$ at the point $a = 1$. Construct the graphs of the given function and its Taylor polynomial of the 3rd degree.

Taylor's formula is widely used for computing the values of a function with a preassigned degree of accuracy. For instance, it is required to evaluate the function $f(x)$ at the point x_0 with an absolute error not exceeding ε if the values of this function and its derivatives are known at the point a. It follows from Taylor's formula that

$$f(x_0) \approx f(a) + \frac{f'(a)}{1!}(x_0 - a) + \ldots + \frac{f^{(n_0)}(a)}{n_0!}(x_0 - a)^{n_0},$$

where n_0 is the minimal of the numbers n for which

$$|R_{n+1}(x_0)| < \varepsilon.$$

Example 7. Compute the number e with an absolute error not exceeding 0.001.

Applying Maclaurin's formula to the function $f(x) = e^x$, we obtain

$$e = f(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{e^{\theta}}{(n+1)!}, \quad 0 < \theta < 1.$$

The least value of n satisfying the condition $\frac{e^{\theta}}{(n+1)!} < 0.0001$, where $0 < \theta < 1$, is $n_0 = 6$. Consequently,

$$e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{6!} = 2.718.$$

3.79. Compute with an absolute error not exceeding 0.001 the approximate values of the following numbers:

(a) $\sin 1$; (b) $\sqrt{e}$; (c) $\ln 1.05$; (d) $\sqrt[5]{33}$.

3.80. Find out the original of the approximate formulas:

(a) $\sqrt{1+x} \approx 1+\frac{1}{2}x-\frac{1}{8}x^2$, $|x|<1$;

(b) $\sqrt[3]{1+x} \approx 1+\frac{1}{3}x-\frac{1}{9}x^2$, $|x|<1$

and estimate their errors.

The remainder in Taylor's formula can be written in *Peano's form*

$$R_{n+1}(x) = o(|x-a|^n),$$

which is very useful in computation of limits.

Example 8. Find $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{5x^2+7x^3}$.

Since $1-\cos^3 x = (1-\cos x)(1+\cos x+\cos^2 x)$, and $5x^2+7x^3 \sim 5x^2$, we get

$$\lim_{x\to 0}\frac{1-\cos^3 x}{5x^2+7x^3} = \lim_{x\to 0}\frac{3(1-\cos x)}{5x^2}.$$

Replacing $\cos x$ by its expansion by Maclaurin's formula $\cos x = 1-\dfrac{x^2}{2!}+o(x^2)$, we obtain

$$\lim_{x\to 0}\frac{1-\cos^3 x}{5x^2+7x^3} = \frac{3}{5}\lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^2)}{x^2} = \frac{3}{5}\lim_{x\to 0}\frac{x^2/2}{x^2},$$

since $\dfrac{x^2}{2!}+o(x^2)\sim\dfrac{x^2}{2}$ as $x\to 0$. Finally,

$$\lim_{x\to 0}\frac{1-\cos^3 x}{5x^2+7x^3} = \frac{3}{10}.$$

Example 9. Find $\lim\limits_{x\to 1}\dfrac{x-1-\sin(2x-2)}{x-1+\sin(3x-3)}$.

By Taylor's formula, we obtain:

$$\sin(2x-2) = \sin 2(x-1) = \frac{2(x-1)}{1!}+o(|x-1|),$$

$$\sin(3x-3) = \frac{3(x-1)}{1!}+o(|x-1|).$$

Consequently,

$$\lim_{x\to 1}\frac{x-1-\sin(2x-2)}{x-1+\sin(3x-3)} = \lim_{x\to 1}\frac{-(x-1)-o(|x-1|)}{4(x-1)+o(|x-1|)}.$$

Rejecting the infinitesimals of higher orders, i.e. passing both in the numerator and denominator to equivalent infinitesimals as $x \to 1$, we obtain

$$\lim_{x \to 1} \frac{x-1-\sin(2x-2)}{x-1+\sin(3x-3)} = \lim_{x \to 1} \frac{-(x-1)}{4(x-1)} = -\frac{1}{4}.$$

3.81. Show that the expansions by Maclaurin's formula for the functions $\sin x$, $\tan x$, $\arcsin x$, $\arctan x$, $e^x - 1$, and $\ln(1+x)$ can be written in the form $x + o(|x|)$ and, as $x \to 0$, all these functions are equivalent to the infinitesimal $\alpha(x) = x$ (and, consequently, equivalent to one another).

3.82. Using the expansion by Maclaurin's formula, compute the given limit:

(a) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$;

(b) $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2+x^3}$;

(c) $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3+x^4}$.

SEC. 5.4.
INVESTIGATION OF BEHAVIOUR OF FUNCTIONS AND CONSTRUCTION OF GRAPHS

1. Increase and Decrease of a Function. Extremum. A function $y = f(x)$ is called an *increasing* (*decreasing*) function in an interval (a, b) if the inequality $x_1 < x_2$, where $x_1, x_2 \in (a, b)$, implies the inequality $f(x_1) < f(x_2)$ ($f(x_1) > f(x_2)$).

If a function $f(x)$ is differentiable on the interval (a, b) and $f'(x) > 0$ for all $x \in (a, b)$, then the function $f(x)$ is increasing in (a, b); and if $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing in this interval.

In simplest cases the domain of definition of a function $y = f(x)$ can be divided into a finite number of *intervals of monotonicity*. Each of these intervals is bounded by *critical points* at which $f'(x) = 0$ or $f'(x)$ is not existent.

If there exists a neighbourhood $U_\delta(x_0)$ of a point x_0 such that for any point $x \neq x_0$ of this neighbourhood the inequality $f(x) > f(x_0)$ (or $f(x) < f(x_0)$) is fulfilled, then the point x_0 is called a *point of minimum* (*of maximum*) of the function $y = f(x)$, and the number $f(x_0)$ is termed a *minimum* (*maximum*) of this function. The generic term for maximum and minimum is *extremum*.

Necessary condition for an extremum. If x_0 is a point of extremum (or an extreme point) of a function $f(x)$, then $f'(x_0) = 0$ or $f'(x_0)$ does not exist, i.e. x_0 is a critical point of this function.

The converse is, generally speaking, false.

Sufficient conditions for an extremum of a continuous function.

(1) Let a function $f(x)$ be differentiable in some neighbourhood $(x_0 - \delta, x_0 + \delta)$ of a critical point x_0 except possibly at x_0. If the derivative $f'(x)$ has opposite signs in the intervals $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$, then x_0 is a point of extremum. The following cases are possible: (a) if $f'(x) > 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) < 0$ for $x \in (x_0, x_0 + \delta)$, then x_0 is a point of maximum, and (b) if $f'(x) < 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) > 0$ for $x \in (x_0 + \delta, x_0)$, then x_0 is a point of minimum. But if $f'(x)$ for $x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$, retains sign, then x_0 is not a point of extremum.

(2) Let the function $f(x)$ be twice differentiable at a critical point x_0 and in some of its neighbourhood. If $f''(x_0) < 0$, then x_0 is a point of maximum of the function $f(x)$, and if $f''(x_0) > 0$, then x_0 is a point of minimum. But if $f''(x_0) = 0$, then some additional investigations are needed.

Example 1. Find the intervals of monotonicity and points of extremum of the function $f(x) = \dfrac{|x-1|}{x^2}$.

We find the derivative:

$$f'(x) = \begin{cases} \dfrac{x-2}{x^3} & \text{for } x \in (-\infty, 0) \cup (0, 1), \\ \dfrac{2-x}{x^3} & \text{for } x \in (1, +\infty). \end{cases}$$

Equating it to zero, we obtain $x = 2$. Thus, the critical points (taking into account the points where the derivative does not exist) are $x_1 = 0$, $x_2 = 1$, $x_3 = 2$. They divide the domain of definition of $f(x)$ into four intervals of monotonicity: $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, +\infty)$. Since $f'(x) > 0$ for $x \in (-\infty, 0) \cup (1, 2)$ and $f'(x) < 0$ for $x \in (0, 1) \cup (2, +\infty)$, $f(x)$ monotonically increases for $x \in (-\infty, 0) \cup (1, 2)$, monotonically decreases for $x \in (0, 1) \cup (2, +\infty)$, reaches a maximum at the point $x_3 = 2$ $\left(f(2) = \dfrac{1}{4}\right)$, and a minimum at the point $x_2 = 1$ $(f(1) = 0)$. It is convenient to tabulate the obtained results as follows:

Table 4.1

x	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, 2)$	2	$(2, +\infty)$
$f(x)$	↗	$+\infty$	↘	0	↗	$\frac{1}{4}$	↘
$f'(x)$	+	Nonexistent	−	Nonexistent	+	0	−

Note that in the example under consideration the first sufficient condition makes it possible to determine the nature of each of the critical points of the given function. At the same time the second sufficient condition is not applicable at the point x_2 since at this point the first derivative does not exist.

4.1*. Prove the following generalization of the second sufficient condition for an extremum. Let x_0 be a critical point of a function $f(x)$ and let the first of the nonzero derivatives of this function at the point x_0 have the order k. If k is an even number, then x_0 is a point of extremum, viz. it is a point of maximum if $f^{(k)}(x_0) < 0$, and a point of minimum if $f^{(k)}(x_0) > 0$. And if k is odd, then there is no extremum at the point x_0.

4.2. Investigate for an extremum at the point x_0 the function $f(x) = (x - x_0)^k \varphi(x)$, where $k \in N$ and $\varphi(x)$ is continuous at the point x_0, and $\varphi(x_0) \neq 0$.

4.3*. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad g(x) = \begin{cases} xe^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that the function $f(x)$ has a minimum at the point $x_0 = 0$, and the function $g(x)$ has no extremum at this point, although

$$f^{(k)}(0) = g^{(k)}(0) = 0, \ k \in \mathbb{N}.$$

In Problems 4.4 to 4.12 find the intervals of increase and decrease and the points of extremum for the indicated functions.

4.4. $y = x\sqrt{1-x^2}$. **4.5.** $y = \dfrac{2x^2-1}{x^4}$.

4.6. $y = \dfrac{x}{\ln x}$. **4.7.** $y = x - 2\sin x$.

4.8. $y = x - 2\ln x$. **4.9.** $y = \ln x - \arctan x$.

4.10. $y = e^x \cos x$. **4.11.** $y = x^x$. **4.12.** $y = \cosh^2 x + 1$.

The *greatest (least) value* of a continuous function $f(x)$ on a given interval $[a, b]$ is attained either at critical points, or at the end points of this interval.

In Problems 4.13 to 4.20 determine the greatest (M) and least (m) values of the given functions on the indicated intervals (if an interval is not indicated, then in the entire domain of definition):

4.13. $y = -3x^4 + 6x^2$; $[-2, 2]$.

4.14. $y = x + 2\sqrt{x}$; $[0, 4]$.

4.15. $y = \frac{x-1}{x+1}$; $[0, 4]$.

4.16. $y = \frac{1-x+x^2}{1+x-x^2}$; $[0, 1]$.

4.17. $y = \sqrt[3]{x+1} - \sqrt[3]{x-1}$; $[0, 1]$.

4.18. $y = \arctan \frac{1-x}{1+x}$; $[0, 1]$.

4.19. $y = \frac{x^2-1}{x^2+1}$. **4.20.** $y = xe^{-x^2/2}$.

In Problems 4.21 to 4.24 prove the given inequalities:

4.21*. $e^x > 1 + x$, $x \neq 0$. **4.22.** $\cos x > 1 - \frac{x^2}{2}$, $x \neq 0$.

4.23. $\frac{e^x + e^{-x}}{2} > 1 + \frac{x^2}{2}$, $x \neq 0$.

4.24. $\sin x + \tan x > 2x$, $x \in (0, \pi/2)$.

4.25. Two bodies move with constant velocities v_1 m/s and v_2 m/s. Their motions are performed in two straight lines which form an angle $\pi/2$, towards the vertex of this angle from which the first body was at a distance a m and the second at a distance b m. In how many seconds after the bodies started will the distance between them be the least?

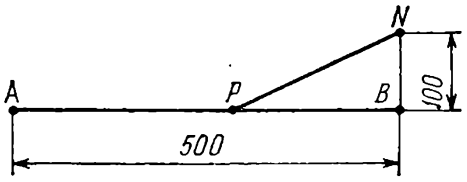

Fig. 49.

4.26. To deliver the products manufactured at factory N to town A, a highway NP is designed to connect the factory to the railway AB passing through A (Fig. 49). The cost of carriage by truck is twice the cost of carriage by rail. At what point P must the highway be joined to the railway so that the total cost of transportation of the products of factory N to town A by truck and by rail is the least?

4.27. A window has the shape of a rectangle completed by a semi-circle (Fig. 50). The perimeter p of this figure is

given. For what dimensions of x and y will the window pass the greatest amount of light?

4.28. Three boards of equal width are given to construct a gutter. For what angle α of inclination of the lateral walls to the bottom of the gutter will its cross section be the greatest?

4.29. Inscribed in a triangle with base a and altitude h is a rectangle whose base lies on the base of the triangle, and two of its vertices on the lateral sides of the triangle. Find the greatest area of the inscribed rectangle.

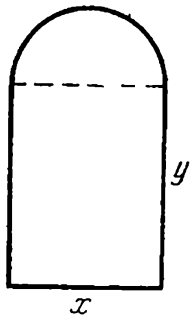

Fig. 50.

4.30. The perimeter of an axial section of a cylinder is equal to $6a$. Find the greatest volume of such a cylinder.

4.31. A cylinder is inscribed in a cone with altitude h and radius of the base r. Find the greatest volume of the inscribed cylinder.

4.32. Find the least volume of the cone circumscribed about a sphere of radius r.

4.33. Find the greatest volume of a cone whose generatix is of length l.

4.34. Determine the greatest area of a rectangle inscribed in a circle of radius r.

4.35. On the parabola $y = x^2$ find the point N least distant from the straight line $y = 2x - 4$.

4.36. A rectangle of the greatest possible area is inscribed in a semi-circle of radius R. Determine the base x and altitude y of the inscribed rectangle.

4.37. Divide a line segment of length a into two parts so that the sum of the areas of the squares constructed on these parts would be the least.

4.38. A conical funnel the radius of whose base is R and whose altitude is H, is filled with water. A ball is submerged into the funnel. Of what radius r must the ball be so that the volume of the water displaced from the funnel by the submerged portion of the ball is the greatest?

4.39. Determine the least height $h = |OB|$ of the door of the vertical tower $ABCD$ so that a rigid rod MN of length l whose end slides along a horizontal line AB could be

brought inside the tower. The width of the tower $|AB| = d < l$ (Fig. 51).

2. Direction of Convexity. Points of Inflection. A graph of a differentiable function $y = f(x)$ is said to be *convex downward* (or *concave upward*) on an interval (a, b) if an arc of a curve on this interval entirely lies above the tangent drawn to the graph of the function $y = f(x)$ at any point $x \in (a, b)$.

And if on the interval (a, b) any tangent lies above an arc of the curve, then the graph of a differentiable function on this interval

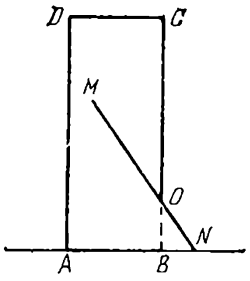

Fig. 51.

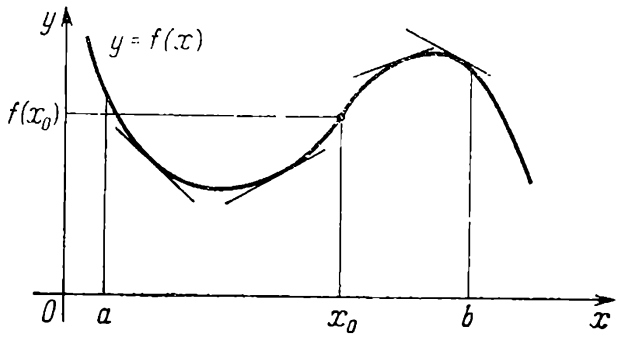

Fig. 52.

is said to be *convex upward* (or *concave downward*) (in Fig. 52 the graph of the function $y = f(x)$ is convex downward on the interval (a, x_0) and convex upward on the interval (x_0, b)).

If a function is twice differentiable on (a, b) and $f''(x) > 0$ ($f''(x) < 0$), then its graph is convex downward (upward) on this interval.

In simplest cases the domain of definition of $f(x)$ can be divided into a finite number of intervals with a constant direction of convexity. Each of these intervals is bounded by points at which $f''(x) = 0$, or $f''(x)$ does not exist. A point $(x_0, f(x_0))$ at which the direction of the convexity of the graph of a function changes is called a *point of inflection* (see Fig. 52).

Sufficient condition for a point of inflection. Let a function $f(x)$ be twice differentiable in some neighbourhood $U_\delta(x_0)$ of the point x_0 at which $f''(x_0)$ is either zero, or does not exist. In this case, if $f''(x)$ has opposite signs in the intervals $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$, then x_0 is a point of inflection.

Example 2. Find the intervals of convexity and the points of inflection of the graph of the function $y = \dfrac{|x-1|}{x^2}$.

We find the second derivative:

$$f''(x) = \begin{cases} \dfrac{2(3-x)}{x^4}, & x \in (-\infty, 0) \cup (0, 1), \\ \dfrac{2(x-3)}{x^4}, & x \in (1, +\infty). \end{cases}$$

Consequently, the critical points of the first derivative are: $x_1 = 0$, $x_2 = 1$, $x_3 = 3$. In addition, at the points x_1 and x_2 the second derivative does not exist (in particular, $f'_-(1) = 4$, and $f'_+(1) = -4$) and at the point x_3 it is equal to zero.

Thus, we have obtained four intervals of convexity: $(-\infty, 0)$, $(0, 1)$, $(1, 3)$ and $(3, +\infty)$. Investigating the sign of the second derivative in each of these intervals, we conclude that the graph of the given function is convex downward on the intervals $(-\infty, 0)$, $(0, 1)$, $(3, +\infty)$ and is convex upward on the interval $(1, 3)$. Consequently, the points x_2 and x_3 are the points of inflection of the graph of the given function, and x_1 is not. The obtained results are conveniently tabulated as follows:

Table 4.2

x	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, 3)$	3	$(3, +\infty)$
$f(x)$	$\cup$	$+\infty$	$\cup$	0	$\cap$	$\frac{2}{9}$	$\cup$
$f''(x)$	$+$	nonexistent	$+$	nonexistent	$-$	0	$+$

In Problems 4.40 to 4.47 find the intervals of convexity of the graph of the given function $y = f(x)$, points of inflection, and slopes k of the tangents at the points of inflection.

4.40. $y = x^7 + 7x + 1$. **4.41.** $y = x^4 + 6x^2$.

4.42. $y = \sqrt[3]{(x-2)^5} + 3$. **4.43.** $y = \sqrt{x+1} - \sqrt[3]{x-1}$.

4.44. $y = \sqrt[3]{(x+1)^2} + \sqrt[3]{(x-1)^2}$. **4.45.** $y = xe^{2x} + 1$.

4.46. $y = x \ln |x|$. **4.47.** $y = x^3 \ln x + 1$.

4.48. For what values of a and b is the point $(1, 3)$ a point of inflection of the curve $y = ax^3 + bx^2$?

4.49. At what choice of the parameter h does the *probability curve*

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \quad h > 0,$$

have points of inflection with abscissas $x = \pm 6$?

4.50. Show that the curve $y = \frac{x+1}{x^2+1}$ has three points of inflection lying on one straight line.

4.51*. Show that the points of inflection of the curve $y = x \sin x$ lie on the curve $y^2 (4 + x^2) = 4x^2$.

3. **Asymptotes.** Let for a function $y = f(x)$ there exist a straight line such that the distance from a point $M(x, f(x))$ on the graph of the function to this straight line tends to zero at an infinite distance of the point M from the origin. Then such a straight line is called an *asymptote* to the graph of the given function.

If the abscissa x of the point M tends to a finite number a, then the half-line $x = a$ ($y > 0$ or $y < 0$) is a vertical asymptote. For a vertical asymptote to exist at the point $x = a$, it is necessary and sufficient that at least one of the limits $\lim\limits_{x \to a \pm 0} f(x)$ be equal to infinity.

Continuous functions have no vertical asymptotes.

And if the abscissa x of the point M tends to $+\infty$ or $-\infty$, then we have an inclined asymptote $y = kx + b$; for an inclined asymptote to exist, it is necessary and sufficient that the following two limits exist:

$$\lim_{x \to \infty} \frac{f(x)}{x} = k \quad \text{and} \quad \lim_{x \to \infty} (f(x) - kx) = b.$$

The indicated limits can be different for $x \to +\infty$ (for a right-hand inclined asymptote) and for $x \to -\infty$ (for a left-hand inclined asymptote).

Example 3. Find the asymptotes to the graph of the function $y = \dfrac{|x - 1|}{x^2}$.

Since the given function is continuous throughout the entire axis except the point $x = 0$, a vertical asymptote can exist only at this point.

We have

$$\lim_{x \to 0+0} \frac{|x-1|}{x^2} = \lim_{x \to 0-0} \frac{|x-1|}{x^2} = +\infty$$

and, consequently, the straight line $x = 0$ is a vertical asymptote. Let us now find the inclined asymptotes. Since

$$\lim_{x \to +\infty} \frac{\dfrac{|x-1|}{x^2}}{x} = 0 = k \quad \text{and} \quad \lim_{x \to +\infty} \left(\frac{|x-1|}{x^2} - 0 \cdot x \right) = 0 = b,$$

the straight line $y = 0 \cdot x + 0 = 0$ is a right-hand inclined (in the present case—horizontal) asymptote.

Analogously, we find that the same straight line $y = 0$ is also a left-hand inclined asymptote.

In Problems 4.52 to 4.59 find the asymptotes to the graphs of the indicated functions.

4.52. $y = \sqrt[5]{\dfrac{x}{x-2}}$. **4.53.** $y = \sqrt[3]{x^3 - x^2}$.

4.54. $y = \sqrt{|x^2-3|}/x$. **4.55.** $y = 3x + \operatorname{arc\,tan} 5x$.

4.56. $y = \dfrac{\ln(x+1)}{x^2} + 2x$. **4.57.** $y \quad \dfrac{\sin x}{x}$.

4.58. $y = x \ln\left(e + \dfrac{1}{x}\right)$. **4.59.** $y = x \operatorname{arc\,sec} x$.

4.60. Prove that the graph of the integral rational function $y = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$, $n \geqslant 2$, has no asymptotes at all.

4. Construction of Graphs of Functions. To construct the graph of a function $y = f(x)$ with a continuous second derivative (everywhere in the domain of definition except possibly at a finite number of points), we first carry out an elementary investigation to determine some peculiarities of the function (if any): symmetry, periodicity, steadiness of sign, zeros, points of intersection with the y-axis, points of discontinuity, etc. Then, using the first and second derivatives, we find the points of extremum and inflection, intervals of monotonicity and convexity, and also the asymptotes.

Example 4. Construct the graph of the function $y = \dfrac{|x-1|}{x^2}$.

The given function is defined and continuous everywhere except at the point $x = 0$, it is nonnegative everywhere and is equal to

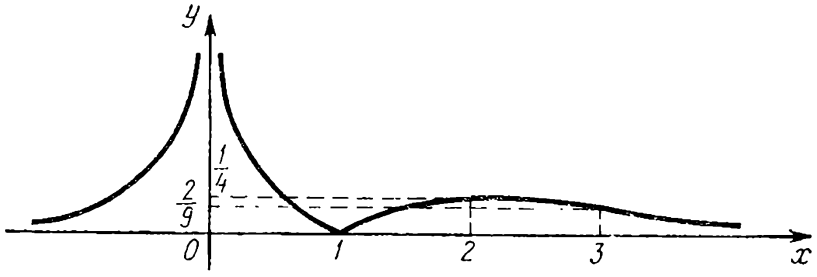

Fig. 53.

zero only at the point $x = 1$. Its investigation is carried out in Examples 1-3. The result of this investigation is useful to bring together, i.e. to combine Tables 4.1 and 4.2 into one table. The graph of the function is given in Fig. 53.

In Problems 4.61 to 4.123 construct the graphs of the given functions.

4.61. $y = \dfrac{(x^2-5)^3}{125}$. **4.62.** $y = \dfrac{1}{4}x^2(x^2-3)^2$.

4.63. $y=\frac{1}{6}x^3(x^2-5)$. 4.64. $y=\frac{x^3}{2(x-1)^2}$.

4.65. $y=\frac{x^4}{x^3-1}$. 4.66. $y=\frac{x^3-3x}{x^2-1}$.

4.67. $y=\frac{x^4}{x^3+1}$. 4.68. $y=\frac{x}{x^3+2}$. 4.69. $y=\frac{x^3}{x^4-1}$.

4.70. $y=\frac{x^2}{x^3-1}$. 4.71. $y=\frac{x}{x^2-4}$. 4.72. $y=\frac{x^3}{x^2-3}$.

4.73. $y=\frac{x}{2-x^3}$. 4.74. $y=\frac{x^2-1}{x^2+1}$. 4.75. $y=\frac{x^3}{x^3+1}$.

4.76. $y=\sqrt[3]{x+1}-\sqrt[3]{x-1}$. 4.77. $y=\sqrt[3]{x^2-2x}$.

4.78. $y=\sqrt[3]{(x+1)^2}+\sqrt[3]{(x-1)^2}$.

4.79. $y=\frac{1}{\sqrt[3]{x+1}}+\frac{1}{\sqrt[3]{x-1}}$. 4.80. $y=\sqrt[3]{1-x^3}$.

4.81. $y=\sqrt[3]{x+1}+\sqrt[3]{x-1}$.

4.82. $y=\sqrt[3]{x^3+1}+\sqrt[3]{x^3-1}$. 4.83. $y=\frac{x^3}{\sqrt{x^4+1}}$.

4.84. $y=\frac{x}{\sqrt{x^2+1}}$. 4.85. $y=\frac{x^3}{3\sqrt[3]{x^3+2}}$.

4.86. $y=\frac{x^2}{\sqrt[3]{x^3-4}}$. 4.87. $y=\frac{x^3}{\sqrt[3]{(x^3+2)^2}}$.

4.88. $y=\frac{x^2}{\sqrt{x^2+1}}$. 4.89. $y=\frac{\sqrt[3]{x^3+2}}{x}$.

4.90. $y=\frac{x}{\sqrt[3]{(x^3+1)^2}}$. 4.91. $y=\frac{\sqrt{|x^2-3|}}{x}$.

4.92. $y=\frac{x^2}{\sqrt{|x^2-1|}}$. 4.93. $y=\sqrt[3]{|x^2-1|}$.

4.94. $y=\sqrt{|x^2-2|^3}$. 4.95. $y=\sin x+\cos x$.

4.96. $y=\frac{1}{\sin x+\cos x}$. 4.97. $y=x\operatorname{arc\,tan} x$.

4.98. $y=\frac{x}{2}+\operatorname{arc\,cot} x$. 4.99. $y=e^{2x-x^2}$.

4.100. $y=xe^{-x^2/2}$. 4.101. $y=\frac{1}{x}e^{-1/x}$.

4.102. $y = \frac{1}{x^2} e^{-1/x^2}$. **4.103.** $y = xe^{1/x}$. **4.104.** $y = \frac{1}{x} e^{-1/x^2}$.

4.105. $y = (x - 2) e^{-1/x}$. **4.106.** $y = (2x - 1) e^{2/x}$.

4.107. $y = (x^2 + 1) e^{-x^2/2}$. **4.108.** $y = x^2 e^{2/x}$.

4.109. $y = x^3 e^{-x^2/2}$. **4.110.** $y = \ln\left(x\sqrt{x^2+1}\right)$.

4.111. $y = \frac{\ln x}{x}$. **4.112.** $y = \frac{1}{x \ln x}$. **4.113.** $y = x^2 \ln x$.

4.114. $y = \frac{\ln x}{x^2}$. **4.115.** $y = x^2 \ln^2 x$.

4.116. $y = x^2/\ln |x|$. **4.117.** $y = x \ln^2 |x|$.

4.118. $y = \ln |x^2 - 1|$. **4.119.** $y = \frac{1}{x^2} \ln^2 |x|$.

4.120. $y = x^x$, $x > 0$. **4.121*.** $y = x^{1/x}$, $x > 0$.

4.122. $y = (1 + x)^{1/x}$, $x > -1$. **4.123*.** $y = \frac{\sin x}{x}$.

In Problems 4.124 to 4.128 construct the curves represented parametrically.

4.124. $x = te^t$, $y = te^{-t}$, $t \in \mathbb{R}$.

Let us carry out auxiliary calculations:

$$x'_t = (1 + t) e^t, \quad y'_t = (1 - t) e^{-t}, \quad y'_x = \frac{1 - t}{1 + t} e^{-2t},$$

$$x''_{tt} = (2 + t) e^t, \quad y''_{tt} = (t - 2) e^{-t}, \quad y''_{xx} = 2 \frac{t^2 - 2}{(1 + t)^3} e^{-3t}.$$

Since $x'_t = 0$ for $t = -1$ and $x''_{tt}(-1) = \frac{1}{e} > 0$, we have $x_{\min} = -\frac{1}{e}$. Since $y'_t = 0$ for $t = 1$ and $y''_{tt}(1) = -\frac{1}{e} < 0$, $y_{\max} = \frac{1}{e}$. Hence it follows that the curve lies in the domain $\left\{(x, y) \mid x \in \left[-\frac{1}{e}, +\infty\right), y \in \left(-\infty, \frac{1}{e}\right]\right\}$. From the expression for the derivative y'_x we determine the critical points $t_1 = 1$ ($y'_x(1) = 0$) and $t_2 = -1$ ($y'_x(-1)$ is nonexistent). The critical points of the first derivative are found from the expression for the second derivative y''_{xx}: $t_3 = \sqrt{2}$ ($y''_{xx}(\sqrt{2}) = 0$), $t_4 = -\sqrt{2}$ ($y''_{xx}(-\sqrt{2}) = 0$) and $t_5 = 1$ ($y''_{xx}(-1)$ does not exist). Consequently, $A(-\sqrt{2}/e^{\sqrt{2}}, -\sqrt{2}e^{\sqrt{2}})$ and $B(\sqrt{2}e^{\sqrt{2}}, \sqrt{2}/e^{\sqrt{2}})$ are points of inflection.

Finally, we find the asymptotes. If $t \to -\infty$, then $x \to 0$, and $y \to -\infty$, i.e. $x = 0$ is a vertical asymptote. Note that, as the points

Table 4.3

t	x	y	y'_x	y''_{xx}	Behaviour of curve
$(-\infty, -\sqrt{2})$	<0	<0	>0	<0	Convex upward, decreases monotonically, $x=0$ is a vertical asymptote
$-\sqrt{2}$	$-\frac{\sqrt{2}}{e^{\sqrt{2}}}$	$-\sqrt{2}e^{\sqrt{2}}$	—	>0	Point of inflection
$(-\sqrt{2}, -1)$	<0	<0	>0	>0	Convex downward, decreases monotonically
-1	$-\frac{1}{e}$	$-e$	nonexistent	nonexistent	Cusp
$(-1, 1)$	—	—	>0	<0	Convex upward; increases monotonically, the point (0, 0) lies on the curve
1	e	$\frac{1}{e}$	0	—	Maximum
$(1, \sqrt{2})$	>0	>0	<0	<0	Convex upward, decreases monotonically
$\sqrt{2}$	$\sqrt{2}\,e^{\sqrt{2}}$	$\frac{\sqrt{2}}{e^{\sqrt{2}}}$		0	Point of inflection
$(\sqrt{2}, +\infty)$	>0	>0	<0	>0	Convex downward, decreases monotonically, $y=0$ is a horizontal asymptote

of the curve approach this asymptote, their abscissa remains negative. If $t \to +\infty$, then $x \to +\infty$, and $y \to 0$, i.e. $y = 0$ is a horizontal asymptote. The points of the curve, approaching this asymptote, have a positive ordinate.

The results of our investigation are then tabulated (Table 4.3) and all necessary conclusions are entered under "Behaviour of Curve".

Note that the seeming contradiction between the positiveness of the first derivative and monotone decrease of the function for $t < -1$

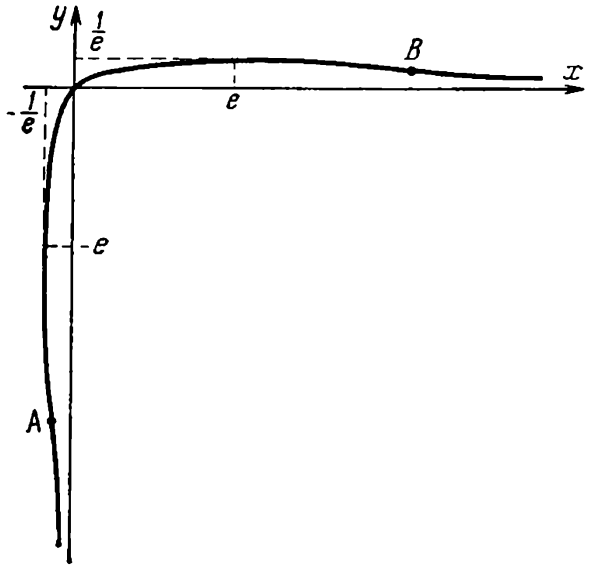

Fig. 54.

is explained by the fact that as the parameter t changes from $-\infty$ to -1 the values of x vary from 0 to $-1/e$ (i.e. decrease). The curve is shown in Fig. 54.

4.125. $x = t^2 - 2t$, $y = t^2 + 2t$, $t \in \mathbb{R}$.

4.126. $x = t + e^{-t}$, $y = 2t + e^{-2t}$, $t \in \mathbb{R}$.

4.127. $x = a \cos^3 t$, $y = a \sin^3 t$, $t \in [0, 2\pi)$.

4.128. $x = t^3 - 3\pi$, $y = t^3 - 6 \arctan t$, $t \in \mathbb{R}$.

In Problems 4.129 to 4.132 construct the indicated curves given in polar coordinates.

4.129. $r = a \sin 3\varphi$. **4.130.** $r = a(1 + \cos \varphi)$.

4.131. $r = \sqrt{\frac{\pi}{\varphi}}$. **4.132.** $r^2 = 2a^2 \cos 2\varphi$.

SEC. 5.5.
VECTOR AND COMPLEX FUNCTIONS OF REAL VARIABLE

1. Definition of Vector Function of Real Variable. If each value of a real variable $t \in D \subset \mathbb{R}$ is associated with a vector $\boldsymbol{a}(t) \in V_3$, then a *vector function* of a real variable $\boldsymbol{a} = \boldsymbol{a}(t)$ is said to be defined on the set D.

To define the vector function $\boldsymbol{a} = \boldsymbol{a}(t)$ is the same as to give three scalar functions $a_x(t)$, $a_y(t)$, $a_z(t)$ which are the coordinates of the vector $\boldsymbol{a}$:

$$\boldsymbol{a} = a_x(t)\,\boldsymbol{i} + a_y(t)\,\boldsymbol{j} + a_z(t)\,\boldsymbol{k},$$

or, in a shorter notation, $\boldsymbol{a} = (a_x(t),\ a_y(t),\ a_z(t))$. If vector $\boldsymbol{a}$ is the radius vector of the point $M(x, y, z)$, then the corresponding vector function is customarily denoted:

$$\boldsymbol{r} = \boldsymbol{r}(t) \quad x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} + z(t)\,\boldsymbol{k}.$$

The *hodograph* of a vector function $\boldsymbol{r} = \boldsymbol{r}(t)$ is a line described in space by the terminus of the vector $\boldsymbol{r}$. Any line in space may be regarded as the hodograph of some vector. The parametric equations of hodograph are:

$$x = x(t),\ y = y(t),\ z = z(t).$$

Example 1. Find the hodograph of the vector function

$$\boldsymbol{r}(t) \quad \frac{1-t^2}{1+t^2}\,\boldsymbol{i} + \frac{2t}{1+t^2}\,\boldsymbol{j} + \boldsymbol{k}, \quad t \in \mathbb{R}.$$

We have the parametric equations of the hodograph

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}, \quad z = 1.$$

Eliminating the parameter t, we obtain

$$x^2 + y^2 = \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} = 1.$$

Hence, the hodograph of the vector function $\boldsymbol{r}(t)$ is the circle

$$x^2 + y^2 = 1,\ z = 1,$$

with the point $(-1, 0, 1)$ excluded. When t changes from $-\infty$ to $+\infty$ the point $M(x, y, z)$ on the hodograph moves from the point $(-1, 0, 1)$ anticlockwise (when viewed from a point situated above the plane $z = 1$); in this case $\lim\limits_{t \to \pm\infty} x = -1$, $\lim\limits_{t \to \pm\infty} y = 0$.

In Problems 5.1 to 5.10 find the hodographs of the given vector functions.

5.1. $\boldsymbol{r} = (2t - 1)\,\boldsymbol{i} + (-3t + 2)\,\boldsymbol{j} + 4t\boldsymbol{k},\ t \in \mathbb{R}.$

5.2. $\boldsymbol{r} = \sqrt{1 - t^2}\,\boldsymbol{i} + \sqrt{1 + t^2}\,\boldsymbol{j},\ t \in [0, 1],$

5.3. $\boldsymbol{r} = 4 \cosh t \cdot \boldsymbol{i} - \boldsymbol{j} + 3 \sinh t \cdot \boldsymbol{k}$, $t \in \mathbb{R}$.

5.4. $\boldsymbol{r} = 3t\boldsymbol{i} + (2t - t^2)\,\boldsymbol{j}$, $t \in \mathbb{R}$.

5.5. $\boldsymbol{r} = \cos t \cdot \boldsymbol{i} + \sin t \cdot \boldsymbol{j} + t\boldsymbol{k}$, $t \in \mathbb{R}$.

5.6. $\boldsymbol{r} = 2 \cos^3 t \cdot \boldsymbol{i} + 2 \sin^3 t \cdot \boldsymbol{j}$, $t \in [0, 2\pi]$.

5.7. $\boldsymbol{r} = t\boldsymbol{i} + t^2\boldsymbol{j} + t^2\boldsymbol{k}$, $t \in \mathbb{R}$.

5.8. $\boldsymbol{r} = \cos^2 t \cdot \boldsymbol{i} + \sin t \cos t \cdot \boldsymbol{j} + \sin t \cdot \boldsymbol{k}$, $t \in [0, 2\pi]$.

5.9. $\boldsymbol{r} = 5 \cos t \cdot \boldsymbol{i} + 4 \sin t \cdot \boldsymbol{j} + 2\boldsymbol{k}$, $t \in [0, 2\pi]$.

5.10. $\boldsymbol{r} = (\sinh t - 1)\,\boldsymbol{i} + \cosh^2 t \cdot \boldsymbol{j} + 3\boldsymbol{k}$, $t \in \mathbb{R}$.

2. Differentiation of a Vector Function. The *derivative of a vector function* $\boldsymbol{a} = \boldsymbol{a}(t)$ with respect to the argument t is again a vector function

$$\frac{d\boldsymbol{a}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{a}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{a}(t + \Delta t) - \boldsymbol{a}(t)}{\Delta t}.$$

If $\boldsymbol{a}(t) = (a_x(t),\ a_y(t),\ a_z(t))$, then

$$\frac{d\boldsymbol{a}}{dt} = \left(\frac{da_x(t)}{dt},\ \frac{da_y(t)}{dt},\ \frac{da_z(t)}{dt}\right).$$

If $\boldsymbol{r} = \boldsymbol{r}(t) = (x(t),\ y(t), z(t))$, then the derivative $\frac{d\boldsymbol{r}}{dt}$ is a vector tangent to the hodograph of the vector function $r(t)$ and going in the direction corresponding to the increase of the argument t.

If t is time, then $\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}$ is the vector of the velocity of the terminus of the vector $\boldsymbol{r}$.

Rules for differentiation of a vector function ($\boldsymbol{a} = \boldsymbol{a}(t)$, $\boldsymbol{b} = \boldsymbol{b}(t)$).

(1) $\frac{d}{dt}(\boldsymbol{a} \pm \boldsymbol{b}) = \frac{d\boldsymbol{a}}{dt} \pm \frac{d\boldsymbol{b}}{dt}$.

(2) $\frac{d}{dt}(\alpha \boldsymbol{a}) = \alpha \frac{d\boldsymbol{a}}{dt}$, where α is a constant scalar.

(3) $\frac{d\boldsymbol{c}}{dt} = 0$, where $\boldsymbol{c}$ is a constant vector.

(4) $\frac{d}{dt}(\varphi \boldsymbol{a}) = \frac{d\varphi}{dt}\boldsymbol{a} + \varphi \frac{d\boldsymbol{a}}{dt}$, where $\varphi = \varphi(t)$ is a scalar function of t.

(5) $\frac{d}{dt}(\boldsymbol{a}, \boldsymbol{b}) = \left(\frac{d\boldsymbol{a}}{dt}, \boldsymbol{b}\right) + \left(\boldsymbol{a}, \frac{d\boldsymbol{b}}{dt}\right)$.

(6) $\frac{d}{dt}[\boldsymbol{a}, \boldsymbol{b}] = \left[\frac{d\boldsymbol{a}}{dt}, \boldsymbol{b}\right] + \left[\boldsymbol{a}, \frac{d\boldsymbol{b}}{dt}\right]$.

(7) $\frac{d}{dt}\boldsymbol{a}(\varphi(t)) = \frac{d\boldsymbol{a}}{d\varphi}\cdot\frac{d\varphi}{dt}$, where $\varphi = \varphi(t)$ is a scalar function of t.

(8) $\left(\boldsymbol{a}, \frac{d\boldsymbol{a}}{dt}\right) = 0$, if $|\boldsymbol{a}| = \text{constant}$.

5.11. Given: the equation of motion $\boldsymbol{r} = 3t\boldsymbol{i} - 4t\boldsymbol{j}$. Determine the path and velocity of motion.

5.12. Given: the equation of motion $\boldsymbol{r} = 3t\boldsymbol{i} + (4t - t^2)\boldsymbol{j}$. Determine the path and velocity of motion. Construct the velocity vectors for the instants: $t = 0$, $t = 1$, $t = 2$, $t = 3$.

5.13. Given: the equation of motion $\boldsymbol{r} = 2(t - \sin t)\boldsymbol{i} + 2(1 - \cos t)\boldsymbol{j}$. Determine the path and velocity of motion. Construct the velocity vectors for the instants: $t = \pi/2$ and $t = \pi$.

5.14. Find the unit tangent vector of the hodograph of the vector function $\boldsymbol{r} = 2^{2t}\boldsymbol{i} - (t + 8)^{4/3}\boldsymbol{j}$ for $t = 0$.

5.15. Find the unit tangent vector of the hodograph of the vector function $\boldsymbol{r} = (t^3 + t)\boldsymbol{i} + t^2\boldsymbol{j}$ for $t = -1$.

5.16. Find the derivatives of the following vector functions:

(a) $\boldsymbol{r} = \sin t\cdot\boldsymbol{i} + \cos^2 t\cdot\boldsymbol{j} + \sin t\cos t\cdot\boldsymbol{k}$;

(b) $\boldsymbol{r} = t\cos t\cdot\boldsymbol{i} + t\sin t\cdot\boldsymbol{j} + t\boldsymbol{k}$;

(c) $\boldsymbol{r} = (t + \cos t)\boldsymbol{i} + t\boldsymbol{j} + \sin t\cdot\boldsymbol{k}$.

5.17. Find the derivatives of the indicated vector functions:

(a) $\boldsymbol{r} = e^t\boldsymbol{i} + \cos t\cdot\boldsymbol{j} + (t^2 + 1)\boldsymbol{k}$ at the point (1, 1, 1);

(b) $\boldsymbol{r} = t^3\boldsymbol{i} + (t + 1)^2\boldsymbol{j} + \sqrt{t^2 + 1}\boldsymbol{k}$ for $t = -2$.

5.18. Find $\frac{d}{dt}(\boldsymbol{a}, \boldsymbol{b})$ if

$\boldsymbol{a} = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + t\boldsymbol{j} + t^2\boldsymbol{k}$.

5.19. Find $\frac{d}{dt}[\boldsymbol{a}, \boldsymbol{b}]$ if $\boldsymbol{a} = \boldsymbol{i} + t\boldsymbol{j} + t^2\boldsymbol{k}$, $\boldsymbol{b} = t\boldsymbol{i} + \boldsymbol{j} + t^2\boldsymbol{k}$.

5.20. Find $\frac{d\boldsymbol{a}}{dt}$ if $\boldsymbol{a} = u\boldsymbol{i} + u^2\boldsymbol{j} + u^3\boldsymbol{k}$, where $u = \sin t$.

3. Tangent to a Space Curve and a Normal Plane. The equation of the tangent to a space curve $x = x(t)$, $y = y(t)$, $z = z(t)$ at point $M_0(x_0, y_0, z_0)$, to which there corresponds the value t_0 of the param-

eter, has the form

$$\frac{x-x_0}{\left.\frac{dx}{dt}\right|_{t=t_0}}=\frac{y-y_0}{\left.\frac{dy}{dt}\right|_{t=t_0}}=\frac{z-z_0}{\left.\frac{dz}{dt}\right|_{t=t_0}},$$

where x, y, z are the moving coordinates of a point on the tangent. The equation of the normal plane at the same point is

$$(x-x_0)\left.\frac{dx}{dt}\right|_{t=t_0}+(y-y_0)\left.\frac{dy}{dt}\right|_{t=t_0}+(z-z_0)\left.\frac{dz}{dt}\right|_{t=t_0}=0.$$

Example 2. Prove that the tangent to the helix $\boldsymbol{r}=(a\cos t,\ a\sin t,\ bt)$ forms a constant angle with the z-axis.

Let us find the vector tangent to the hodograph of the vector $\boldsymbol{r}$:

$$\frac{d\boldsymbol{r}}{dt}=(-a\sin t,\ a\cos t,\ b).$$

Hence

$$\cos\gamma=\frac{z'(t)}{\left|\frac{d\boldsymbol{r}}{dt}\right|}=\frac{b}{\sqrt{a^2+b^2}},$$

i.e. $\gamma=$ constant.

Example 3. Write the equations of the tangent line and the normal plane to the curve $x=t^2-1$, $y=t+1$, $z=t^3$ at the point $M_0\ (0,\ 2,\ 1)$.

To the given point there corresponds the value of the parameter $t=1$. We have

$$\frac{dx}{dt}=2t,\quad \frac{dy}{dt}=1,\quad \frac{dz}{dt}=3t^2.$$

Substituting the value $t=1$, we get

$$\left.\frac{dx}{dt}\right|_{t=1}=2,\quad \left.\frac{dy}{dt}\right|_{t=1}=1,\quad \left.\frac{dz}{dt}\right|_{t=1}=3.$$

The desired equation of the tangent line:

$$\frac{x}{2}=\frac{y-2}{1}=\frac{z-1}{3}.$$

The equation of the normal plane:

$$2(x-0)+1\cdot(y-2)+3(z-1)=0,$$

or

$$2x+y+3z-5=0.$$

In Problems 5.21 to 5.25 write the equations of the tangent line and the normal plane to the given curve at the indicated point.

5.21. $x = 4 \sin^2 t$, $y = 4 \sin t \cos t$, $z = 2 \cos^2 t$ for $t = \pi/4$.

5.22. $x = \frac{1}{2} t^2$, $y = \frac{1}{3} t^3$, $z = \frac{1}{4} t^4$ for $t = 2$.

5.23. $x = a \cosh t$, $y = a \sinh t$, $z = at$ for $t = 0$.

5.24. $x^2 + y^2 = 10$, $y^2 + z^2 = 25$ at the point M_0 (1, 3, 4).

5.25. $2x^2 + 3y^2 + z^2 = 9$, $3x^2 + y^2 - z^2 = 0$ at the point $M_0 = (1, -1, 2)$.

4. Second Derivative of a Vector Function. If

$$\boldsymbol{r} = \boldsymbol{r}(t) = (x(t), y(t), z(t)),$$

then

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt}\right) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right).$$

If t is time, then $\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\boldsymbol{v}}{dt} = \boldsymbol{w}$ is the acceleration vector of the terminus of the vector $\boldsymbol{r}$.

Let a curve in the xy-plane be a hodograph of the vector function $\boldsymbol{r} = \boldsymbol{r}(s) = (x(s), y(s))$, where s is the arc length of the curve.

The *curvature* of the curve at the point M_0 is defined as the number

$$K = \left| \lim_{M \to M_0} \frac{\varphi}{\Delta s} \right|,$$

where φ is the angle of rotation of the tangent corresponding to the arc $\overline{M_0M}$ (Fig. 55) of the given curve, and Δs is the length of this arc. The quantity $R = 1/K$ is referred to as the *radius of curvature*.

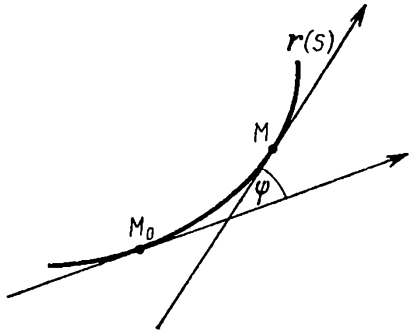

Fig. 55.

The curvature K is defined by the relation $K = \left|\frac{d^2\boldsymbol{r}}{ds^2}\right|$.

Given below are the formulas for computing the curvature of a curve:

(1) if a curve is defined by an equation in the explicit form $y = f(x)$, then

$$K = \left| \frac{y''}{(1 + y'^2)^{3/2}} \right|;$$

(2) if a curve is given by an equation in the implicit form* $F(x, y) = 0$, then

$$K = \left| \frac{\begin{vmatrix} F''_{xx} & F''_{xy} & F'_x \\ F''_{xy} & F''_{yy} & F'_y \\ F'_x & F'_y & 0 \end{vmatrix}}{(F'^2_x + F'^2_y)^{3/2}} \right|;$$

(3) if a curve is represented by the parametric equations $x = x(t)$, $y = y(t)$, then

$$K = \left| \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2 + y'^2)^{3/2}} \right|;$$

(4) if a curve is given in polar coordinates by the equation $r = r(\varphi)$, then

$$K = \left| \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} \right|.$$

The *osculating circle* (or the *circle of curvature*) *of a curve* at its point M is defined as the limiting position of the circle drawn through the point M and two other points of the curve P and Q when $P \to M$ and $Q \to M$.

The radius of the circle of curvature is equal to the radius of curvature, and the centre of the circle of curvature (the *centre of curvature*), corresponding to the point M, is found on the normal to the curve drawn at the point M in the direction of the concavity of the curve.

The coordinates X and Y of the centre of curvature are

$$X = x - \frac{y'(1+y'^2)}{y''}, \qquad Y = y + \frac{1+y'^2}{y''}.$$

The *evolute of a curve* is defined as a line described by the centre of curvature as a point moves in the curve. The formulas for the coordinates of the centre of curvature are determined by the parametric equations of the evolute.

Example 4. Find the equation of the evolute of the parabola $y^2 = 2(x + 1)$.

We have $2yy' = 2$, i.e. $y' = \frac{1}{y}$. After repeated differentiation we obtain $y'^2 + yy'' = 0$, whence $y'' = -\frac{y'^2}{y} = -\frac{1}{y^3}$. Now we

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* Here we make use of partial derivatives of a function of two variables; for their definition see Chapter 7, Sec. 7.1, subsection 3.

find the coordinates of the centre of curvature:

$$X = x - \frac{y'(1+y'^2)}{y''} = \frac{y^2}{2} - 1 - \frac{\frac{1}{y}\left(1+\frac{1}{y^2}\right)}{-1/y^3} = \frac{3}{2}y^2,$$

$$Y = y + \frac{1+y'^2}{y''} = y + \frac{1+\frac{1}{y^2}}{-1/y^3} = -y^3;$$

we have thus found the parametric equations of the evolute:

$$X = \frac{3}{2}y^2, \quad Y = -y^3.$$

Eliminating the parameter y, we find the equation of the evolute in the form

$$Y^2 = \frac{8}{27}X^3.$$

5.26. Find the second derivatives of the given vector functions:

(a) $\boldsymbol{r} = \cos t \cdot \boldsymbol{i} + e^t \boldsymbol{j} + (t^2 + 1)\boldsymbol{k}$,

(b) $\boldsymbol{r} = t\boldsymbol{i} + t\cos t \cdot \boldsymbol{j} + t \sin t \cdot \boldsymbol{k}$

for an arbitrary t and for $t = 0$.

5.27. Given the equation of motion: $\boldsymbol{r} = 2(t - \sin t)\,\boldsymbol{i} + 2(1 - \cos t)\,\boldsymbol{j}$. Determine the acceleration of motion. Construct the acceleration vectors for the instants $t = \pi/2$, $t = \pi$.

5.28*. Given the equation of motion: $\boldsymbol{r} = 3t\boldsymbol{i} + (4t - t^2)\,\boldsymbol{j}$. Determine the acceleration $\boldsymbol{w}$ of motion and its tangential w_τ and normal w_n components for an arbitrary instant t and for $t = 0$.

5.29. Given the equation of motion: $\boldsymbol{r} = {}^1/_2\, t^2\boldsymbol{i} + {}^1/_3\,(2t + 1)^{3/2}\,\boldsymbol{j}$. Determine the acceleration of motion and its tangential and normal components for an arbitrary instant t and for $t = 0$.

In Problems 5.30 to 5.36 compute the curvature of the given curves.

5.30. $y = x^2$ at the origin and at the point M (1, 1).

5.31. $x^2 + 9y^2 = 9$ at the vertices of the ellipse A (3, 0) and B (0, 1).

5.32. $x^2 - xy + y^2 = 1$ at the point M (1, 1).

5.33. $x = t^2$, $y = t - {}^1/_3\, t^3$ for $t = 1$.

5.34. $x = {}^1/_2\, t^2$, $y = {}^1/_3\, t^3$ at the point M (1/2, 1/3).

5.35. $r = a\,(1 - \cos\varphi)$ at an arbitrary point and for $\varphi = \pi$.

5.36. $r^2 = a^2 \sin 2\varphi$ for $\varphi = \pi/4$.

In Problems 5.37 to 5.43 find the radius of curvature (at any point) of the given curve.

5.37. y $\sqrt[3]{x}$. **5.38.** $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ 1.

5.39. $x^{2/3} + y^{2/3} = a^{2/3}$. **5.40.** $x = a\cos t$, $y = b\sin t$.

5.41. $x = a\,(t - \sin t)$, $y = a\,(1 - \cos t)$.

5.42. $r^2 = a^2 \cos 2\varphi$. **5.43.** $r = a\varphi$.

5.44*. The *vertex* of a curve is its point at which curvature has either a maximum or a minimum. Find the vertex of the curve $y = e^{-x}$.

5.45. Find the vertex of the curve $y = \ln x$.

In Problems 5.46 to 5.50 compute the coordinates of the centre of curvature and write the equation of the circle of curvature of the given curve at the indicated point.

5.46. $y = \frac{a^3}{a^2 + x^2}$ at the point M (0, a).

5.47. $y = e^{-x^2}$ at the point M (0, 1).

5.48. $y = xe^x$ at the point M (—1, —1/e).

5.49. $y = \sin x$ at the point M ($\pi/2$, 1).

5.50. $x = a\,(t - \sin t)$, $y = a\,(1 - \cos t)$ at the point M (πa, $2a$).

In Problems 5.51 to 5.55 find the evolutes of the given curves.

5.51. $y = x^3$. **5.52.** $x^2 - y^2 = a^2$. **5.53.** $x^{2/3} + y^{2/3} = a^{2/3}$.

5.54. $x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$.

5.55. x $2t$, $y = t^2 - 2$.

5. **Differential Characteristics of Space Curves.** At any nonsingular point M (x, y, z) of a space curve r $r(t)$ it is possible to construct three mutually perpendicular vectors:

(1) $\boldsymbol{T} = \frac{d\boldsymbol{r}}{dt}$ (direction vector of the *tangent*);

(2) $\boldsymbol{B}$ $\left[\frac{d\boldsymbol{r}}{dt}, \frac{d^2\boldsymbol{r}}{dt^2}\right]$ (direction vector of the *binormal*);

(3) $N = [B, T]$ (direction vector of the *principal normal*) or corresponding to them basic *unit vectors*:

$$\tau = \frac{T}{|T|}, \quad \beta = \frac{B}{|B|}, \quad \nu = \frac{N}{|N|},$$

which can also be computed by the formulas:

$$\tau = \frac{dr}{ds}, \quad \nu = \frac{d\tau}{ds} \Big/ \left| \frac{d\tau}{ds} \right|, \quad \beta = [\tau, \nu].$$

A trihedron with vertex at a point M_0 whose edges are the tangent, principal normal, and binormal is called the *natural trihedron* of a space curve. Its faces are: the *osculating plane* (passes through the vectors T and N), the *normal plane* (passes through the vectors N and B), and the *rectifying plane* (passes through the vectors B and T).

The equations of the principal normal have the form

$$\frac{x - x_0}{N_x} = \frac{y - y_0}{N_y} = \frac{z - z_0}{N_z}.$$

where x, y, z are the moving coordinates of a point of the principal normal, and N_x, N_y, N_z are the coordinates of the vector N.

The equations of the binormal have the form

$$\frac{x - x_0}{B_x} = \frac{y - y_0}{B_y} = \frac{z - z_0}{B_z}.$$

The equation of the osculating plane is

$$B_x (x - x_0) + B_y (y - y_0) + B_z (z - z_0) = 0.$$

The equation of the rectifying plane is

$$N_x (x - x_0) + N_y (y - y_0) + N_z (z - z_0) = 0.$$

Example 5. Find the basic unit vectors τ, ν, and β of the curve $x = 1 - \sin t$, $y = \cos t$, $z = t$ at the point M to which there corresponds the value of the parameter $t = 0$. Write the equations of the tangent, principal normal, and binormal at this point.

We have

$$r = (1 - \sin t)\, i + \cos t \cdot j + tk,$$

$$\frac{dr}{dt} = -\cos t \cdot i - \sin t \cdot j + k,$$

$$\frac{d^2r}{dt^2} = \sin t \cdot i - \cos t \cdot j.$$

For $t = 0$ we obtain:

$$T \quad \frac{dr}{dt} = -i + k, \quad \frac{d^2r}{dt^2} = -j,$$

$$B=\left[\frac{dr}{dt},\ \frac{d^2r}{dt^2}\right]=\begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}=i+k,$$

$$N=[B,\ T]=\begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix}=-2j.$$

Consequently,

$$\tau=\frac{-i+k}{\sqrt{2}},\quad \nu=-j,\quad \beta=\frac{i+k}{\sqrt{2}}.$$

Since for $t=0$ we have $x=1$, $y=1$, $z=0$, then

$\frac{x-1}{-1}=\frac{y-1}{0}=\frac{z}{1}$ are the equations of the tangent;

$\frac{x-1}{0}\quad\frac{y-1}{1}=\frac{z}{0}$ are the equations of the principal normal;

$\frac{x-1}{1}=\frac{y-1}{0}=\frac{z}{1}$ are the equations of the binormal.

If a space curve is given as the intersection of two surfaces

$$\begin{cases} F(x,\ y,\ z)=0, \\ G(x,\ y,\ z)=0, \end{cases}$$

then, instead of the vectors $\frac{dr}{dt}$ and $\frac{d^2r}{dt^2}$ it is more convenient to consider the vectors $dr=(dx,\ dy,\ dz)$ and $d^2r=(d^2x,\ d^2y,\ d^2z)$, regarding one of the variables x, y, z as an independent variable and its second differential to be equal to zero.

Example 6. Write the equations of the osculating, normal, and rectifying planes of the curve

$$\begin{cases} x^2+y^2+z^2=6, \\ x^2-y^2+z^2=4 \end{cases}$$

at its point $M(1,\ 1,\ 2)$.

Differentiating the given equations and regarding x as an independent variable, we obtain:

$$\begin{aligned} x\,dx+y\,dy+z\,dz&=0, \\ x\,dx-y\,dy+z\,dz&=0 \end{aligned}$$

and

$$\begin{aligned} dx^2+dy^2+y\,d^2y+dz^2+z\,d^2z&=0, \\ dx^2-dy^2-y\,d^2y+dz^2+z\,d^2z&=0. \end{aligned}$$

For $x = 1$, $y = 1$, $z = 2$ we have:

$$dy = 0, \quad dz = -\frac{1}{2}dx,$$

$$d^2y = 0, \quad d^2z = -\frac{3}{8}dx^2.$$

Consequently,

$$d\boldsymbol{r} = \left(dx,\ 0,\ -\frac{1}{2}dx\right), \quad d^2\boldsymbol{r} = \left(0,\ 0,\ -\frac{3}{8}dx^2\right).$$

Replacing these vectors by the collinear vectors (2, 0, —1) and (0, 0, —1), respectively, we obtain

$$\boldsymbol{T} = (2,\ 0,\ -1),$$

$$\boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2\boldsymbol{j}, \quad \boldsymbol{N} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{vmatrix} = 2(-\boldsymbol{i} - 2\boldsymbol{k}).$$

Hence we find:

$y - 1 = 0$ is the equation of the osculating plane;
$2x - z = 0$ is the equation of the normal plane;
$x + 2z - 5 = 0$ is the equation of the rectifying plane.

In Problems 5.56 to 5.59 find the basic unit vectors τ, ν, β and write the equations of the tangent, principal normal, and binormal of the given curves.

5.56. $x = e^t$, $y = e^{-t}$, $z = t$ for $t = 0$.

5.57. $x = t - \sin t$, $y = 1 - \cos t$, $z = 4 \sin \frac{t}{2}$ for $t = \pi$.

5.58. $x = 2t$, $y = \ln t$, $z = t^2$ for $t = 1$.

5.59. $y = x$, $z = 2x^2$ at the point $x = 1$.

5.60. Write the equations of the planes forming the natural trihedron of the curve $x = t^2 + 1$, $y = \cos t$, $z = e^t$ at the point (1, 1, 1).

5.61. Write the equations of the planes forming the natural trihedron of the curve $x = t/\sqrt{2}$, $y = t/\sqrt{2}$, $z = \ln \sin t$ for $t = \pi/2$.

5.62. Find the vectors τ, ν, β and write the equations of all the edges and planes forming the natural trihedron of the curve $x = (t + 1)^2$, $y = t^3$, $z = \sqrt{t^2 + 1}$ at the point (1, 0, 1).

5.63. Find the vectors τ, ν, β and write the equations of all the edges and planes forming the natural trihedron

of the curve $\begin{cases} x^2 + y^2 + z^2 = 14, \\ x + 2y - z = 2 \end{cases}$ at the point (1, 2, 3).

The *curvature* of a space curve is defined analogously to that of a plane curve. If a curve is given by the equation $\boldsymbol{r} = \boldsymbol{r}(s)$, then

$$K = \frac{1}{R} = \left| \frac{d^2\boldsymbol{r}}{ds^2} \right|.$$

If a curve is represenred parametrically, then we have

$$K = \frac{1}{R} = \frac{\left| \left[\frac{d\boldsymbol{r}}{dt}, \frac{d^2\boldsymbol{r}}{dt^2} \right] \right|}{\left| \frac{d\boldsymbol{r}}{dt} \right|^3}.$$

The *torsion* (*second curvature*) of a space curve at a point M is defined as the number

$$\sigma = \frac{1}{\rho} = \lim_{N \to M} \frac{\theta}{\Delta s},$$

where θ is the angle of rotation of the binormal corresponding to the arc $\overset{\frown}{MN}$. The quantity ρ is called the *radius of torsion* or the *radius of second curvature*.

If $\boldsymbol{r} = \boldsymbol{r}(s)$, then

$$\sigma = \mp \left| \frac{d\beta}{ds} \right| = \frac{\frac{d\boldsymbol{r}}{ds} \cdot \frac{d^2\boldsymbol{r}}{ds^2} \cdot \frac{d^3\boldsymbol{r}}{ds^3}}{\left| \frac{d^2\boldsymbol{r}}{ds^2} \right|^2},$$

where the minus sign is taken if the vectors $\frac{d\beta}{ds}$ and $\boldsymbol{\nu}$ are in the same direction and the plus sign if otherwise.

If $\boldsymbol{r} = \boldsymbol{r}(t)$, where t is an arbitrary parameter, then

$$\sigma = \frac{\frac{d\boldsymbol{r}}{dt} \cdot \frac{d^2\boldsymbol{r}}{dt^2} \cdot \frac{d^3\boldsymbol{r}}{dt^3}}{\left| \left[\frac{d\boldsymbol{r}}{dt}, \frac{d^2\boldsymbol{r}}{dt^2} \right] \right|^2}.$$

Example 7. Find the curvature and torsion of the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$ at an arbitrary point.

We have

$$\boldsymbol{r} = (e^t \cos t,\ e^t \sin t,\ e^t),$$

$$\frac{d\boldsymbol{r}}{dt} = (e^t (\cos t - \sin t),\ e^t (\sin t + \cos t),\ e^t),$$

$$\frac{d^2\boldsymbol{r}}{dt^2} = (-2e^t \sin t,\ 2e^t \cos t,\ e^t),$$

$$\frac{d^3\boldsymbol{r}}{dt^3} = (-2e^t (\sin t + \cos t),\ 2e^t (\cos t - \sin t),\ e^t).$$

Hence

$$\left[\frac{dr}{dt}, \frac{d^2r}{dt^2}\right] = \begin{vmatrix} i & j & k \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & e^t \\ -2e^t \sin t & 2e^t \cos t & e^t \end{vmatrix}$$

$$= e^{2t}(\sin t - \cos t, \ -(\sin t + \cos t), \ 2)$$

$$\frac{dr}{dt} \cdot \frac{d^2r}{dt^2} \cdot \frac{d^3r}{dt^3} = \begin{vmatrix} e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & e^t \\ -2e^t \sin t & 2e^t \cos t & e^t \\ -2e^t(\sin t + \cos t) & 2e^t(\cos t - \sin t) & e^t \end{vmatrix} = 2e^{3t}.$$

Consequently,

$$K = \frac{e^{2t}\sqrt{(\sin t - \cos t)^2 + (\sin t + \cos t)^2 + 4}}{e^{3t}\sqrt{((\sin t - \cos t)^2 + (\sin t + \cos t)^2 + 1)^3}} = \frac{\sqrt{2}}{3} e^{-t},$$

$$\sigma = \frac{2e^{3t}}{e^{4t}((\sin t - \cos t)^2 + (\sin t + \cos t)^2 + 4)} = \frac{e^{-t}}{3}.$$

In Problems 5.64 to 5.69 compute the curvature and torsion of the given curve.

5.64. $x = e^t$, $y = e^{-t}$, $z = t\sqrt{2}$ at an arbitrary point and for $t = 0$.

5.65. $x = t$, $y = t^2$, $z = t^3$ at an arbitrary point and for $t = 0$.

5.66. $x = 3t - t^3$, $y = 3t^2$, $z = 3t + t^3$ at an arbitrary point and for $t = 1$.

5.67. $x = 2t$, $y = \ln t$, $z = t^2$ at an arbitrary point and for $t = 1$.

5.68. $y = \frac{x^2}{2}$, $z = \frac{x^3}{3}$ for $x = 1$.

5.69. $2x = y^2$, $z = x^2$ at an arbitrary point and for $y = 1$.

5.70*. Given an equation of motion: $r = ti + t^2 j + \frac{2}{3} t^3 k$. Determine the acceleration w of the given motion, tangential w_τ and normal w_ν components of the acceleration at an arbitrary instant t and for $t = 1$.

6. Complex Functions of a Real Variable. If each value of the real variable $t \in D \subset \mathbb{R}$ is associated with a certain complex number $z = x + iy$, then $z(t)$ is called a *complex function of the real variable* t with the domain of definition D:

$$z = z(t) = x(t) + iy(t).$$

To specify a complex function $z = z(t)$ is the same as to define two real functions $x = x(t)$, $y = y(t)$ or to give a vector function $r(t) = (x(t), y(t))$.

The *derivative* of a complex function $z(t)$ is defined as the complex function $z'(t) = \lim\limits_{\Delta t \to 0} \dfrac{\Delta z(t, \Delta t)}{\Delta t} = x'(t) + iy'(t)$. Complex functions of a real variable are differentiated with the aid of the ordinary differentiation rules (see Sec. 5.1, subsection 1).

Example 8. Find the curve defined by the function $z = t^2 + it$, $t \in (-\infty, +\infty)$, and the derivative of this function.

If $z = x + iy$, then $x = t^2$, $y = t$. The required curve is a parabola $y^2 = x$. We fin the derivative of the given function $z' = 2t + i$.

In Problems 5.71 to 5.78 construct the curves given by the equations $z = z(t)$, and find $z'(t)$.

5.71*. $z = 1 - i + te^{i\frac{\pi}{4}}$, $t \in (-\infty, +\infty)$.

5.72. $z = 2e^{it}$, $t \in [0, \pi]$.

5.73. $z = 3e^{it} + e^{-it}$, $t \in (-\infty, +\infty)$.

5.74. $z = (2 + i)e^t + (2 - i)e^{-t}$, $t \in (-\infty, +\infty)$.

5.75. $z = t^2 + it^4$, $t \in (-\infty, +\infty)$.

5.76. $z = t + i - ie^{-it}$, $t \in [0, 2\pi]$.

5.77. $z = ae^{it}(1 - it)$, $a \in \mathbb{R}$, $t \in (-\infty, +\infty)$.

5.78. $z = e^{(\alpha+i\beta)t}$, $\alpha, \beta \in \mathbb{R}$, $t \in (-\infty, +\infty)$.

5.79 *. As is known, $z = z(t)$ defines the law of motion of a point in the plane. Find the components of velocity and acceleration in the direction of the tangent to the curve $z = z(t)$ and in the direction perpendicular to the tangent.

5.80*. A point z traverses the circle $|z| = R$ with a constant angular velocity equal to unity. Find the velocity vector of point w moving together with z according to the law $w = f(z)$.

SEC. 5.6.
NUMERICAL METHODS OF FUNCTIONS OF ONE VARIABLE

1. Numerical Solution of Equations. The root $\xi \in (a, b)$ of the equation $f(x) = 0$ *is isolated* on the closed interval $[a, b]$ if this interval contains no other roots of the indicated equation. In this case the interval $[a, b]$ is said to be a *root isolating interval.*

(1) *Method of chords.* Let on the interval $[a, b]$ of isolation of the root of the equation $f(x) = 0$ the following conditions be fulfilled:

(a) the functions $f(x)$, $f'(x)$, and $f''(x)$ are continuous;

(b) $f(a) \cdot f(b) < 0$;

(c) the functions $f'(x)$ and $f''(x)$ do not change sign.

Let us define the numbers x_n $(n = 1, 2, 3, \ldots)$ by the equalities

$$x_n = \begin{cases} x_{n-1} - \dfrac{(x_{n-1} - a) f(x_{n-1})}{f(x_{n-1}) - f(a)}, & x_0 = b, \text{ if } f(a) \cdot f(x_1) < 0, \\ x_{n-1} - \dfrac{(b - x_{n-1}) f(x_{n-1})}{f(b) - f(x_{n-1})}, & x_0 = a, \text{ if } f(a) \cdot f(x_1) \geqslant 0. \end{cases}$$

Then the sequence $(x_n)_{n \in \mathbb{N}}$ converges to the root ξ as $n \to$ and for all natural n the following inequalities are fulfilled:

$$|x_n - \xi| \leqslant \frac{|f(x_n)|}{m},$$

$$|x_n - \xi| \leqslant \frac{M - m}{m} |x_n - x_{n-1}|,$$

where $m = \min\limits_{a \leqslant x \leqslant b} |f'(x)|$ and $M = \max\limits_{a \leqslant x \leqslant b} |f'(x)|$.

Example 1. Find the roots of the equation

$$x \cdot \operatorname{arc\,tan} x - 1 = 0$$

with an accuracy to 0.0001, applying the method of chords.

On constructing the graphs of the functions $y = \operatorname{arc\,tan} x$ and $y = 1/x$, we conclude by the points of intersection of these curves that the indicated equation has two roots ξ_1 and ξ_2 equal by absolute value but having opposite signs. Let us find the positive root ξ_1 choosing the interval $[1, \sqrt{3}]$ for its isolation. For the function $f(x) = x \cdot \operatorname{arc\,tan} x - 1$ we have

$$f'(x) = \operatorname{arc\,tan} x + \frac{x}{1 + x^2}, \quad f''(x) = \frac{1}{(1 + x^2)^2}$$

and

$$f(1) \cdot f(\sqrt{3}) = \left(\frac{\pi}{4} - 1\right)\left(\frac{\pi}{\sqrt{3}} - 1\right) = -0.2146019 \cdot 0.8137992 < 0,$$

therefore the conditions (a), (b) and (c) are fulfilled. Since $f''(x) > 0$ for $x \in [1, \sqrt{3}]$, we have $m \leqslant f'(x) \leqslant M$, where

$$m = f'(1) = \frac{\pi}{4} + \frac{1}{2} = 1.2853981,$$

$$M = f'(\sqrt{3}) = \frac{\pi}{3} + \frac{\sqrt{3}}{4} = 1.4802102,$$

and $\frac{M-m}{m}=0.1515577$. To determine the sign of the product $f(1)\cdot f(x_1)$, let us find x_1. Since

$$x_1=1-\frac{(\sqrt{3}-1)f(1)}{f(\sqrt{3})-f(1)}=1.1527608,$$

the numbers x_n should be computed by the formula

$$x_n=x_{n-1}-\frac{(\sqrt{3}-x_{n-1})f(x_{n-1})}{f(\sqrt{3})-f(x_{n-1})}.$$

We then tabulate the results of computation:

n	x_{n-1}	$f(x_{n-1})$	$-(x_n-x_{n-1})$	x_n	$\frac{M-m}{m}(x_n-x_{n-1})$
1	2	−0.2146019	−0.1527608	1.1527608	0.023152
2	1.1527608	−0.0129601	−0.0090807	1.1618415	0.0013762
3	1.1618415	−0.0006758	−0.000473	1.1623145	0.0000716

The last column determines the limiting absolute error. Hence, $\xi_1 = 1.1623 \pm 0.0001$ and $\xi_2 = -1.1623 \pm 0.0001$.

(2) *Method of tangents.* Let on the interval $[a, b]$ for isolating the root ξ of the equation $f(x)=0$ the above indicated conditions (a), (b), (c) be fulfilled and let the numbers x_n $(n=1, 2, 3, \ldots)$ be determined by the equality

$$x_n=x_{n-1}-\frac{f(x_{n-1})}{f'(x_{n-1})},$$

and

$$x_0=\begin{cases} a \text{ if } f(a)\cdot f(c)<0, \\ b \text{ if } f(a)\cdot f(c)>0, \\ c \text{ if } f(c)=0, \end{cases}$$

where

$$c=a-\frac{(b-a)f(a)}{f(b)-f(a)}.$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to the root ξ as $n\to\infty$, and for all natural n the following inequalities are fulfilled:

$$|x_n-\xi|\leqslant\frac{|f(x_n)|}{m} \text{ and } |x_n-\xi|\leqslant\frac{M_1}{2m}(x_n-x_{n-1})^2,$$

where

$$m = \min_{a \leqslant x \leqslant b} |f'(x)|, \quad M_1 = \max_{a \leqslant x \leqslant b} |f''(x)|.$$

Example 2. Find the positive root of the equation $x \cdot \arctan x - 1 = 0$ to within 0.0001, using the method of tangents.

As in the preceding example, we take $[1, \sqrt{3}]$ for the root isolating interval. Since for the function $f(x) = x \cdot \arctan x - 1$ we have $c = 1 - \dfrac{(\sqrt{3} - 1) f(1)}{f(\sqrt{3}) - f(1)} = 1.1527608 > 0$, the numbers x_n are computed by the formula

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \qquad x_0 = \sqrt{3}.$$

The functions $f'(x)$, $f''(x)$ and the value $m = 1.2853981$ are found in Example 1. Further, $M_1 = f''(1) = 0.25$ because $f'''(x) = -\dfrac{4x}{(1 + x^2)^3} < 0$ on the isolation interval. Finally, $\dfrac{M_1}{2m} = 0.0972461$.

The results of computations are then tabulated:

n	x_{n-1}	$f(x_{n-1})$	$f'(x_{n-1})$	$-(x_n - x_{n-1})$	x_n	$\frac{M_1}{2m}(x_n - x_{n-1})^2$
1	1.7320508	0.8137992	1.4802102	0.5497862	1.1822646	0.0534645
2	1.1822646	0.0270628	1.3617976	0.0198728	1.1623918	0.0000384

Consequently, the root of the given equation is $\xi = 1.16239 \pm 0.00004$.

In Problems 6.1 to 6.5 make certain that the given equation has no real roots.

6.1. $2^x - x - 1/2 = 0$. **6.2.** $x^2 - \arctan x + 1 = 0$.

6.3. $(x^2 + 2x + 2)^2 = 0$. **6.4.** $\sqrt{2x - 1} + \log \dfrac{1}{x} = 0$.

6.5. $x^4 - x^2 + 1 = 0$.

6.6*. The root ξ of the function $f(x) = 0$ is isolated on the interval $[a, b]$, the function $f(x)$ is continuous, and $f(a) \cdot f(b) < 0$. Dividing successively one indicated interval, find the root isolating interval reduced by $\dfrac{1}{2^n}$ as compared with the original one.

6.7. Solve the equation $x^3 + x^2 - 3 = 0$, using a combined method: applying the method of chords and the method of tangents, and comparing the results obtained.

Constructing the graphs of the functions $y = x^3$ and $y = 3 - x^2$, we arrive at the conclusion that the given equation has one real root ξ on the interval [1, 2]. Let us reduce the isolation interval to $\frac{1}{4}$ of its length, using the half-division method. For $f(x) = x^3 + x^2 - 3$ we have $f(1) = -1 < 0$ and $f(2) = 9 > 0$. We find $f(1.5) = \frac{21}{8} > 0$, therefore [1, 1.5] is a more narrow isolation interval. Finding $f(1.25) = 0.515625 > 0$, we obtain the interval [1, 1.25]. Since

$$c = 1 - \frac{(1.25 - 1) f(1)}{f(1.25) - f(1)} = 1.1649484 > 0,$$

when applying the method of chords, it is necessary to use the formula

$$\bar{x}_n = \bar{x}_{n-1} - \frac{(1.25 - \bar{x}_{n-1}) f(\bar{x}_{n-1})}{f(1.25) - f(\bar{x}_{n-1})} \quad (n = 1, 2, 3, \ldots)$$

and, when applying the method of tangents, the formula

$$\tilde{x}_n = \tilde{x}_{n-1} - \frac{f(\tilde{x}_{n-1})}{f'(\tilde{x}_{n-1})} \quad (n = 1, 2, 3, \ldots), \quad \tilde{x}_0 = 1.25.$$

The results of computations are then tabulated as follows:

(a) for the methods of chords:

n	$\bar{x}_{n-1}$	$f(\bar{x}_{n-1})$	$-(\bar{x}_n - \bar{x}_{n-1})$	$\bar{x}_n$
1	1	−1	−0.1649484	1.1649484
2	1.1649484	−0.0619384	−0.0091209	1.1740693
3	1.1740693	−0.0031786	−0.0004651	1.1745344

(b) for the method of tangents:

n	$\tilde{x}_{n-1}$	$f(\tilde{x}_{n-1})$	$f'(\tilde{x}_{n-1})$	$-(\tilde{x}_n - \tilde{x}_{n-1})$	$\tilde{x}_n$
1	1.25	0.515625	7.1875	0.0717391	1.1782609
2	1.1782609	0.0240767	6.5214179	0.0036919	1.1745690

When computing by the method of chords, we obtained an increasing sequence $(\bar{x}_n)$ of approximations of the root ξ:

$$1 < 1.1649484 < 1.1740693 < 1.1745344 < \ldots < \xi,$$

and when applying the method of tangents—a decreasing sequence $(\tilde{x}_n)$:

$$\xi < \ldots < 1.1745690 < 1.1782609 < 1.25.$$

The coinciding digits of the terms in both sequences are correct digits for the root ξ. Given the limiting absolute error ε, the value n for which the desired accuracy is obtained is found from the inequality

$$|\bar{x}_n - \tilde{x}_n| < \varepsilon,$$

where $\xi = \frac{1}{2}(\bar{x}_n + \tilde{x}_n) \pm \varepsilon$. Thus,

$$\xi = 1.17455 \pm 0.00003.$$

In Problems 6.8 to 6.37 compute to within 0.0001 the real roots of the given equations, applying one of the indicated methods: (a) method of chords, (b) method of tangents, (c) combined method.

6.8. $x^3 + 2x - 8 = 0$. **6.9.** $x^3 + x + 1 = 0$.

6.10. $x^4 - 3x^2 + 4x - 1 = 0$. **6.11.** $x^3 + 2x - 30 = 0$.

6.12. $x^6 - 3x^2 + x - 1 = 0$. **6.13.** $x^3 - 2x - 5 = 0$.

6.14. $x^3 - 5x + 1 = 0$. **6.15.** $2x^3 - 5x^2 + 7x - 2 = 0$.

6.16. $(x + 1)^3 - x = 0$. **6.17.** $x^4 - 2x - 2 = 0$.

6.18. $x^4 - 4x + 1 = 0$. **6.19.** $x^5 + x + 1 = 0$.

6.20. $x = \sqrt[3]{5 - x}$. **6.21.** $x = 2 + \sqrt[4]{x}$.

6.22. $x^3 + 60x - 80 = 0$. **6.23.** $x^5 - x - 2 = 0$.

6.24. $x = 10 \log x$. **6.25.** $x = 2 - \log x$.

6.26. $x^2 = -\ln x$. **6.27.** $x^2 = \ln (x + 1)$.

6.28. $4x = 2^x$. **6.29.** $x^2 = e^x + 2$.

6.30. $x + \sin x - 1 = 0$. **6.31.** $x - \cos x = 0$.

6.32. $x^2 = \cos x$. **6.33.** $x = \arctan \sqrt[3]{x}$.

6.34. $\ln x = \arctan x$. **6.35.** $x^2 + \ln x - 4 = 0$.

6.36. $x^2 \cdot \arctan - 1 = 0$. **6.37*.** $e^{x-2} - x = 0$.

2. **Interpolation of Functions.** Let a function $y = f(x)$ attain the values $f(x_k)$ y_k at the *points of interpolation* $x_k \in [a, b]$, $k = 0, 1, \ldots, n$ then the *divided differences* are defined by the equalities

$$\Delta y(x_k, x_{k+1}) = \frac{y_k - y_{k+1}}{x_k - x_{k+1}},$$

$$\Delta y(x_k, x_{k+1}, x_{k+2}) = \frac{\Delta y(x_k, x_{k+1}) - \Delta y(x_{k+1}, x_{k+2})}{x_k - x_{k+2}},$$

. . . . . .

$$\Delta y(x_k, x_{k+1}, \ldots, x_{k+l-1}, x_{k+1}) = \frac{\Delta y(x_k, x_{k+1}, \ldots, x_{k+l-1}) - \Delta y(x_{k+1}, \ldots, x_{k+l})}{x_k - x_{k+l}},$$

and the *interpolation polynomial* of the function $f(x)$ on the interval $[a, b]$ has the form

$$p_n(x) = y_0 + \sum_{k=1}^{n} (x-x_0)(x-x_1)\ldots(x-x_{k-1})\,\Delta y(x_0, x_1, \ldots, x_k); \quad (1)$$

and if there exists a continuous derivative $f^{(n+1)}(x)$ on $[a, b]$ then the equality

$$|f(x) - p_n(x)| \leqslant \frac{M_{n+1}}{(n+1)!}\left|\prod_{k=0}^{n}(x-x_k)\right|, \quad (2)$$

is fulfilled, where

$$M_{n+1} = \max_{a \leqslant x \leqslant b} |f^{(n+1)}(x)|.$$

Example 3. Find $\sqrt{2}$ to within 0.0001, by constructing an interpolation polynomial for the function $f(x) = \sqrt{x}$ on the interval [1.69, 2.25].

Let us choose $n = 2$ and the interpolation points $x_0 = 1.69$, $x_1 = 1.96$, $x_2 = 2.25$. We estimate the accuracy by formula (2). Since $f^{(IV)}(x) = -\frac{15}{16}x^{-7/2} < 0$, the function $f'''(x) = \frac{3}{8}x^{-5/2}$ is decreasing on the interval $I = [1.69, 2.25]$; therefore

$$M_3 = \max_{x \in I} f'''(x) = f'''(1.69) = \frac{3}{8} \cdot \frac{1}{(1.69)^2 \cdot 1.3} = 0.1009984.$$

Then for the difference $r_2(x) = f(x) - p_2(x)$ we shall obtain the inequality

$$|r_2(x)| < \frac{M_3}{3!}|(x-1.69)(x-1.96)(x-2.25)|,$$

whence it follows the fulfilment of the inequality

$$|r_2(x)| < \frac{0.1009984}{6} 0.31 \cdot 0.04 \cdot 0.25 = 0.0000521$$

and the preassigned accuracy is achieved.

Let us find the coefficients of the interpolation polynomial, computing the divided differences and tabulating the computation results:

k	x_k	y_k	$\Delta y(x_k, x_{k+1})$	$\Delta y(x_k, x_{k+1}, x_{k+2})$
0	1.69	1.3	$\frac{1.3-1.4}{1.69-1.96} = 0.3703703$	$\frac{0.3703703-0.3448275}{1.69-2.25}$
1	1.96	1.4	$\frac{1.4-1.5}{1.96-2.25} = 0.3448275$	$= -0.0456121$
2	2.25	1.5		

The polynomial has the form

$$p_2(x) = 1.3 + 0.3703703\,(x - 1.69) - 0.0456121\,(x - 1.69) \times (x - 1.96),$$

$$p_2(2) = 1.3 + 0.3703703 \cdot 0.31 - 0.0456121 \cdot 0.31 \cdot 0.04 = 1.3 + 0.1148147 - 0.0005655 = 1.4142492.$$

Hence

$$\sqrt{2} = 1.4142 \pm 0.0001.$$

The *final differences* $\Delta^k y_i$ ($k = 1, 2, \ldots$; $i = 0, 1, 2, \ldots$) are defined by the equalities

$$\Delta^1 y_i = \Delta y_i = y_{i+1} - y_i,$$
$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i,$$
$$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i.$$

For *equally spaced nodes* $x_k = x_0 + kh$ ($k = 0, 1, \ldots, n$) with an interpolation step $h > 0$ interpolation polynomial (1) takes the form

$$p_n(x) = y_0 + \sum_{k=0}^{n} \frac{t(t-1)\ \ldots\ (t-k+1)}{k!} \Delta^k y_0, \tag{3}$$

where $t = \frac{x - x_0}{h}$ and $\Delta^k y_0$ are finite differences of the kth order, and inequality (2) takes the form

$$|f(x) - p_n(x)| \leqslant \frac{M_{n+1}}{(n+1)!} h^{n+1} \left| \prod_{k=0}^{n} (t-k) \right|, \tag{4}$$

Example 4. The function $y = f(x)$ is given by the table

x	1.0	1.1	1.2	1.3
y	2.7854	2.8330	2.8761	2.9151

Find the analytical expression by which the given function can be represented on the interval [1, 1.3] and compute $f(1.15)$.

The analytical expression enabling us to compute the values of the function $f(x)$ not given in the table will be sought for in the form of a polynomial whose values coincide with the given values of the function, i.e. in the form of the polynomial $p_3(x)$ satisfying the relations $p_3(x_k) = f(x_k)$ for $k = 0, 1, 2, 3$. The only polynomial with such properties is the interpolation polynomial $p_3(x)$ defined by equality (3). We find the finite differences, tabulating the results of the computations as follows:

k	x_k	y_k	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$
0	1	2.7854			
			0.0476		
1	1.1	2.8330		−0.0045	
			0.0431		0.0004
2	1.2	2.8761		−0.0041	
			0.0390		
3	1.3	2.9151			

Applying formula (3) for $h = 0.1$, $n = 3$, and $x_0 = 1$, we obtain

$$p_3(x) = 2.7854 + 0.476\,(x - 1) - 0.225\,(x - 1)\,(x - 1.1) + 0.0666\,(x - 1)\,(x - 1.1)\,(x - 1.2).$$

Then

$$p_3(1.5) = 2.7854 + 0.476 \cdot 0.15 - 0.225 \cdot 0.15 \cdot 0.05 + 0.0666 \cdot 0.15 \cdot 0.05\,(-0.05) = 2.7854 + 0.0714 - 0.0017 + 0.0000 = 2.8551.$$

To compute $f(1.15)$, let us note that $f(1.15) = p_3(1.15)$, and, if the derivative $f^{(n+1)}(x)$ is unknown, the modulus of the last of the terms entering into sum (3) is regarded as the limiting absolute error of the equality $f(x) = p_n(x)$. Therefore $f(1.15) = 2.8551$.

6.38. Prove the equality

$$\Delta^k y_i = \sum_{\nu=0}^{k} C_k^\nu (-1)^\nu y_{k+i-\nu},$$

where

$$C_k^\nu = \frac{k!}{\nu!\,(k-\nu)!}, \quad 0! = 1.$$

6.39*. Prove the equality

$$\Delta y\,(x_1,\ \ldots,\ x_k) = \sum_{\nu=1}^{k} \frac{y_\nu}{w_k'(x_\nu)},$$

where

$$w_k = \prod_{i=1}^{k} (x - x_i).$$

6.40. For the function $f(x) = \cos \frac{\pi}{12} x$ construct the interpolation polynomial taking $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ for interpolation points. Find $\cos \frac{\pi}{10}$ and estimate the accuracy of the computation.

6.41. For the function $f(x) = \ln x$ construct the interpolation polynomial choosing the points $x_0 = 9$, $x_1 = 10$, $x_2 = 12$, $x_3 = 15$ and using the values $\ln 2 = 0.693147$, $\ln 3 = 1.098613$, and $\ln 5 = 1.609438$. Find $\ln 11$ and estimate the accuracy of the computation.

In Problems 6.42 to 6.51 the function $y = f(x)$ is represented in a tabular form. Find the values of this function for the indicated values x_1 and x_2 of the argument x which are not found in the given table.

6.42.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
y	1.042	1.061	1.087	1.119	1.160	1.212	1.274	1.350

$x_1 = 1.26$, $x_2 = 1.58$.

6.43.

x	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5
y	1.958	2.107	2.268	2.443	2.632	2.841	3.071	3.324

$x_1 = 1.89$, $x_2 = 2.43$.

6.44.

x	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10
y	0.742	0.789	0.835	0.880	0.924	0.967	1.008	1.046

$x_1 = 0.83$, $x_2 = 0.97$.

6.45.

x	1.70	1.75	1.80	1.85	1.90	1.95	2.00	2.05
y	1.2322	1.2097	1.1789	1.1389	1.0888	1.0281	0.9558	0.8713

$x_1 = 1.74$, $x_2 = 1.97$.

6.46.

x	2.70	2.75	2.80	2.85	2.90	2.95	3.00	3.05
y	1.5827	1.4865	1.3721	1.2383	1.0838	0.9071	0.7069	0.4817

$x_1 = 2.72$, $x_2 = 2.93$.

6.47.

x	10	15	20	25	30	35	40	45
y	0.985	0.966	0.940	0.906	0.866	0.819	0.766	0.707

$x_1 = 23$, $x_2 = 41$.

6.48.

x	1.1	1.6	2.1	2.6	3.1	3.6	4.1	4.6
y	1.029	1.389	1.649	1.800	1.852	1.822	1.739	1.632

$x_1 = 1.3$, $x_2 = 4.0$.

6.49.

x	0.13	0.18	0.23	0.28	0.33	0.38	0.43	0.48
y	0.1296	0.1790	0.2280	0.2764	0.3242	0.3712	0.4173	0.4626

$x_1 = 0.20$, $x_2 = 0.41$.

6.50.

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	0.1198	0.0897	0.0660	0.0477	0.0339	0.0236	0.0162	0.0109

$x_1 = 1.25$, $x_2 = 1.76$.

6.51.

x	50	55	60	65	70	75	80	85
y	0.285	0.319	0.223	0.042	−0.148	−0.273	−0.283	−0.178

$x_1 = 58$, $x_2 = 79$.

6.52. Compute the values of the *sine integral* $\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}\,dt$ for $x = 0.26$ and $x = 0.45$ using the table of its values:

x	0.17	0.22	0.27	0.32	0.37	0.42	0.47	0.52
$\mathrm{Si}\,(x)$	0.16973	0.21941	0.26891	0.31819	0.36720	0.41591	0.46427	0.51225

6.53. Compute the values of the *probability integral*

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \text{ for } x = 0.27 \text{ and } x = 0.58$$

using the table of its values:

x	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75
$\Phi(x)$	0.05637	0.16800	0.27633	0.37938	0.47548	0.56332	0.64203	0.71116

6.54. Making use of interpolation, solve the equation

$$x \cdot \ln x - 1 = 0.$$

On the root isolating interval $I = [1.6, 1.9]$ we have for the function $y = x \ln x - 1$:

x	1.6	1.7	1.8	1.9
y	−0.2479952	−0.0979324	0.0580148	0.2195226

On the interval I the function $y = x \ln x - 1$ is increasing, since $y' = \ln x + 1 > 0$ for $x \in I$. Consequently, there exists an inverse function $x = \varphi(y)$ for which we shall construct the interpolation polynomial $x_3(y)$, regarding now y as the argument and x as the value of the function. This technique is known as *inverse interpolation*. The results of computation are then tabulated:

k	y	x	$\Delta x\,(y_k, y_{k+1})$	$\Delta x\,(y_k, y_{k+1}, y_{k+2})$	$\Delta x\,(y_k, y_{k+1}, y_{k+2}, y_{k+3})$
0	−0.2479952	1.6			
1	−0.0979324	1.7	0.6663876	−0.0821705	0.0270049
2	0.0580148	1.8	0.6412426	−0.0695452	
3	0.2195226	1.9	0.6191651		

Hence, the desired polynomial has the form

$$x_3(y) = 1.6 + 0.6663876\,(y + 0.2479952) - 0.0821705\,(y + 0.2479952)\,(y + 0.0979324) + 0.0270049\,(y + 0.2479952)\,(y + 0.0979324) \times (y - 0.0580148).$$

Setting $y = 0$, we obtain the root

$$x_3(0) = 1.6 + 0.1652609 - 0.0019956 - 0.000038 - 1.7632273.$$

Consequently, the root is equal to 1.76323 ± 0.00004, where the limiting absolute error is set to be equal to the absolute value of the last term in the expression for $x_3(0)$.

6.55. Using the table of values of the function $y = f(x)$, find the value x_0 for which $f(x_0) = 0.569$:

x	1.50	1.55	1.60	1.65	1.70	1.75	1.80	1.85
y	−1.125	−0.926	−0.704	−0.458	−0.187	0.109	0.432	0.782

6.56. Using the table of values of the function $y = f(x)$, find the value x_0 for which $f(x_0) = 4.498$:

x	1.1	1.2	1.3	1.4	1.5	1.6
y	2.431	2.928	3.497	4.144	4.875	5.696

6.57. Using the given table, solve the equation $\sinh x = 4.9370$, applying the method of inverse interpolation:

x	2	2.2	2.4	2.6
y	3.6269	4.4571	5.4662	6.6947

6.58. Using the given table, solve the equation $\tan x = 1.767$, applying the method of inverse interpolation:

x	60°	61°	62°
y	1.732	1.804	1.881

3. Numerical Differentiation. The formulas for numerical differential are obtained as a result of differentiation of the interpolation formulas:

$$f'(x) \approx p_n'(x) = \Delta y(x_0, x_1) + ((x - x_0) + (x - x_1)) \times \Delta y(x_0, x_1, x_2) + ((x - x_0)(x - x_1) + (x - x_0)(x - x_2) + (x - x_1)(x - x_2)) \times \Delta y(x_0, x_1, x_2, x_3) + \ldots,$$

the error of the approximate equality $f'(x) = p_n'(x)$ being equal to the derivative of the error $r_n(x) = f(x) - p_n(x)$.

In the case of the equally spaced interpolation points $x_k = x_{k-1} + h$ $(k = 1, \ldots, n)$, $x_k \in [a, b]$ and $f(x_k) = y_k$ the following relations hold true:

$$f'(x) \approx \frac{1}{h}\left(\Delta y_0 + \frac{2t-1}{2}\Delta^2 y_0 + \frac{3t^2-6t+2}{6}\Delta^3 y_0 + \frac{2t^3-9t^2+11t-3}{12}\Delta^4 y_0 + \ldots\right), \quad (5)$$

$$f''(x) \approx \frac{1}{h^2}\left(\Delta^2 y_0 + (t-1)\Delta^3 y_0 + \frac{6t^2-18t+11}{12}\Delta^4 y_0 + \ldots\right), \quad (6)$$

where $t = \frac{1}{h}(x - x_0)$. Formulas (5) and (6) contain n and $n-1$ terms, respectively.

Example 5. A material point M performs rectilinear motion. The law of this motion $s = f(\tau)$ is given with the aid of the below table (τ—time in seconds, s—path in metres):

τ	0	1	2	3	4	5	6
s	0	2	10	30	68	130	222

Find the velocity v and acceleration w of the point M at the instant of time $\tau = 3.5$.

We first compile the table of finite differences of the function $s = f(\tau)$:

τ	s	Δs	$\Delta^2 s$	$\Delta^3 s$	$\Delta^4 s$
0	0				
		2			
1	2		6		
		8		6	
2	10		12		0
		20		6	
3	30		18		0
		38		6	
4	68		24		0
		62		6	
5	130		30		
		92			
6	222				

Taking $\tau = 3$ (the nearest to $\tau = 3.5$) for the starting instant, we have $t = \frac{3.5 - 3}{1} = 0.5$. Applying formulas (5) and (6), we obtain:

$$v = f'(3.5) = \frac{1}{1}\left(38 + \frac{2 \cdot 0.5 - 1}{2} 24 + \frac{3 \cdot (0.5)^2 - 6 \cdot 05 + 2}{6} \cdot 6\right) = 37.75 \text{ m/s},$$

$$w = f''(3.5) = \frac{1}{1^2}\left(24 + (0.5 - 1) \cdot 6 + \frac{6 \cdot (0.5)^2 - 18 \cdot 0.5 + 11}{12} \cdot 0\right) = 21 \text{ m/s}^2.$$

In Problems 6.59 to 6.63 the function $f(x)$ is given in a tabular form. Evaluate the derivative $f'(x)$ at the two indicated points x_1 and x_2:

6.59.

x	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5
$(f)\,x$	1.44013	1.54722	1.67302	1.81973	1.98970	2.18547	2.40978	2.66557

$x_1 = 2.03$, $x_2 = 2.22$.

6.60.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$(f)\,x$	1.0083	1.1134	1.2208	1.3310	1.4449	1.5634	1.6876	1.8186

$x_1 = 1.14$, $x_2 = 1.42$.

6.61.

x	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5
$(f)\,x$	3.92847	4.41016	4.93838	5.51744	6.15213	6.84782	7.61045	8.44671

$x_1 = 3.02$, $x_2 = 3.31$.

6.62.

x	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10
$(f)\,x$	0.2803	0.3186	0.3592	0.4021	0.4472	0.4945	0.5438	0.5952

$x_1 = 0.82$, $x_2 = 1.03$.

6.63.

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$(f)\,x$	0.8802	0.9103	0.9340	0.9523	0.9661	0.9764	0.9838	0.9891

$x_1 = 1.34$, $x_2 = 1.65$.

In Problems 6.64 and 6.65 evaluate $f'(x)$ and $f''(x)$ at the indicated point:

6.64.

x	1	2	3	4	5	6
$(f)\,x$	1	5	21	55	113	201

$x = 2$.

6.65.

x	0	1	2	3	4	5
$(f)\,x$	1	3	19	85	261	631

$x = 2.5$.

Answers

1.1. 0.331. **1.2.** 0.5. **1.3.** -1. **1.5.** $(\Delta x)^2 - 2\Delta x$. **1.6.** $e\,(e^{\Delta x} - 1)$. **1.7.** $\log_2 (1 + \Delta x)$. **1.9.** $-\frac{2}{x^3}$. **1.10.** $\frac{1}{2\sqrt{x}}$. **1.11.** $2^x \ln 2$. **1.12.** $\frac{1}{x}\log_2 e$. **1.14.** *Hint.* Take advantage of the identity: $xf(x_0) - x_0 f(x) = (xf(x_0) - x_0 f(x)) - (x_0 f(x) - x_0 f(x_0))$. **1.15.** $f'_-(-1) = -2$, $f'_+(-1) = f'_-(1) = 0$, $f'_+(1) = 2$. **1.17.** $f'_-(0) = f'_+(0) = 0$. **1.18.** $f'_-(0) = -1$, $f'_+(0) = 1$. **1.19.** $f'_-(0) = 1$, $f'_+(0) = 0$. **1.20.** *Hint.* The function $y = \sin\frac{1}{x}$ has no limit as $x \to 0$. **1.21.** $-2 + \frac{8}{3}x^3$. **1.22.** $-\frac{25x^4}{a^2}$. **1.23.** $-\frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4}$. **1.24.** $\frac{2}{(x+1)^2}$. **1.25.** $\frac{1-4x^2-x^4}{(x^3-x)^2}$. **1.26.** $2x\,(3x^4 + 12x^2 - 31)$. **1.27.** $\frac{6}{\sqrt{2\pi}}x$. **1.28.** $-\frac{3(x^2+1)}{(x^3+3x-1)^2}$. **1.29.** $-\frac{3}{5}a\frac{1}{\sqrt[5]{x^8}} + \frac{2}{3b\sqrt[5]{x}}$. **1.30.** $\frac{bc-ad}{(c+dx)^2}$. **1.31.** $\frac{1-4x}{x^2(2x-1)^2}$. **1.32.** $\frac{2}{\sqrt{x}(2-\sqrt{x})^2}$.

1.33. $\frac{x+1}{2x\sqrt{x}}$. 1.34. $\frac{2}{\sqrt[3]{x}}-\frac{1}{2\sqrt[4]{x}}$. 1.35. $2\sqrt[3]{x^2}(3\sqrt[3]{x}+5)$.

1.36. $\frac{1}{x}\left(\frac{2}{\sqrt[3]{x^2}}-\frac{3}{\sqrt[4]{x^3}}\right)$. 1.37. $3x^2\cot x-\frac{x^3}{\sin^2 x}=\frac{x^2(3\sin 2x-2x)}{2\sin^2 x}$.

1.38. $\frac{1}{\sqrt[3]{x^2}\cos^2 x}-\frac{2\tan x}{\sqrt[3]{x^5}}=\frac{x-\sin 2x}{x\sqrt[3]{x^2}\cos^2 x}$. 1.39. $-\frac{1}{1+\sin x}$.

1.40. $\frac{1}{2\sqrt{x}}\sin x+\sqrt{x}\cos x$.

1.41. $\frac{5}{3}\sqrt[3]{x^2}(2\ln x-3^x)+\sqrt[3]{x^5}\left(\frac{2}{x}-3^x\ln 3\right)$.

1.42. $x^2\left(3\log_2 x+\frac{1}{\ln 2}\right)+\frac{2x-x^2}{e^x}$. 1.43. $\frac{2\cos^3 x-3}{\cos^2 x}$.

1.44. $\frac{1}{\sin^2\left(x+\frac{\pi}{4}\right)}$. 1.45. $\frac{\sqrt{x}(19x^5+9a)}{6\sqrt[3]{(x^5+a)^2}}$.

1.46. $-\frac{2x}{(1+x^2)\sqrt{1-x^4}}$. 1.47. $\frac{3}{2}\cos\frac{3x}{2}$. 1.48. $-4\sin\frac{2x}{3}$.

1.49. $24x(1+4x^2)^2$. 1.50. $\frac{9x}{2\sqrt[4]{1+3x^2}}$. 1.51. $\frac{1}{2}\sin x$

1.52. $\frac{2\cos^4 x}{|\cos 4x|}\left(\sqrt{1-\sin^4 x}+\sqrt{1+\sin 4x}\right)$.

1.53. $\arcsin\ln x+\frac{1}{\sqrt{1-(\ln x)^2}}$. 1.54. $\frac{1}{2}\cos x$.

1.55. $\frac{3\sin 2x}{4\sqrt[4]{1+\sin^2 x}}$. 1.56. $2xe^{-2x}(1-x)$.

1.57. $\frac{1}{3}e^{\frac{x}{3}}\left(\cos^2\frac{x}{3}-\sin\frac{2x}{3}\right)$. 1.58. $\sqrt{x^2+a}$. 1.59. $\frac{1}{\cos x}$.

1.60. $\frac{2x}{1-x^4}$. 1.61. $-\frac{1}{(x^2+4)\sqrt{\operatorname{arc}\cot\frac{x}{2}}}$.

1.62. $\frac{x^2-1}{3x^2\cos^2\left(x+\frac{1}{x}\right)\sqrt[3]{\left(1+\tan\left(x+\frac{1}{x}\right)\right)^2}}$.

1.63. $-\frac{1}{3}\cos\frac{x}{3}\sin\left(2\sin\frac{x}{3}\right)$. 1.64. $\frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$.

1.65. $\frac{1}{2(1+x^2)}$. **1.66.** $\frac{\sqrt{a^2-b^2}}{a+b\cos x}\operatorname{sgn}(\sin x)$.

1.67. $\frac{e^{x/2}}{2\sqrt{x}}(1+x)$. **1.68.** $-e^{-x^2}\frac{1+2x^2}{2x^2}$.

1.69. $2^{\frac{x}{\ln x}}\cdot\frac{\ln x-1}{\ln^2 x}\ln 2$. **1.70.** $2^{\sqrt{\sin^2 x}}\ln 2\cdot\cos x\cdot\operatorname{sgn}(\sin x)$.

1.71. $3^{2^x}\cdot 2^x\ln 3\cdot\ln 2$. **1.72.** $\frac{1}{x}\log\left(\frac{x^2}{19}\right)$. **1.73.** $\frac{1}{x^{\ln 2\cdot\ln 2x}}$.

1.74. $\frac{e^{\sqrt{\ln(ax^2+bx+c)}}(2ax+b)}{2\sqrt{\ln(ax^2+bx+c)}\,(ax^2+bx+c)}$.

1.75. $\frac{x}{(2+x^2)\sqrt{1+x^2}\operatorname{arc\,tan}\sqrt{1+x^2}}$. **1.76.** $\frac{1}{\sqrt{a^2+x^2}}$.

1.77. $\cosh x$. **1.78.** $\sinh x$. **1.79.** $\frac{1}{\cosh^2 x}$. **1.80.** $-\frac{1}{\sinh^2 x}$.

1.81. $\frac{(x-3)(19x-17)}{(x+1)^4}$. **1.82.** $\frac{10-2x-2x^2}{3x^2\sqrt[3]{x^2(x+2)^2(x-1)}}$.

1.83. $-\frac{2x^2+9x+1}{2\sqrt{x+2}\sqrt[3]{(x-1)^5(2x+1)^4}}$.

1.84. $\frac{11x^5-7x^4-58x^3+48x^2}{4\sqrt{x-1}\sqrt{(x+2)^3}\sqrt[4]{(x-5)^5}}$. **1.85.** $x^x(\ln x-1)$.

1.86. $x^{2x}\cdot 2^x\left(\frac{1}{x}+\ln x\cdot\ln 2\right)$. **1.87.** $(\sqrt{x})^{\sqrt[3]{x}}\frac{3+\ln x}{6\sqrt[3]{x^2}}$.

1.88. $(\ln x)^{\frac{1}{x}}\frac{1-\ln x\cdot\ln\ln x}{x^2\ln x}$.

1.89. $(\sin x)^{\operatorname{arc\,sin} x}\left(\frac{\ln\sin x}{\sqrt{1-x^2}}+\operatorname{arc\,sin} x\cdot\cot x\right)$.

1.90. $x^{x^{\infty}}\cdot x^{x-1}(1+x\ln x(\ln x-1))$.

1.91. $\frac{(\ln x)^x}{x^{\ln x}}\left(\ln\ln x+\frac{1}{\ln x}-\frac{2\ln x}{x}\right)$.

1.92. $x^{x^2+1}(1+\ln x^2)+x^{2x}\cdot 2^x\left(\frac{1}{x}+\ln 2\cdot\ln x\right)+2^{x^x}\ln 2\cdot x^x(\ln x+1)$, $x>0$. *Hint.* Find the derivative of each term.

1.93. $-\sin 2x\,\frac{1+2\sqrt{1+2\cos^2 x}}{2\sqrt{1+\cos^2 x}\,(\cos^2 x+\sqrt{1+\cos^2 x})}$.

Hint. Take $u = \cos^2 x$ for an intermediate variable and then take advantage of the rule for differentiating a composite function.

1.94. $-\dfrac{\arccos x}{\sqrt{1-x^2}}(2\ln \arccos x + 1)$.

1.95. $-2xe^{-x^2}\dfrac{\arcsin e^{-x^2} + e^{-x^2}(1-e^{-2x^2})^{1/2}}{(1-e^{-2x^2})^{3/2}}$.

1.96. $-\dfrac{a^{-x}\ln a}{(1+a^{-2x})^2}(4a^{-x}\arctan a^{-x} + a^{-2x} - 1)$.

1.97. $a = 2$, $b = 0$. *Hint.* The conditions of continuity $\lim\limits_{x\to -0} f(x) = \lim\limits_{x\to +0} f(x)$ and differentiability $f'_-(0) = f'_+(0)$ form a system of two equations with respect to a and b. **1.98.** $a = -\dfrac{1}{2}$, $b = \dfrac{3}{2}$.

1.99. $\dfrac{2x^2}{\sqrt[3]{(1-x^6)^2(1-x^3)^2}}$.

1.100. $\dfrac{1+2\sqrt{x}+4\sqrt{x}\sqrt{x+\sqrt{x}}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$.

1.101. $\dfrac{1}{m+n}\left(n\left(\dfrac{1-x}{1+x}\right)^{\frac{m}{m+n}} - m\left(\dfrac{1+x}{1-x}\right)^{\frac{n}{m+n}}\right)$.

1.102. $-\sin 2x \cos(\cos 2x)$. **1.103.** $\dfrac{m\sin mx}{\cos^{n+1} mx} = \dfrac{m\tan mx}{\cos^n mx}$.

1.104. $\left(\dfrac{a}{b}\right)^x\left(\dfrac{b}{x}\right)^a\left(\dfrac{a}{x}\right)^b\left(\dfrac{b-a}{x} + \ln\dfrac{a}{b}\right)$, $x > 0$. **1.105.** $\dfrac{n}{x\ln mx}$.

1.106. $\dfrac{1}{3x^2-2}$, $|x| < \sqrt{\dfrac{2}{3}}$. **1.107.** $2\pi \log_2 e \cot\left(2\pi x + \dfrac{\pi}{2}\right)$.

1.108. $\dfrac{2\tan x(1+\tan^2 x)}{1+\tan^4 x}$. **1.109.** $-\dfrac{1}{x\ln^2 x}$.

1.110. $(\sin x)^{\cos x}(\cot x \cos x - \sin x \ln \sin x)$.

1.111. $\dfrac{1}{2}(\sqrt{x})^{\sin^2 x}\left(\sin 2x \ln x + \dfrac{\sin^2 x}{x}\right)$.

1.112. $-\dfrac{\sin x}{2\sqrt{\cos x}}\, a^{\sqrt{\cos x}}(1 + \sqrt{\cos x}\ln a)$.

1.113. $\tanh^3 x\left(1 + \dfrac{1}{2\sinh x}\right)$. **1.114.** $\dfrac{1}{\cosh 2x}$.

1.115. $e^{-x}(a\cosh ax - \sinh ax)$. **1.116.** $\dfrac{\operatorname{sgn}(\sinh x)}{\cosh x}$. **1.118.** $-\dfrac{1}{x\sqrt{x^2-1}}$.

1.119. $(\cos x - \pi k)$ if $x \in (\pi k, \pi(k+1))$, and if $x = \pi k$ then $y'_-(\pi k) =$

$= -1$, $y'_+(\pi k) = 1$, $k \in \mathbb{Z}$. **1.120.** $\frac{1}{1+x^2}$, $x > 0$, $-\frac{1}{1+x^2}$, $x < 0$; $y'_-(0) = -1$, $y'_+(0) = +1$. **1.122.** -1, $x \leqslant 0$; $-e^{-x}$, $x > 0$. **1.123.** 1, $x \leqslant 0$; $\frac{1}{1+x}$, $x > 0$. **1.124.** 0, $|x| \geqslant 1$; $2xe^{-x^2}(1-x^2)$, $|x| < 1$.

1.125. $\frac{2x^4+4x^3-36x^2+54}{3(1-x)^2(9-x^2)}\sqrt{\frac{3-x}{(3+x)^2}}$.

1.126. $y \sum_{i=1}^{n} \frac{\alpha_i}{x-a_i}$. **1.127.** $a^{x^a} \cdot x^{a-1} \cdot a \ln a$.

1.128. $(\log_x a)^x \left(-\frac{1}{\ln x} - \ln \log_a x\right)$.

1.129. $\cos x \cos(\sin x) \cos(\sin(\sin x))$. **1.130.** $\left(\frac{1}{x}\right)^{\frac{1}{x}} \frac{\ln x - 1}{x^2}$.

1.131. $-\frac{\sin x}{3^x}(1+\ln^2 3)$.

1.132. $\frac{a\cos ax \cos bx + b \sin ax \sin bx}{\cos^2 bx}\left(3^{\frac{\sin ax}{\cos bx}} \ln 3 + \frac{\sin^2 ax}{\cos^2 bx}\right)$.

1.135. $\varphi(x_0)$. *Hint.* Take advantage of the definition of the derivative. **1.136.** $\frac{\varphi(x)\varphi'(x)+\psi(x)\psi'(x)}{\sqrt{\varphi^2(x)+\psi^2(x)}}$. **1.137.** $\frac{\varphi'(x)\psi(x)-\varphi(x)\psi'(x)}{\varphi^2(x)+\psi^2(x)}$.

1.138. $\psi(x)^{\varphi(x)}\left(\varphi'(x)\ln\psi(x) + \frac{\varphi(x)\psi'(x)}{\psi(x)}\right)$. **1.140.** $\frac{f'(\ln x)}{x}$.

1.141. $\frac{f'(x)}{f(x)}$. **1.143.** $f'(f(x))f'(x)$. **1.144.** $\frac{4}{3}$. **1.145.** $-\frac{1}{e}$.

1.146. $-\frac{b^2x}{a^2y}$. **1.147.** $\frac{x(y^2-2x^2)}{y(2y^2-x^2)}$. **1.148.** $-\sqrt{\frac{y}{x}}$. **1.149.** $\frac{1}{2(1+\ln y)}$.

1.150. $\frac{e^x \sin y + e^y \sin x}{e^y \cos x - e^x \cos y}$. **1.151.** $-\frac{y}{x}$. **1.152.** $\frac{2^x(2^y-1)}{2^y(1-2^x)} = -2^{y-x}$.

1.153. $\sqrt{\frac{1-y^2}{1-x^2}} \cdot \frac{1-\sqrt{1-x^2}}{1-\sqrt{1-y^2}}$. **1.154.** $\frac{x+y}{x-y}$. **1.155.** $\frac{y}{x}\frac{1-x^2-y^2}{1+x^2+y^2}$.

1.156. $\frac{y}{x} \cdot \frac{x\ln y - y}{y\ln x - x}$. **1.157.** $\frac{y}{x}$. **1.160.** $\pm\frac{1}{\sqrt{x^2-1}}$. *Hint.* The function $y = \cosh x$, $x \in (-\infty, +\infty)$, has no inverse, therefore one should consider two intervals $(-\infty, 0)$ $(0, +\infty)$, on each of which the given function is monotone and, hence, has an inverse.

1.161. $\log_2 e \cdot \cot x$. **1.162.** $\frac{1}{\sqrt{1+8x}}$. **1.164.** $\frac{1}{1+e^{\alpha(x)}}$.

1.165. $\frac{2}{1+6\alpha^2(x)}$. 1.166. $\frac{\alpha(x)}{\alpha(x)+\log_2 e}$. 1.167. $\frac{1}{1+\ln \alpha(x)}$.

1.168. $3t-\frac{5}{2}$. 1.169. $\frac{1}{3t}$. 1.170. $-\frac{2t}{t+1}$. 1.171. -2^{3t+1}.

1.172. $-\frac{b}{a}\cot\varphi$. 1.173. $2\cos^2 t\,(\cos 2t-2\sin 2t)$. 1.174. 1.

1.175. $\frac{t}{2}$. 1.176. $\frac{2}{3}\ln 2\cot 2t$. 1.177. $-\frac{\sqrt{2-t^2}}{\sqrt{1-4t^2}}$.

1.178. $\sqrt[6]{\frac{(1-\sqrt{t})^4}{t\,(1-\sqrt[3]{t})^3}}$. 1.179. $\frac{b}{a}\tanh t$. 1.180. 1. 1.181. -1.

1.182. $2+\sqrt{3}$. 1.183. $-\frac{4}{3}$. 1.184. $-2\cos 2x$. 1.185. $\frac{2-6x^4}{(1+x^4)^2}$.

1.186. $-\frac{2}{3\ln 2}\,\frac{x^2+1}{2(x^2+1)^2}$. 1.187. $2e^{-x^2}(2x^2-1)$. 1.188. $\frac{3x}{(1-x^2)^2}+$

$+\frac{(1+2x^2)\arcsin x}{(1-x^2)^{5/2}}$. 1.189. $x^{\sqrt{x}-1}\,(2+\ln x)\left(\frac{1}{4}\ln x+\frac{1}{2}-\frac{1}{4\sqrt{x}}+\right.$

$\left.+\frac{1}{2\sqrt{x}\,(2+\ln x)}\right)$. *Hint.* Use the logarithmic derivative.

1.190. $y'(0)=3$, $y''(0)=12$, $y'''(0)=9$. 1.191. 2. 1.192. 6. 1.193. $y(0)=$

$=1$, $y'(0)=\ln 2$, $y''(0)=\ln^2 2-1$. 1.194. $y'=-\frac{2}{x^3}f'\left(\frac{1}{x^2}\right)$,

$y''=\frac{6}{x^4}f'\left(\frac{1}{x^2}\right)+\frac{4}{x^6}f''\left(\frac{1}{x^2}\right)$. 1.195. $y'=\frac{e^x f'(e^x)}{f(e^x)}$, $y''=$

$=e^{2x}\left(\frac{f'(e^x)}{e^x f(e^x)}+\frac{f''(e^x)}{f(e^x)}-\frac{f'^2(e^x)}{f^2(e^x)}\right)$. 1.197. $y'=\frac{uu'+vv'}{\sqrt{u^2+v^2}}$,

$y''=\frac{(u^2+v^2)(uu''+vv'')+(u'v-uv')^2}{(u^2+v^2)^{3/2}}$. 1.198. $y'=\frac{u'}{u}-\frac{v'}{v}$,

$y''=\frac{uu''-u'^2}{u^2}-\frac{vv''-v'^2}{v^2}$. 1.199. $\frac{m!}{(m-n)!}x^{m-n}$ if $n\leqslant m$; 0 if $n>m$.

1.200. $(k\ln a)^n a^{kx}$. 1.201. $\sin\left(x+n\frac{\pi}{2}\right)$. *Hint.* $y'=\cos x=$

$=\sin\left(x+\frac{\pi}{2}\right)$, $y''=\left(\sin\left(x+\frac{\pi}{2}\right)\right)'=\cos\left(x+\frac{\pi}{2}\right)=\sin(x+\pi)$

and so on. 1.202. $(-1)^{n-1}\frac{(n-1)!}{x^n}$. 1.203. $2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)$.

Hint. Make use of the formula: $\cos^2 x=\frac{1+\cos 2x}{2}$. 1.204. $\frac{2n!}{(1-x)^{n+1}}$.

1.206. $\frac{(x-1)^{50}-(x-2)^{50}}{(x^2-3x+2)^{51}}$. 1.207. $\frac{1\cdot 3\cdot 5\ldots 37\,(79-x)}{2^{20}(1-x)^{20}\sqrt{1-x}}$. *Hint.* Take

advantage of the equality $1 + x = 2 - (1 - x)$. **1.208.** $\cos x\,(209 - x - x^2) - 15 \sin x\,(2x + 1)$. **1.209.** $e^x\,(x^2 + 39x + 360)$. **1.210.** $\sqrt{2} \sin\left(x - \frac{\pi}{4}\right) e^{-x}$. **1.211.** $\frac{8! \log_2 e}{x^9}$. **1.212.** $x \sinh x + 100 \cosh x$. **1.213.** *Hint.* Carry out the proof using the method of mathematical induction.

1.216. $-\frac{24c^3(ad-bc)}{d^5}$. **1.217.** -48. **1.223.** $-\frac{p^2}{y^3}$. **1.224.** $e^{2y}\frac{2-xe^y}{(1-xe^y)^3}$. **1.225.** $-\frac{2(1+y^2)}{y^5}$. **1.226.** $\frac{y((1+y)^2+(x-1)^2)}{x^2(1+y)^3}$. **1.227.** $-\frac{f''_{xx}}{(f'_x)^3}$. **1.230.** $-\cot^3 t$ or $\frac{1}{(x^2-1)^{3/2}}$, $x \in (1, +\infty)$. **1.231.** $-\frac{2}{1-t^2}$ or $-2\sec^2 x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **1.232.** $2(1+t^2)$ or $2\sec^2 x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **1.233.** $\frac{1}{3a\cos^1 t \sin t}$ or $\frac{a^{2/3}}{3x^{4/3}\sqrt{a^{2/3}-x^{2/3}}}$, $x \in (0, a)$. **1.235.** $7x + y - 3 = 0$, $x - 7y + 71 = 0$. **1.236.** $y - 5 = 0$, $x + 2 = 0$. **1.237.** $x - 4y + 4 = 0$, $4x + y - 18 = 0$. **1.238.** $y - 2x = 0$, $2y + x = 0$. **1.239.** $x - y - 1 = 0$, $x + y - 1 = 0$. **1.240.** $2x - y + 3 = 0$, $x + 2y - 1 = 0$. **1.241.** $7x - 10y + 6 = 0$, $10x + 7y - 34 = 0$. **1.242.** $y = 0$, $(\pi + 4)x + (\pi - 4)y - \pi^2\sqrt{2}/4 = 0$. **1.243.** $5x + 6y - 13 = 0$, $6x - 5y + 21 = 0$. **1.244.** $x + y - 2 = 0$. **1.245.** $\arctan\frac{2}{e}$. **1.246.** $M_0\,(1/8, -1/16)$. **1.247.** $y = x^2 - x + 1$. **1.249.** $2x - y - 1 = 0$. **1.250.** $4x - 4y - 21 = 0$. **1.251.** 3.75. **1.254.** At the point $M_1(0, 0)$ the angle is equal to 0 (the parabolas touch each other) and at the point $M_2(1, 1)$ the angle is $\arctan\frac{1}{7}$. **1.255.** $\arctan\frac{8}{15}$. **1.256.** $\arctan 2\sqrt{2}$. **1.257.** $\pi/4$ and $\pi/2$. **1.260.** $2/\sqrt{5}$. **1.262.** If a curve is given by the equation $r = r(\varphi)$, then the Cartesian coordinates of points M of this curve, as functions of the angle φ, are given by the expressions

$$x = r(\varphi)\cos\varphi,\quad y = r(\varphi)\sin\varphi.$$

Hence $\overrightarrow{OM} = r(\varphi)\cos\varphi \cdot \boldsymbol{i} + r(\varphi)\sin\varphi \cdot \boldsymbol{j}$, i.e. the vector $\boldsymbol{\rho}(1, \tan\varphi)$ is collinear with $\overrightarrow{OM}$; the vector $\boldsymbol{\tau}(1, y'_x)$ is the direction vector of the tangent TT', and since

$$y'_x = \frac{y'_\varphi}{x'_\varphi} = \frac{r'\sin\varphi + r\cos\varphi}{r'\cos\varphi - r\sin\varphi} = \frac{r + r'\tan\varphi}{r' - r\tan\varphi},$$

the vector $e\,(r' - r\tan\varphi,\ r + r'\tan\varphi)$ is collinear with τ. Consequently,

$$\cos\theta = \frac{(\rho,\ e)}{|\rho|\cdot|e|} = \frac{r'}{\sqrt{r^2+(r')^2}},$$

whence $\tan\theta = \sqrt{\dfrac{1-\cos^2\theta}{\cos^2\theta}} = \dfrac{r}{r'}$. **1.263.** $\theta = \operatorname{arc}\tan\dfrac{1}{k}$.

1.264. $\theta = \dfrac{\pi}{2} + 2\varphi$.

1.265. (a) $t_1 = 0$, $t_2 = 8$; (b) $t \in (0,\ 4) \cup (8,\ +\infty)$; (c) $t_1 = \dfrac{4}{3}(3 + \sqrt{3})$, $t_2 = \dfrac{4}{3}(3 - \sqrt{3})$. **1.266.** $-a\omega\sin\omega t$. **1.267.** 242. **1.268.** $\dfrac{7}{18}\pi$. **1.269.** $a\omega e^{a\varphi}$. **1.270.** $v_x = -2a\omega\sin 2\varphi$, $v_y = -2a\omega\cos 2\varphi$. **1.271.** At points (3, 16/3) and (−3, −16/3). **1.272.** $4\pi r^2 v$ and $8\pi r v$. **1.273.** 2π rad/s.

2.2. $(\Delta y)_1 = 1.261$, $(dy)_1 = 1.2$, $(\Delta y)_2 = 0.120601$, $(dy)_2 = 0.12$. **2.3.** $\Delta s = 2x\Delta x + \Delta x^2$, $ds = 2x\,\Delta x$. **2.4.** $ds = f'(t_1)\,\Delta t$ is the path which would be covered by the point M during the time interval Δt if it performs uniform motion with velocity $f'(t_1)$. **2.5.** $ds = 0.1$, $\Delta s = 0.08$. **2.6.** (a) 0; (b) $\dfrac{\pi}{2} + k\pi$. **2.7.** Equalities (a) and (c) are impossible; equality (b) is possible in case of a linear function (see Problem 2.1). **2.8.** 2 cm. **2.9.** 3 cm. **2.10.** $2\sqrt{a^2 - x^2}\,dx$. **2.11.** $x\sin x\,dx$. **2.12.** $\operatorname{arc}\tan x\,dx$. **2.13.** $\ln x\,dx$. **2.14.** $\operatorname{arc}\sin x\,dx$. **2.15.** $\dfrac{2x\,dx}{1+5y^2}$. **2.16.** $\dfrac{x+y}{x-y}dx$. **2.17.** $\dfrac{dx}{e^y - 1}$. **2.18.** (a) 0.05; (b) 0.805; (c) 0.2. **2.19.** 2.93. **2.20.** 1.2. **2.21.** $\Delta V \approx 2\pi r h\,\Delta r$. Since h is constant, v is a function of only one argument r: $v = \pi h r^2$. **2.22.** $\Delta V \approx -\dfrac{RT}{p^2}\Delta p$. Since T is constant, the volume V is a function of only one argument p; $V = RT\dfrac{1}{p}$. **2.23.** $-ab^2\sin(bx + c)\,dx^2 = -b^2 y\,dx^2$.

2.24. $3^{-x^2}\ln 9\,(2x^2\ln 3 - 1)\,dx^2$. **2.25.** $\dfrac{(2 - x^2)\sin x - 2x\cos x}{x^3}dx^2$.

2.26. $2a\,dx^2$. **2.27.** $\dfrac{2dx^2}{(x+2y)^3}$. **2.28.** $-\dfrac{R^2\,dx^2}{(y-b)^3}$.

2.29. $6\,\dfrac{x(1+3y^2)\,dx^2}{(1-3y^2)^3}$. **2.30.** $\dfrac{(x-y)\,dx^2}{(1-a\cos y)^3}$.

3.1. $f(x)$ is discontinuous at $x = 0 \in [-1,\ 1]$. **3.2.** 0. **3.4.** *Hint.* Carry out the proof by contradiction after having determined that the derivative of the left-hand member of the equation has only one real root $x = 1$. **3.5.** *Hint.* Carry out the proof by contradiction on

having determined that the derivative of the left-hand member of the equation has no real roots. **3.6.** *Hint.* $F(b) = F(a)$. **3.7.** $\xi = 1/\sqrt{3}$. **3.11.** *Hint.* Take advantage of the result obtained in Problem 3.8. **3.13.** $\xi_1 = 1/2$, $\xi_2 = 5/3$. **3.14.** 0. **3.15.** 1/3. **3.16.** $\frac{m}{n} a^{m-n}$. **3.17.** $\frac{\ln a - \ln b}{\ln c - \ln d}$. **3.18.** 1. **3.19.** 2/3. **3.20.** $\frac{2}{3\sqrt[6]{5}}$. **3.21.** a^2/b^2. **3.22.** 2. **3.23.** 2/3. **3.24.** $-1/2$. **3.25.** 2. **3.26.** 9/50. **3.27.** 1/2. **3.28.** 1/2. **3.29.** 0. **3.30.** 1/2. **3.31.** $-\infty$. **3.32.** $\cos 3$. **3.33.** -2. **3.34.** 1. **3.35.** 0. **3.36.** 0. **3.37.** 0. **3.38.** 2. **3.39.** 0. **3.40.** $+\infty$. **3.41.** $1/\pi$. **3.42.** a. **3.43.** 0. **3.44.** -1. **3.45.** 0. **3.46.** 1/12. **3.47.** -1. **3.48.** 2/3. **3.49.** 1. **3.50.** 1. **3.51.** 1. **3.52.** e. **3.53.** 1. **3.54.** 2. **3.55.** $1/e$. **3.56.** 1. **3.57.** $1/e$. **3.58.** 1. **3.59.** e^{-6}. **3.60.** e^2. **3.61.** $-9 + 17(x+1) - 9(x+1)^2 + 2(x+1)^3$. **3.62.** $7 + 11(x-1) + 10(x-1)^2 + 4(1+\theta(x-1))(x-1)^3$; (a) $\theta = 1/4$; (b) θ—any real number; (c) $\theta = 1/4$. **3.63.** $P(-1) = 143$, $P'(0) = -60$, $P''(1) = 26$. **3.64.** $1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!} x^{n+1}$. **3.65.** $\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{\frac{n-1}{2}} \frac{x^n}{n!} + \frac{\sin\left(\theta x + (n+1)\frac{\pi}{2}\right)}{(n+1)!} x^{n+1}$, n is odd; $\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \ldots + (-1)^{\frac{n}{2}} \frac{x^{n-1}}{(n-1)!} + \frac{\sin\left(\theta x + (n+1)\frac{\pi}{2}\right)}{(n+1)!} x^{n+1}$, n is even. **3.66.** $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^{\frac{n-1}{2}} \frac{x^{n-1}}{(n-1)!} + \cos\frac{\theta x + (n+1)\frac{\pi}{2}}{(n+1)!} x^{n+1}$, n is odd; $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \ldots + (-1)^{\frac{n}{2}} \frac{x^n}{n!} + \frac{\cos\left(\theta x + (n+1)\frac{\pi}{2}\right)}{(n+1)!} x^{n+1}$, n is even. **3.67.** $x - \frac{x^2}{2} + \ldots + (-1)^{n-1} \frac{x^n}{n} + \frac{x^{n+1}}{(n+1)(1+\theta x)^{n+1}}$, $x > -1$. **3.68.** $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{\frac{n-1}{2}} \frac{x^n}{n!} + R_{n+1}(x)$, n is odd; $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{\frac{n}{2}} \frac{x^{n-1}}{n-1} + R_{n+1}(x)$, n is even. *Hint.* Write the

remainder in the general form. **3.69.** $1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+\ldots$

$$\ldots+\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n+$$

$$+\frac{\alpha(\alpha-1)\ldots(\alpha-n)(1+\theta x)^{\alpha-n-1}}{(n+1)!}x^{n+1},\quad x>-1.$$ **3.70.** $1-\frac{x^2}{2!}+\frac{x^4}{2^2\cdot 2!}-\frac{x^6}{2^3\cdot 3!}+\ldots+(-1)^n\frac{x^{2n}}{2^n\cdot n!}$. *Hint.* Set $u=-\frac{x^2}{2}$ in the expansion of e^u by Maclaurin's formula (see Problem 3.64). **3.71.** $\frac{1}{2}\left[\frac{(2x)^2}{2!}-\frac{(2x)^4}{4!}+\ldots+(-1)^{n-1}\frac{(2x)^{2n}}{(2n)!}\right]$. *Hint.* Write $\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos^2 x$ and take advantage of the result obtained in Problem 3.66. **3.72.** $\frac{5x}{2!}-\frac{(5x)^3}{2^3\cdot 3!}+\ldots+(-1)^n\frac{(5x)^{2n+1}}{2^{2n+1}(2n+1)!}$. **3.73.** $2\ln 2+\frac{x^2}{4}-\frac{x^4}{4^2\cdot 2}+\frac{x^6}{4^3\cdot 3}+\ldots+(-1)^{n+1}\frac{x^{2n}}{4^n\cdot n}$. *Hint.* Write $\ln(4+x)^2$ in the form $\ln(4+x^2)=2\ln 2+\ln\left(1+\frac{x^2}{4}\right)$ and use the result obtained in Problem 3.67. **3.74.** $2\left[1+\frac{1}{2}\times\right.$

$$\times\frac{x^2}{8}-\frac{1}{2^2\cdot 2!}\frac{x^4}{8^2}+\frac{1\cdot 3}{2^3\cdot 3!}\cdot\frac{x^6}{8^3}+\ldots+(-1)^{n-1}\times\frac{1\cdot 3\ldots(2n-3)}{2^n\cdot n!}\times$$

$$\left.\times\frac{x^{2n}}{8^n}\right].$$ **3.75.** $2-(x-2)+(x-2)^2-(x-2)^3+\frac{(x-2)^4}{(1+\theta(x-2))^5}$. **3.76.** $x+\frac{x^3}{3}\cdot\frac{1+2\sin^3\theta x}{\cos^4\theta x}$. **3.77.** $x+\frac{x^3}{6}+\frac{x^4}{4!}\,\frac{9\theta x+6\theta^3x^3}{(1-\theta^2x^2)^{7/2}}$. **3.78.** $1-\frac{(x-1)}{2}+\frac{3}{8}(x-1)^2-\frac{5}{16}(x-1)^3+\frac{35}{128}\frac{(x-1)^4}{(1+\theta(x-1))^{9/2}}$. **3.79.** (*a*) 0.842; (b) 1.648; (c) 0.049; (d) 2.012. **3.80.** Error: (a) $\frac{1}{16}\frac{x^3}{(1+\theta x)^{5/2}}$; (b) $\frac{5}{81}\frac{x^3}{(1+\theta x)^{8/3}}$. **3.82.** (a) 1; (b) 1/2; (c) 1/2.

4.1. *Hint.* Take advantage of the expansion of the function by Taylor's formula in the neighbourhood of the point x_0 up to the term of order k inclusively. **4.2.** $f(x_0)=0$ is a minimum if $\varphi(x_0)>0$ and n is even; $f(x_0)=0$ is a maximum if $\varphi(x_0)<0$ and n is even; there is no extremum if n is odd. **4.3.** *Hint.* Make use of the first sufficient condition for an extremum. **4.4.** On $(-1,\ -1/\sqrt{2})\cup(1/\sqrt{2},\ 1)$ decreases, on $(-1/\sqrt{2},\ 1/\sqrt{2})$ increases; $y_{\min}=y(-1/\sqrt{2})=-1/\sqrt{2}$, $y_{\max}=y(1/\sqrt{2})=1/2$. **4.5.** On $(-\infty,\ -1)\cup(0,\ 1)$ increases, on $(-1,\ 0)\cup(1,\ +\infty)$ decreases; $y_{\max}=y(-1)=y(1)=1$. **4.6.** On

$(0, 1) \cup (1, e)$ decreases, on $(e, +\infty)$ increases; $y_{\min} = y(e) = e$. **4.7.** On $\left(\frac{\pi}{3}(6k-1), \frac{\pi}{3}(6k+1)\right)$ decreases, on $\left(\frac{\pi}{3}(6k+1), \frac{\pi}{3}(6k+5)\right)$ increases; $y_{min} = y\left(2k\pi + \frac{\pi}{3}\right) = 2k\pi + \left(\frac{\pi}{3} - \sqrt{3}\right) \approx \approx 2k\pi - 0.685$, $y_{\max} = y\left(2k\pi + \frac{5\pi}{3}\right) = 2(k+1)\pi - \left(\frac{\pi}{3} - \sqrt{3}\right) \approx \approx 2(k+1)\pi + 0.685$, $k \in \mathbb{Z}$. **4.8.** On $(0, 2)$ decreases, on $(2, +\infty)$ increases; $y_{\min} = y(2) = 2(1 - \ln 2) \approx 0.61$. **4.9.** Increases in the entire domain of definition. **4.10.** On $\left(\frac{\pi}{4}(8k-3), \frac{\pi}{4}(8k+1)\right)$ increases, on $\left(\frac{\pi}{4}(8k+1), \frac{\pi}{4}(8k+5)\right)$ decreases; $y_{\max} = y\left(2k\pi + \frac{\pi}{4}\right) = e^{2k\pi}\left(e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2}\right) \approx \approx 1.55e^{2k\pi}$, $y_{\min} = y\left(2k\pi + \frac{5\pi}{4}\right) = -e^{2k\pi}\left(e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2}\right) \approx -1.55e^{2k\pi}$, $k \in \mathbb{Z}$. **4.11.** On $(0, 1/e)$ decreases, on $(1/e, +\infty)$ increases; $y_{\min} = y(1/e) = (1/e)^{\frac{1}{e}} \approx 0.69$. **4.12.** On $(-\infty, 0)$ decreases, on $(0, +\infty)$ increases; $y_{\min} = y(0) = 2$. **4.13.** $M = 3$, $m = -24$. **4.14.** $M = 8$, $m = 0$. **4.15.** $y = 0.6$, $m = -1$. **4.16.** $M = 1$, $m = 0.6$. **4.17.** $M = 2$, $m = \sqrt[3]{2} \approx 1.26$. **4.18.** $M = \pi/4$, $m = 0$. **4.19.** $M = 1$, $m = -1$. **4.20.** $M = 1/\sqrt{e} \approx 0.61$, $m = -1/\sqrt{e} \approx -0.61$. **4.21.** *Hint.* Consider the function $y = e^x - (1+x)$ and show that it has only one minimum: $y_{\min} = y(0) = 0$. **4.25.** $\frac{av_1 + bv_2}{v_1^2 + v_2^2}$ s. **4.26.** $|AP| = \left(500 - \frac{100}{\sqrt{3}}\right)$ km $\approx \approx 442.3$ km. **4.27.** $x + \frac{2p}{4+\pi}$, $y = \frac{1}{2}\left(p - x - \frac{\pi x}{2}\right)$. **4.28.** $\alpha = \frac{2\pi}{3}$. **4.29.** $\frac{ah}{4}$. **4.30.** πa^3. **4.31.** $\frac{4}{27}\pi r^2 h$. **4.32.** $\frac{8}{3}\pi r^3$. **4.33.** $\frac{2\pi}{9\sqrt{3}}e^3$. **4.34.** $2r^2$. **4.35.** $N(1, 1)$. **4.36.** $x = R\sqrt{2}$, $y = R/\sqrt{2}$. **4.37.** Divide the line segment into two. **4.38.** $r = \frac{RH\sqrt{R^2+H^2}}{(\sqrt{R^2+H^2} - R)(\sqrt{R^2+H^2} + 2R)}$. **4.39.** $h = (e^{2/3} - d^{2/3})^{3/2}$. **4.40.** On $(-\infty, 0)$ the graph is convex upward, on $(0, +\infty)$ it is convex downward, $M(0, 1)$ is a point of inflection, $k = 7$. **4.41.** The graph is convex downward everywhere. **4.42.** On $(-\infty, 2)$—convex upward, on $(2, +\infty)$—convex downward, $M(2, 0)$—point of inflection, $k = 0$. **4.43.** On $(-\infty, -1) \cup (1 + \infty)$—convex downward, on $(-1, 1)$—convex upward, $M_1(-1, \sqrt[3]{2})$ and $M_2(1, \sqrt[3]{2})$—points of inflection, $k_1 = k_2 = \infty$. **4.44.** The graph is

convex upward everywhere. 4.45. On $(-\infty, -1)$—convex upward, on $(-1, +\infty)$—convex downward, $M(-1, 1-e^{-2})$—a point of inflection, $k=-e^{-2}\approx-0.14$. 4.46. On $(-\infty, 0)$—convex upward, on $(0, +\infty)$—convex downward, $M(0, 0)$—a point of inflection, $k=\infty$. 4.47. On $(0, e^{-5/6})$—convex upward, on $(e^{-5/6}, +\infty)$—convex downward, $M\left(e^{-5/6}, 1-\frac{5}{6}e^{-5/6}\right)$—point of inflection, $k=-\frac{3}{2}e^{-5/3}\approx$ ≈-0.28. 4.48. $a=-\frac{3}{2}$, $b=\frac{9}{2}$. 4.49. $\frac{1}{\sigma\sqrt{2}}$. 4.51. *Hint.* If x_0 is the abscissa of the point of inflection, then $x_0 \tan x_0=2$. Then $y_0^2=y^2(x_0)=x_0^2\sin^2 x_0=\frac{4x_0^2}{4+x_0^2}$. 4.52. $x=2$, $y=1$. 4.53. $y=x-\frac{1}{3}$. 4.54. $x=0$, $y=1$ (right), $y=-1$ (left). 4.55. $y=3x+\frac{\pi}{2}$ (right), $y=3x-\frac{\pi}{2}$ (left). 4.56. $x=0$, $y=2x$, $x=-1$ (right). 4.57. $y=0$. 4.58. $x=-\frac{1}{e}$, $y=x+\frac{1}{e}$. 4.59. $y=\frac{\pi}{2}x-1$. 4.61. $y_{\min}=y(0)=$ $=-1$; $\left(\pm 1, -\frac{64}{125}\right)$ and $(\pm\sqrt{5}, 0)$—points of inflection. 4.62. $y_{\max}=y(\pm 1)=1$, $y_{\min}=y(\pm\sqrt{3})=y(0)=0$; $\left(\pm\sqrt{\frac{6-\sqrt{21}}{5}}, \frac{6-\sqrt{21}}{20}\left(\frac{6-\sqrt{21}}{5}-3\right)^2\right)$ and $\left(\pm\sqrt{\frac{6+\sqrt{21}}{5}}, \sqrt{\frac{6+\sqrt{21}}{20}}\times\right.$ $\left.\times\left(\frac{6+\sqrt{21}}{5}-3\right)^2\right)$—points of inflection. 4.63. $y_{\max}=y(-\sqrt{3})=$ $=\sqrt{3}$, $y_{\min}=y(\sqrt{3})=-\sqrt{3}$; $(0; 0)$ and $\left(\pm\frac{\sqrt{6}}{2}, \mp\frac{7\sqrt{6}}{16}\right)$—points of inflection. 4.64. $y_{\min}=y(3)=\frac{27}{8}$; $(0, 0)$—point of inflection; $x=1$ and $y=\frac{x+2}{2}$—asymptotes. 4.65. $y_{\max}=y(0)=0$, $y_{\min}=$ $=y(\sqrt[3]{4})=\frac{4}{3}\sqrt[3]{4}$; $\left(-\sqrt[3]{2}, -\frac{2}{3}\sqrt[3]{2}\right)$—point of inflection; $x=1$ and $y=x$—asymptotes. 4.66. $(0, 0)$—point of inflection; $x=\pm 1$ and $y=x$—asymptotes. 4.67. $y_{\max}=y(\sqrt[3]{4})=-\frac{4}{3}\sqrt[3]{4}$, $y_{\min}=y(0)=0$; $\left(\sqrt[3]{2}, \frac{2}{3}\sqrt[3]{2}\right)$—point of inflection; $x=-1$ and $y=x$—asymptotes. 4.68. $y_{\max}=y(1)=\frac{1}{3}$, $\left(\sqrt[3]{4}, \frac{1}{6}\sqrt[3]{4}\right)$—point of inflection; $x=$

$= -\sqrt[3]{2}$ and $y = 0$—asymptotes. **4.69.** (0, 0)—point of inflection; $x = \pm 1$ and $y = 0$—asymptotes. **4.70.** $y_{max} = y(0) = 0$, $y_{min} = y(-\sqrt[3]{2}) = -\frac{\sqrt[3]{4}}{3}$; $\left(-\sqrt[3]{\frac{7+\sqrt{45}}{2}}, -\frac{\sqrt[3]{2(7+\sqrt{45})^2}}{9+\sqrt{45}}\right)$ and $\left(-\sqrt[3]{\frac{7-\sqrt{45}}{2}}, -\frac{\sqrt[3]{2(7-\sqrt{45})^2}}{9-\sqrt{45}}\right)$—points of inflection; $x = 1$ and $y = 0$—asymptotes. **4.71.** (0, 0)—points of inflection; $x = -2$, $x = 2$, $y = 0$—asymptotes. **4.72.** $y_{max} = y(-3) = -4.5$, $y_{min} = y(3) = 4.5$; (0, 0)—points of inflection; $x = -\sqrt{3}$, $x = \sqrt{3}$, $y = x$—asymptotes. **4.73.** $y_{min} = y(-1) = -1/3$; $(-\sqrt[3]{4}, -\sqrt[3]{4/6})$—point of inflection; $x = \sqrt[3]{2}$ and $y = 0$—asymptotes. **4.74.** $y_{min} = y(0) = -1$, $(\pm\sqrt{3}/3, -1/2)$—points of inflection; $y = 1$—asymptote. **4.75.** (0, 0) and $(\sqrt[3]{4}/2, 1/3)$—points of inflection; $x = -1$ and $y = 1$—asymptotes. **4.76.** $y_{max} = y(0) = 2$, $(\pm 1, \sqrt[3]{2})$—points of inflection; $y = 0$—asymptote. **4.77.** $y_{min} = y(1) = -1$; (0, 0) and (2, 0)—points of inflection. **4.78.** $y_{max} = y(0) = 2$, $y_{min} = y(\pm 1) = \sqrt[3]{4}$. **4.79.** (0,0)—point of inflection; $x = -1$, $x = 1$, $y = 0$—asymptotes. **4.80.** (0, 1) and (1, 0)—points of inflection; $y = -x$—asymptote. **4.81.** (0, 0) $(\pm 1, \pm\sqrt[3]{2})$—points of inflection. **4.82.** (0, 0), $(\pm 1, \pm\sqrt[3]{2})$—points of inflection; $y = 2x$—asymptote. **4.83.** (0, 0), $(\pm 1, \pm\sqrt[3]{2})$—points of inflection; $y = x$—asymptote. **4.84.** (0, 0)—point of inflection; $y = -1$—left-hand asymptote, $y = 1$—right-hand asymptote. **4.85.** $y_{min} = y(-\sqrt[3]{3}) = 1$; (0, 0)—point of inflection; $x = -\sqrt[3]{2}$—asymptote. **4.86.** $y_{max} = y(0) = 0$, $y_{min} = y(2) = \sqrt[3]{16}$; $(-\sqrt[3]{4}, -\sqrt[3]{2})$—point of inflection; $x = \sqrt[\infty]{4}$ and $y = x$—asymptotes. **4.87.** $y_{max} = y(-\sqrt[3]{6}) = -3/\sqrt[3]{2}$; (0, 0) and $(\sqrt[3]{3}, 3/\sqrt[3]{25})$—point of inflection; $x = -\sqrt[3]{2}$ and $y = x$—asymptotes. **4.88.** $y_{min} = y(0) = 0$, $(\pm\sqrt{2}, 2/\sqrt{3})$—points of inflection; $y = x$—right-hand asymptote, $y = -x$—left-hand asymptote. **4.89.** $(-\sqrt[3]{2}, 0)$ and $(-1, -1)$—points of inflection; $x = 0$ and $y = 1$—asymptotes. **4.90.** $y_{max} = y(1) = 1/\sqrt[3]{4}$, $(\sqrt[3]{4}, \sqrt[3]{-0.16}$ point of inflection; $x = -1$, $y = 0$—asymptotes. **4.91.** $y_{max} = y(-\sqrt{3}) = 0$, $y_{min} = y(\sqrt{3}) = 0$; $(\sqrt{2}, 1/\sqrt{2})$, $(-\sqrt{2}, -1/\sqrt{2})$—points of inflection; $x = 0$—asymptote, $y = 1$—right-hand asymptote, $y = -1$—left-hand asymptote. **4.92.** $y_{min} = y(0 = 0)$, $y_{min} = y(\pm\sqrt{2}) = 2$; $x = \pm 1$—asymptotes, $y = x$—right-hand asymptote, $y = -x$—left-hand asymptote. **4.93.** $y_{max} = y(0) = 1$, $y_{min} = y(\pm 1) = 0$. **4.94.** $y_{max} = y(0) = 2\sqrt{2}$, $y_{min} = y(\pm\sqrt{2}) = 0$; $(\pm 1, 1)$—points of inflection. **4.95.** $y_{min} = y\left(\frac{5\pi}{4} + 2k\pi\right) = -\sqrt{2}$, $y_{max} = y\left(\frac{\pi}{4} + 2k\pi\right) = \sqrt{2}$, $\left(\frac{3\pi}{4} +\right.$

$+k\pi, 0\Big)$—points of inflection, $k \in \mathbb{Z}$. **4.96.** $y_{\min} = y\left(\frac{\pi}{4} + 2k\pi\right) = -\frac{\sqrt{2}}{2}$, $y_{\max} = y\left(-\frac{3\pi}{4} + 2k\pi\right) = -\frac{\sqrt{2}}{2}$, $x = \frac{3\pi}{4} + k\pi$—asymptotes, $k \in \mathbb{Z}$. **4.97.** $y_{\min} = y(0) = 0$, $y = -\frac{\pi x}{2} - 1$—left-hand asymptote, $y = \frac{\pi x}{2} - 1$—right-hand asymptote. **4.98.** $y_{\min} = y(1) = \frac{1}{2} + \frac{\pi}{4}$, $y_{\max} = y(-1) = -\frac{1}{2} + \frac{3\pi}{4}$, $\left(0, \frac{\pi}{2}\right)$—points of inflection; $y = \frac{x}{2} + \pi$—left-hand asymptote, $y = \frac{x}{2}$—right-hand asymptote. **4.99.** $y_{\max} = y(1) = e$, $\left(\frac{2 \pm \sqrt{2}}{2}, e^{1/2}\right)$—points of inflection; $y = 0$—asymptote. **4.100.** $y_{\max} = y(1) = \frac{1}{\sqrt{e}}$, $y_{\min} = x(-1) = -\frac{1}{\sqrt{e}}$; $(0, 0)$, $\left(\pm\sqrt{3}, \pm\frac{\sqrt{3}}{e\sqrt{e}}\right)$—points of inflection; $y = 0$—asymptote. **4.101.** $y_{\max} = y(1) = \frac{1}{e}$, $\left(1 \pm \frac{\sqrt{2}}{2}, (2 \mp \sqrt{2}) \times e^{-(2 \mp \sqrt{2})}\right)$—points of inflection; $x = 0$—left-hand asymptote, $y = 0$—asymptote. **4.102.** $y_{\max} = y(\pm 1) = \frac{1}{e}$, $y_{\min} = y(0) = 0$; $\left(\pm\frac{\sqrt{3}}{3}, 3e^{-3}\right)$ and $\left(\pm\sqrt{2}, \frac{1}{2}e^{-1/2}\right)$—points of inflection; $y = 0$—asymptote. **4.103.** $y_{\min} = y(1) = e$; $y = x + 1$—asymptote, $x = 0$—right-hand asymptote. **4.104.** $y_{\max} = y(\sqrt{2}) = \frac{1}{\sqrt{2e}}$, $y_{\min} = y(-\sqrt{2}) = -\frac{1}{\sqrt{2e}}$, $\left(\pm\sqrt{\frac{5 - \sqrt{17}}{2}}, \pm\frac{1}{2}\sqrt{5 + \sqrt{17}}e^{-\frac{1}{4}(5 + \sqrt{17})}\right)$ and $\left(\pm\sqrt{\frac{5 + \sqrt{17}}{2}}, \pm\frac{1}{2}\sqrt{5 - \sqrt{17}}e^{-\frac{1}{4}(5 - \sqrt{17})}\right)$—points of inflection; $y = 0$—asymptote. **4.105.** $y_{\max} = y(-2) = -4\sqrt{e}$, $y_{\min} = y(1) = -1/e$; $(0, 4, -1.6e^{-5/2})$—point of inflection; $x = 0$—left-hand asymptote, $y = x - 3$—asymptote. **4.106.** $(1, e^2)$—point of inflection, $x = 0$—right-hand asymptote, $y = 2x + \sqrt{3}$—asymptote. **4.107.** $y_{\max} = y(\pm 1) = 2/\sqrt{e}$, $y_{\min} = y(0) = 1$;

$\left(\pm\sqrt{2-\sqrt{3}}, (3-\sqrt{3})e^{-\frac{2-\sqrt{3}}{2}}\right)$ and $\left(\pm\sqrt{2+\sqrt{3}}, (3+\sqrt{3})e^{-\frac{2+\sqrt{3}}{2}}\right)$ —points of inflection, $y=0$—asymptote. **4.108.** $y_{\min}=y(1)=e^2$, $x=0$—right-hand asymptote. **4.109.** $y_{\max}=y(\sqrt{3})=3\sqrt{3}e^{-3/2}$, $y_{\min}=y(-\sqrt{3})=-3\sqrt{3}e^{-3/2}$; $(0, 0)$, $(\pm 1, \pm e^{-1/2})$, $(\pm\sqrt{6}, \pm\pm\sqrt{6}e^{-3})$—points of inflection; $y=0$—asymptote. **4.110.** $(0,0)$—point of inflection. **4.111.** $y_{\max}=y(e)=\frac{1}{e}$, $\left(e\sqrt{e}, \frac{3}{2e\sqrt{e}}\right)$—point of inflection; $x=0$ and $y=0$—right-hand asymptotes. **4.112.** $y_{\max}=y\left(\frac{1}{e}\right)=-e$; $x=1$—asymptote, $x=0$, and $y=0$—right-hand asymptotes. **4.113.** $y_{\min}=y\left(\frac{1}{\sqrt{e}}\right)=-\frac{1}{2e}$; $\left(\frac{1}{e\sqrt{e}}, -\frac{3}{2e^3}\right)$—point of inflection. **4.114.** $y_{\max}=y(\sqrt{e})=\frac{1}{2e}$; $\left(\sqrt[6]{e^5}, \frac{5}{6\sqrt[3]{e^5}}\right)$—point of inflection, $x=0$ and $y=0$—right-hand asymptotes. **4.115.** $y_{\max}=y(1/e)=1/e^2$, $y_{\min}=y(1)=0$; $\left(e^{-1.5-\sqrt{1.25}}, \frac{7+3\sqrt{5}}{2}e^{-3-\sqrt{5}}\right)$ and $\left(e^{-1.5+\sqrt{1.25}}, \frac{7-3\sqrt{5}}{2}e^{-3+\sqrt{5}}\right)$—points of inflection. **4.116.** $y_{\max}=y(0)=0$, $y_{\min}=y(\pm\sqrt{e})=2e$; $x=\pm 1$—asymptotes. **4.117.** $y_{\max}=y(1/e^2)=4/e^2$, $y_{\max}=y(-1)=0$, $y_{\min}=y(-1/e^2)=-4/e^2$; $(0, 0)$, $(\pm 1/\sqrt{e}, \pm 1/\sqrt{e})$—points of inflection. **4.118.** $y_{\max}=y(0)=0$; $x=\pm 1$—asymptotes. **4.119.** $y_{\max}=y(\pm e)=1/e^2$, $y_{\min}=y(\pm 1)=0$; $\left(\pm e^{\frac{5-\sqrt{13}}{6}}, \left(\frac{5-\sqrt{13}}{6}\right)^2 e^{\frac{5-\sqrt{13}}{2}}\right)$, $\left(\pm e^{\frac{5+\sqrt{13}}{6}}, \left(\frac{5+\sqrt{13}}{6}\right)^2 e^{-\frac{5+\sqrt{13}}{2}}\right)$—points of inflection; $x=0$ and $y=0$—asymptotes. **4.120.** $y_{\min}=y(1/e)=(1/e)^{1/e}\approx 0.69$, convex downward, $y\to 1$ as $x\to+0$, i.e. $M(+0, 1)$—end point. **4.121.** $y_{\max}=y(e)=e^{1/e}\approx 1.44$; (0.58, 0.12) and (4.35, 1.4)—points of inflection; $M(+0, 0)$—end point; $y=1$—asymptote. *Hint.* The points of inflection may be not found, it suffices to show that they are found from the equation $\ln^2\frac{x}{e}+2x\ln\frac{x}{e}-x=0$. **4.122.** $x=0$—point of removable discontinuity ($y_-(0)=y_+(0)=e$), the function is decreasing, convex downward, $x=-1$—vertical asymptote, $y=1$—asymptote. **4.123.** $x=0$—point of removable discontinuity, $y=0$—asymptote. The points of extrema satisfy the equation $\tan x=x$. The points of inflection sat-

isfy the equation $\tan x = \frac{2x}{2-x^2}$. *Hint.* It is not necessary to find the points of extrema and inflections. **4.125.** $x_{\min} = -1$ for $t = 1$ ($y(1) = 3$), $y_{\min} = -1$ for $t = -1$ ($x(-1) = 3$); a parabola with vertex at the origin, whose axis is a straight line $y = x$ ($x > 0$, $y > 0$). **4.126.** $x_{\min} = y_{\min} = 1$ for $t = 0$ (a cusp); $y = 2x$—asymptote for $t \to +\infty$. **4.127.** Astroid (see Chapter 2, Sec. 2.3, Fig. 27). **4.128.** $\left(-1-3\pi, -1+\frac{3\pi}{2}\right)$—a maximum, $\left(1-3\pi, 1-\frac{3\pi}{2}\right)$—a minimum, $(-3\pi, 0)$—point of inflection, $y = x$ and $y = x + 6\pi$—asymptotes. **4.129.** Three-leafed rose; $D = [0, \pi/3] \cup [2\pi/3, \pi] \cup [4\pi/3, 5\pi/3]$; extrema for $\varphi = \pi/6$, $\varphi = 5\pi/6$, $\varphi = 3\pi/2$. **4.130.** Cardioid, the pole is a cusp, $r_{\max} = r(0) = 2a$, $r_{\min} = r(\pi) = 0$. **4.131.** $D = (0, +\infty)$; the curve represents a spiral twining about the pole approaching it asymptotically; $(\sqrt{2\pi}, 1/2)$—point of inflection; the polar axis ($\varphi = 0$) is a horizontal asymptote. **4.132.** Bernouli's lemniscate (see Chapter 2, Sec. 2.3, Fig. 21).

5.1. Straight line $\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z}{4}$. **5.2.** In the xy-plane the arc of the circle $x^2 + y^2 = 2$ between the points $(1, 1)$ and $(0, \sqrt{2})$ traversed anticlockwise. **5.3.** The right-hand branch of the hyperbola $\frac{x^2}{16} - \frac{z^2}{9} = 1$, $y = -1$, traversed from below upward when observed from the origin. **5.4.** In the xy-plane the parabola $y = \frac{1}{9}(6x - x^2)$, traversed from left to right. **5.5.** Helical line $x = \cos t$, $y = \sin t$, $z = t$. **5.6.** Astroid $x^{2/3} + y^{2/3} = 2^{2/3}$, $z = 0$. **5.7.** The line of intersection of the cylinders $y = x^2$, and $z = x^3$ traversed from below upward. **5.8.** *Viviani's curve*—the line of intersection of a sphere and a circular cylinder: $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 = x$. **5.9.** Ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $z = 2$. **5.10.** Twice traversed parabola, $y = x^2 + x$, $z = 3$. **5.11.** Straight line $4x + 3y = 0$, $z = 0$, $v = 3i - 4j$. **5.12.** Parabola (in the xy-plane) $y = \frac{1}{9}(12x - x^2)$; $v = 3i + (4 - 2t)j$, $v|_{t=0} = 3i + 4j$, $v|_{t=1} = 3i + 2j$, $v|_{t=2} = 3i$, $v|_{t=3} = 3i - 2j$. **5.13.** Cycloid (in the xy-plane) $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$; $v = 2(1 - \cos t)\,i + 2 \sin t \cdot j$; for $t = \frac{\pi}{2}$ $v = 2(i + j)$, for $t = \pi$ $v = 4i$. **5.14.** $0.6i - 0.8j$. **5.15.** $\frac{1}{\sqrt{5}}(2i - j)$. **5.16.** (a) $\cos t \cdot i - \sin 2t \cdot j + \cos 2t \cdot k$; (b) $(\cos t - t \sin t)\,i + (\sin t + t \cos t)\,j + k$; (c) $(1 - \sin t)\,i + j + \cos t \cdot k$. **5.17.** (a) i; (b) $12i - 2j - \frac{2}{\sqrt{5}}k$. **5.18.** $1 + 3t^2 + 5t^4$. **5.19.** $(3t^2 - 2t)\,i + (3t^2 - 2t)\,j - 2tk$. **5.20.** $\cos t\,(i + 2uj + 3u^2k)$. **5.21.** $x + 2z = 4$, $y = 2$ (tangent); $2x - z = 3$ (normal plane). **5.22.**

$\frac{x-2}{1}=\frac{y-\frac{8}{3}}{2}=\frac{z-4}{4}$ (tangent); $3x+6y+12z-70=0$ (normal plane). 5.23. $y=z$, $x=a$ (tangent); $y+z=0$ (normal plane). 5.24. $\frac{x-1}{12}=\frac{y-3}{-4}=\frac{z-4}{3}$ (tangent); $12x-4y+3z=12$ (normal plane). 5.25. $\frac{x-1}{8}=\frac{y+1}{10}=\frac{z-2}{7}$ (tangent); $8x+10y+7z=12$ (normal plane). 5.26. (a) $\frac{d^2\boldsymbol{r}}{dt^2}=-\cos t\cdot\boldsymbol{i}+e^t\boldsymbol{j}+2\boldsymbol{k}$, $\left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_{t=0}=-\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$; (b) $\frac{d^2\boldsymbol{r}}{dt^2}=-(2\sin t+t\cos t)\,\boldsymbol{j}+(2\cos t-t\sin t)\,\boldsymbol{k}$; $\left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_{t=0}=2\boldsymbol{k}$. 5.27. $\boldsymbol{w}=2\sin t\cdot\boldsymbol{i}+2\cos t\cdot\boldsymbol{j}$; $\boldsymbol{w}|_{t=\pi/2}=2\boldsymbol{i}$; $\boldsymbol{w}|_{t=\pi}=-2\boldsymbol{j}$. 5.28. $\boldsymbol{w}=-2\boldsymbol{j}$, $w_\tau=\frac{4(t-2)}{\sqrt{4t^2-16t+25}}$, $w_n=\frac{6}{\sqrt{4t^2-16t+25}}$; for $t=0$, $w_\tau=-1.6$, $w_n=1.2$. *Hint.* $w_\tau=\frac{dv}{dt}$, $w_n=\sqrt{w^2-w_\tau^2}$. 5.29. $\boldsymbol{w}=\boldsymbol{i}+\frac{1}{\sqrt{2t+1}}\boldsymbol{j}$, $w_\tau=1$, $w_n=\frac{1}{\sqrt{2t+1}}$; for $t=0$ $\boldsymbol{w}=\boldsymbol{i}+\boldsymbol{j}$, $w_n=1$. 5.30. $K|_{x=0}=2$, $K|_{x=1}=\frac{2}{5\sqrt{5}}$. 5.31. $K_A=3$, $K_B=1/9$. 5.32. $3/\sqrt{2}$. 5.33. $1/2$. 5.34. $\frac{1}{2\sqrt{2}}$. 5.35. $K=\frac{1}{4a\sin\frac{\varphi}{2}}$, $K|_{\varphi=\pi}=\frac{3}{4a}$. 5.36. $\frac{3}{a}$. 5.37. $\frac{(9x^{4/3}+1)^{3/2}}{6x^{1/3}}$. 5.38. $\frac{(b^4x^2+a^4y^2)^{3/2}}{a^4b^4}$. 5.39. $\sqrt[3]{|axy|}$. 5.40. $\frac{(b^4x^2+a^4y^2)^{3/2}}{a^4b^4}=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}$. 5.41. $4a\left|\sin\frac{t}{2}\right|$. 5.42. $\frac{a^2}{3r}$. 5.43. $\frac{(a^2+r^2)^{3/2}}{2a^2+r^2}$. 5.44. $\left(\frac{\ln 2}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. *Hint.* Write the expression for curvature K and find its point of extremum. 5.45. $\left(\frac{1}{\sqrt{2}},-\frac{\ln 2}{2}\right)$. 5.46. $\left(0,\frac{a}{2}\right)$; $x^2+\left(y-\frac{a}{2}\right)^2=\frac{a^2}{4}$. 5.47. $\left(0,\frac{1}{2}\right)$; $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$. 5.48. $\left(-1,e-\frac{1}{e}\right)$; $(x+1)^2+\left(y-e+\frac{1}{e}\right)^2=e^2$. 5.49. $\left(\frac{\pi}{2},0\right)$; $\left(x-\frac{\pi}{2}\right)^2+y^2=1$.

5.50. $(\pi a, -2a)$; $(x-\pi a)^2+(y+2a)^2=16a^2$. 5.51. $X=\dfrac{x-9x^3}{2}$, $Y=\dfrac{15x^4+1}{6x}$. 5.52. $X^{2/3}-Y^{2/3}=(2a)^{2/3}$. 5.53. $(X+Y)^{2/3}+(X-Y)^{2/3}=2a^{2/3}$. 5.54. $Y=a\cosh\dfrac{X}{a}$. 5.55. $X^2=\dfrac{4}{27}Y^3$.

5.56. $\boldsymbol{\tau}=\dfrac{1}{\sqrt{3}}(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})$, $\boldsymbol{\nu}=\dfrac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})$, $\boldsymbol{\beta}=\dfrac{1}{\sqrt{6}}(-\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k})$; $x-1=-(y-1)=z$ (tangent); $x=y$, $z=0$ (principal normal); $\dfrac{x-1}{-1}=\dfrac{y-1}{1}=\dfrac{z}{2}$ (binormal). 5.57. $\boldsymbol{\tau}=\boldsymbol{i}$, $\boldsymbol{\nu}=-\dfrac{1}{\sqrt{2}}(\boldsymbol{j}+\boldsymbol{k})$, $\boldsymbol{\beta}=\dfrac{1}{\sqrt{2}}(\boldsymbol{j}-\boldsymbol{k})$, $y=2$, $z=4$ (tangent); $y-z+2=0$, $x=\pi$ (principal normal); $y+z=6$, $x=\pi$ (binormal). 5.58. $\boldsymbol{\tau}=\dfrac{1}{3}(2\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k})$, $\boldsymbol{\nu}=\dfrac{1}{3}(-\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k})$, $\boldsymbol{\beta}=\dfrac{1}{3}(2\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k})$; $\dfrac{x-2}{2}=\dfrac{y}{1}=\dfrac{z-1}{2}$ (tangent); $\dfrac{x-2}{-1}=\dfrac{y}{-2}=\dfrac{z-1}{2}$ (principal normal); $\dfrac{x-2}{2}=\dfrac{y}{-2}=\dfrac{z-1}{-1}$ (binormal). 5.59. $\boldsymbol{\tau}=\dfrac{1}{\sqrt{18}}(\boldsymbol{i}+\boldsymbol{j}+4\boldsymbol{k})$, $\boldsymbol{\nu}=-\dfrac{1}{3}(2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})$, $\boldsymbol{\beta}=\dfrac{1}{\sqrt{2}}(\boldsymbol{i}-\boldsymbol{j})$; $x-1=y-1=\dfrac{z-2}{4}$ (tangent); $\dfrac{x-1}{2}=\dfrac{y-1}{2}=\dfrac{z-2}{-1}$ (principal normal); $\dfrac{x-1}{1}=\dfrac{y-1}{-1}=\dfrac{z-2}{0}$ (binormal). 5.60. $x+2y=3$ (osculating plane); $z=1$ (normal plane); $2x-y=1$ (rectifying plane). 5.61. $y=x$ (osculating plane); $x+y=\dfrac{\pi}{\sqrt{2}}$ (normal plane); $z=0$ (rectifying plane). 5.62. $\boldsymbol{\tau}=\boldsymbol{i}$, $\boldsymbol{\nu}=\boldsymbol{k}$, $\boldsymbol{\beta}=\boldsymbol{j}$, $y=0$, $z=1$ (tangent); $x=1$, $y=0$ (principal normal); $x=1$, $z=1$ (binormal); $y=0$ (osculating plane); $x=1$ (normal plane); $z=1$ (rectifying plane). 5.63. $\boldsymbol{\tau}=\dfrac{1}{\sqrt{5}}(2\boldsymbol{i}-\boldsymbol{j})$, $\boldsymbol{\nu}=-\dfrac{1}{\sqrt{30}}(\boldsymbol{i}+2\boldsymbol{j}+5\boldsymbol{k})$, $\boldsymbol{\beta}=\dfrac{1}{\sqrt{6}}(\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})$; $\dfrac{x-1}{2}=\dfrac{y-2}{-1}=\dfrac{z-3}{0}$ (tangent); $\dfrac{x-1}{1}=\dfrac{y-2}{2}=\dfrac{z-3}{5}$ (principal normal); $\dfrac{x-1}{1}=\dfrac{y-2}{2}=\dfrac{z-3}{-1}$ (binormal); $x+2y-z-2=0$ (osculating plane); $2x-y=0$ (normal plane); $x+2y+5z-20=0$ (rectifying plane). 5.64. $K=\dfrac{\sqrt{2}}{(x+y)^2}$, $\sigma=-\dfrac{\sqrt{2}}{(x+y)^2}$; for

$t=0$ $K=\frac{\sqrt{2}}{4}$, $\sigma=-\frac{\sqrt{2}}{4}$. **5.65.** $K=2\sqrt{\frac{9t^4+9t^2+1}{(9t^4+4t^2+1)^3}}$, $\sigma=\frac{3}{9t^4+9t^2+1}$; for $t=0$ $K=2$, $\sigma=3$. **5.66.** $K=\sigma=\frac{1}{3(t^2+1)^2}$; for $t=1$ $K=\sigma=\frac{1}{12}$. **5.67.** $K=\frac{2t}{(2t^2+1)^2}$, $\sigma=-\frac{2t}{(2t^2+1)^2}$; for $t=1$ $K=\frac{2}{9}$, $\sigma=-\frac{2}{9}$. **5.68.** $K=\frac{\sqrt{2}}{3}$, $\sigma=\frac{1}{3}$.

5.69. $K=\sqrt{\frac{9y^4+4y^6+1}{(y^6+y^2+1)^3}}$, $\sigma=-\frac{6y}{9y^4+4y^6+1}$; for $y=1$ $K=-\frac{\sqrt{14}}{3\sqrt{3}}$, $\sigma=-\frac{3}{7}$. **5.70.** $\boldsymbol{w}=2\boldsymbol{j}+4t\boldsymbol{k}$, $w_\tau=4t$, $w_\nu=2$; $w_\tau|_{t=1}=4$. *Hint.* $w_\tau=\frac{dV}{dt}$, $w_\nu=\frac{v^2}{R}$.

5.71. Straight line $x-y=2$; $z'(t)=e^{i\frac{\pi}{4}}$. *Hint.* Use Euler's formula $e^{i\varphi}=\cos\varphi+i\sin\varphi$. **5.72.** Upper semi-circle $y=\sqrt{4-x^2}$; $z'(t)=2ie^{it}$. **5.73.** Ellipse $x=4\cos t$, $y=2\sin t$; $z'(t)=i(3e^{it}-e^{-it})$ **5.74.** Right-hand branch of the hyperbola $\frac{x^2}{16}-\frac{y^2}{4}=1$; $z'(t)=(2+i)e^t-(2-i)e^{-t}$. **5.75.** Twice traversed "right-hand" branch of the parabola $y=x^2$; $z'(t)=2t+4it^3$. **5.76.** Arch of the cycloid $x=t-\sin t$, $y=1-\cos t$; $z'(t)=1-e^{-it}$. **5.77.** Involute of the circle $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$; $z'(t)=ate^{it}$. **5.78.** Logarithmic spiral $r=e^{\frac{\alpha}{\beta}\varphi}$, $\alpha\beta\neq 0$; if $\alpha=0$, then the function $r=1$; if $\beta=0$, then the ray $\varphi=0$; $z'(t)=(\alpha+i\beta)e^{(\alpha+i\beta)t}$. **5.79.** r', $r\varphi'$; $r''-r\varphi'^2$, $2r'\varphi'+r\varphi''$. *Hint.* Represent the law of motion in the exponential form $z=re^{i\varphi}$ and find the derivatives z' and z''. The sought-for quantities are the coefficients of $e^{i\varphi}$ and $ie^{i\varphi}$. **5.80.** Velocity $v=izf'(z)$. *Hint.* Take advantage of the exponential form of complex number: $z=Re^{i\varphi}$ and find the derivative $\frac{dw}{dt}=\frac{dw}{dz}\cdot\frac{dz}{dt}$.

6.6. By hypothesis $f(a)\cdot f(b)<0$. Let us determine the numbers x_n $(n=0, 1, \ldots)$ by the equality

$$x_n=(a_n+b_n)/2,$$

where $a_0=a$, $b_0=b$, $p_n=f(a_{n-1})\cdot f(x_{n-1})$, $n=1, 2, \ldots$, and

$$a_n=\begin{cases} a_{n-1}, & \text{if } p_n<0,\\ x_{n-1}, & \text{if } p_n\geqslant 0.\end{cases} \qquad b_n=\begin{cases} x_{n-1}, & \text{if } p_n\leqslant 0,\\ b_{n-1}, & \text{if } p_n>0.\end{cases}$$

We obtain $[a_n, b_n]\subset[a_{n-1}, b_{n-1}]$, $n=1, 2, \ldots$, where $[a_n, b_n]$ is the root isolating interval whose length is $\frac{1}{2^n}$ the length of the original interval. In particular, $a_n=b_n=x_{n-1}$ if $f(x_{n-1})=0$. **6.8.** 1.6702.

6.9. -0.6823. **6.10.** -2.2340, 0.3276. **6.11.** 2.8931. **6.12.** -1.4305, 1.2963. **6.13.** 2.0946. **6.14.** -2.3300, 0.2016, 2.1284. **6.15.** 0.3684. **6.16.** -2.3247. **6.17.** -0.7976, 1.4945. **6.18.** 0.2510, 1.4934. **6.19.** -0.7549. **6.20.** 1.5160. **6.21.** 3.3532. **6.22.** 1.2970. **6.23.** 1.2672. **6.24.** 1.3713. **6.25.** 1.7556. **6.26.** 0.6529. **6.27.** 0, 0.7469. **6.28.** 4, 0.3099. **6.29.** -1.4916. **6.30.** 0.5110. **6.31.** 0.7391. **6.32.** ± 0.8241. **6.33.** ± 0.7339, 0. **6.34.** 3.6926. **6.35.** 1.8411. **6.36.** 1.0967. **6.37.** 0.1586, 3.1462. *Hint.* Analysing the behaviour of the function $f(x) = e^{x-2} - x$ and its derivative $f'(x) = e^{x-2} - 1$, we conclude that the equation $e^{x-2} - x = 0$ has two roots: one on the interval $[0, 0.3]$, the other on $[3, 3.2]$. Since $f''(x) = e^{x-2} > 0$, the first derivative $f'(x)$ increases and the following inequalities are fulfilled: $-0.864665 = e^{-2} - 1 \leqslant f'(x) \leqslant e^{-1.7} - 1 = -0.817316$ for $x \in [0, 0.3]$, $1.718281 = e - 1 \leqslant f'(x) \leqslant e^{1.2} - 1 = 2.320116$ for $x \in [3, 3.2]$. Therefore, $|f'(x)| > 0.8173$ in the first interval and $|f'(x)| > 1.7182$ in the second which enables us to carry out the estimation of accuracy. **6.38** and **6.39.** *Hint.* Make use of the method of mathematical induction. **6.40.** $f\left(\frac{12}{10}\right) = \cos\frac{\pi}{10} = 0.9511 \pm 0.0001$. **6.41.** $\ln 11 = 2.3979 \pm 0.0003$. *Hint.* To find the values of the function at the points of interpolation, use the equalities $\ln 9 = 2 \ln 3$, $\ln 10 = \ln 5 + \ln 2$, $\ln 12 = 2 \ln 2 + \ln 3$, $\ln 15 = \ln 5 + \ln 3$. **6.42.** $f(1.26) = 1.105$, $f(1.58) = 1.261$. **6.43.** $f(1.89) = 2.092$, $f(2.43) = 3.144$. **6.44.** $f(0.83) = 0.817$, $f(0.97) = 0.942$. **6.45.** $f(1.74) = 1.2148$, $f(1.97) = 1.0007$. **6.46.** $f(2.72) = 1.5463$, $f(2.93) = 0.9805$. **6.47.** $f(23) = 0.921$, $f(41) = 0.755$. **6.48.** $f(1.3) = 1.184$, $f(4.0) = 1.758$. **6.49.** $f(0.20) = 0.1987$, $f(0.41) = 0.3990$. **6.50.** $f(1.25) = 0.0771$, $f(1.76) = 0.0128$. **6.51.** $f(58) = 0.275$, $f(79) = -0.291$. **6.52.** $\mathrm{Si}(0.26) = 0.25903$, $\mathrm{Si}(0.45) = 0.44497$. **6.53.** $\Phi(0.27) = 0.29742$, $\Phi(0.58) = 0.58792$. **6.55.** 1.82. **6.56.** 1.45. **6.57.** 2.3. **6.58.** $60^\circ 30'$. **6.59.** $f'(2.03) = 1.42249$, $f'(2.22) = 1.87640$. **6.60.** $f'(1.14) = 1.0704$, $f'(1.42) = 1.1698$. **6.61.** $f'(3.02) = 5.63133$, $f'(3.31) = 7.34833$. **6.62.** $f'(0.82) = 0.8077$, $f'(1.03) = 0.9914$. **6.63.** $f'(1.34) = 0.1873$, $f'(1.65) = 0.0741$. **6.64.** $f'(2) = 9$, $f''(2) = 12$. **6.65.** $f'(2.5) = 63.5$, $f''(2.5) = 75$.

Chapter 6

INTEGRAL CALCULUS: FUNCTIONS OF ONE VARIABLE

SEC. 6.1. BASIC METHODS OF COMPUTING THE INDEFINITE INTEGRAL

1. Antiderivative (Primitive) and Indefinite Integral. A function $F(x)$ is called an *antiderivative* (a *primitive*) of the function $f(x)$ defined on some set X if $F'(x) = f(x)$ for all $x \in X$. If $\Phi(x)$ and $F(x)$ are two antiderivatives (two primitive functions) of one and the same function $f(x)$, then

$$\Phi(x) = F(x) + C,$$

where C is a constant. Conversely, if $F(x)$ is an antiderivative of the function $f(x)$, the $\{F(x) + C \mid C \in \mathbb{R}\}$ is the totality of all of its primitive functions called the *indefinite integral* of the function $f(x)$ and denoted by the symbol $\int f(x)\,dx$. Thus, by definition,

$$\int f(x)\,dx = \{F(x) + C\}, \tag{1}$$

where $F(x)$ is one of the antiderivatives of the function $f(x)$, and the constant C takes on real values.

By virtue of the established tradition, equality (1) is written without denoting explicitly the set on the right, i.e. in the form

$$\int f(x)\,dx = F(x) + C,$$

where C is called an arbitrary constant.

Properties of the indefinite integral:

1. $\left(\int f(x)\,dx\right)' = f(x)$.
2. $\int f'(x)\,dx = f(x) + C$.
3. $\int af(x)\,dx = a\int f(x)\,dx, \quad a \neq 0$.
4. $\int (f_1(x) + f_2(x))\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx$.

The table of basic indefinite integrals:

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1).$
2. $\int \frac{dx}{x} = \ln|x| + C.$
3. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1); \quad \int e^x\,dx = e^x + C.$
4. $\int \sin x\,dx = -\cos x + C.$
5. $\int \cos x\,dx = \sin x + C.$
6. $\int \frac{dx}{\cos^2 x} = \tan x + C.$
7. $\int \frac{dx}{\sin^2 x} = -\cot x + C.$
8. $\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| + C.$
9. $\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|.$
10. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C \quad (a \neq 0).$
11. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C.$
12. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C, \quad |x| < |a|.$
13. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2-a^2}| + C, \quad |x| > |a|.$
14. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) + C.$
15. $\int \sinh x\,dx = \cosh x + C.$
16. $\int \cosh x\,dx = \sinh x + C.$
17. $\int \frac{dx}{\cosh^2 x} = \tanh x + C.$
18. $\int \frac{dx}{\sinh^2 x} = -\coth x + C.$

In Problems 1.1 to 1.16 find the antiderivatives of the given functions.

1.1. $2x^7$. **1.2.** $4\sqrt[3]{x}$. **1.3.** $\frac{3}{x}+\frac{5}{x^2}$.

1.4. $\frac{x^3+5x^2-1}{x}$. **1.5.** $\frac{(\sqrt{x}+1)^2}{x\sqrt{x}}$.

1.6. $1-2\sin^2\frac{x}{2}$. **1.7.** $\frac{1}{\sqrt{a+bx}}$. **1.8.** e^{2-3x}. **1.9.** $\frac{1}{\sqrt[3]{5^x}}$.

1.10. $\frac{1}{\cos^2 4x}$. **1.11.** $\frac{x^3+1}{x-1}$. **1.12.** $1-8\sin^2 2x\cos^2 2x$.

1.13. $\left(\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}-\sin^2\frac{x}{2}\right)^2$.

1.14. $\cos(\alpha+x)\cos(\alpha-x)+\sin(\alpha+x)\sin(\alpha-x)$.

1.15. $\int\left(3x^2+2x+\frac{1}{x}\right)dx$. **1.16.** $\int\frac{2x+3}{x^1}\,dx$.

The finding of an indefinite integral with the aid of the table of integrals and identical transformations is called direct integration.

Example 1. Compute $\int\frac{dx}{x^2-x^4}$.

$$\int\frac{dx}{x^2-x^4}=\int\frac{dx}{x^2(1-x^2)}=\int\frac{1-x^2+x^2}{x^2(1-x^2)}\,dx=$$
$$=\int\frac{dx}{x^2}+\int\frac{dx}{1-x^2}=-\frac{1}{x}+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+C.$$

In Problems 1.17 to 1.43 find the indicated integrals, making use of the table of basic integrals.

1.17. $\int\sqrt{mx}\,dx$. **1.18.** $\int\frac{dx}{\sqrt[n]{x}}$.

1.19. $\int\left(\frac{1}{\sqrt[3]{x^2}}-\frac{x+1}{\sqrt[4]{x^3}}\right)dx$.

1.20. $\int\frac{(\sqrt{a}+\sqrt{x})^2}{\sqrt{ax}}\,dx$.

1.21. $\int\frac{x^3+2}{x}dx$. **1.22.** $\int 2^x e^x\,dx$.

1.23. $\int 2^x(1+3x^2 2^{-x})\,dx$.

1.24. $\int (2x+3\cos x)\,dx$. **1.25.** $\int \frac{2-\sin x}{\sin^2 x}\,dx$.

1.26. $\int \frac{3-2\cot^2 x}{\cos^2 x}\,dx$.

1.27. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\,dx$. **1.28.** $\int \sin^2 \frac{x}{2}\,dx$.

1.29*. (a) $\int \tan^2 x\,dx$; (b) $\int \tanh^2 x\,dx$.

1.30. $\int \frac{dx}{\cos 2x+\sin^2 x}$.

1.31. $\int (\arcsin x+\arccos x)\,dx$.

1.32. $\int \left(\sin \frac{x}{2}-\cos \frac{x}{2}\right)^2 dx$.

1.33. $\int \frac{dx}{x^2+4}$. **1.34.** $\int \frac{dx}{5-x^2}$. **1.35.** $\int \frac{dx}{\sqrt{3-x^2}}$.

1.36. $\int \frac{\sqrt{x^2-3}-\sqrt{x^2+3}}{\sqrt{x^2-9}}\,dx$. **1.37.** $\int \frac{(1+x)^2}{x(1+x^2)}\,dx$.

1.38. $\int (x+a)(x+b)\,dx$. **1.39.** $\int (a^{1/3}+x^{1/3})^3\,dx$.

1.40. $\int \frac{\cos^2 x+3\cos x-2}{\cos^2 x}\,dx$.

1.41. (a) $\int \cot^2 x\,dx$; (b) $\int \coth^2 x\,dx$.

1.42. $\int \frac{dx}{\sqrt{x^2-7}}$. **1.43.** $\int \frac{x^2-9}{x^2-8}\,dx$.

2. **Integration by Change of Variable.** There are two variants of the technique:

(a) *The method of placing under the differential sign.* Let it be required to compute the integral $\int f(x)\,dx$. Let us assume that there exist a differentiable function $u=\varphi(x)$ and a function $g(u)$ such that the integrand $f(x)\,dx$ can be written in the form

$$f(x)\,dx = g(\varphi(x))\,\varphi'(x)\,dx = g(u)\,du$$

(this transformation is known as placing $u=\varphi(x)$ under the differential sign). Note that the following relation is fulfilled:

$$\int f(x)\,dx = \int g(\varphi(x))\,\varphi'(x)\,dx = \int g(u)\,du\Big|_{u=\varphi(x)}.$$

Therefore the computation of the integral $\int f(x)\,dx$ is reduced to computing the integral $\int g(u)\,du$ (which may turn out to be simpler than the original one) followed by the substitution $u = \varphi(x)$.

Example 2. Compute the integral $\int \sin^3 x \cos x\,dx$.

We have:

$$\int \sin^3 x \cos x\,dx = \int \sin^3 x\,d(\sin x)$$

$$= \int u^3\,du = \frac{u^4}{4}\Big|_{u=\sin x} + C = \frac{\sin^4 x}{4} + C.$$

Example 3. Compute the integral $\int \frac{2x+1}{x^2+x-3}\,dx$.

We have:

$$\int \frac{2x+1}{x^2+x-3}\,dx = \int \frac{d(x^2+x-3)}{x^2+x-3} = \int \frac{du}{u}$$

$$= \ln|u|\,|_{u=x^2+x-3} + C = \ln|x^2+x-3| + C.$$

The operation of placing the function $\varphi(x)$ under differential sign is equivalent to replacing the variable x by a new variable $u = \varphi(x)$.

Example 4. Compute the integral $\int \frac{dx}{\sqrt[3]{(3x+1)^2}}$.

Let us change variable by the formulas

$$u = 3x + 1.$$

Then $du = 3dx$, i.e. $dx = \frac{1}{3}du$ and

$$\int \frac{dx}{\sqrt[3]{(3x+1)^2}} = \frac{1}{3}\int \frac{du}{u^{2/3}} = u^{1/3}\Big|_{u=3x+1} + C = \sqrt[3]{3x+1} + C.$$

The transformation performed is equivalent to placing the function $u = 3x + 1$ under the differential sign.

In Problems 1.44 to 1.80 compute the given integral with the aid of a suitable substitution.

1.44. $\int \sqrt{3+x}\,dx$. **1.45.** $\int (3-4\sin x)^{1/3}\cos x\,dx$.

1.46. $\int \cosh x \sinh x\,dx$. **1.47.** $\int \frac{\sec^2 x}{\tan^4 x}\,dx$.

1.48. $\int \frac{dx}{x\ln^2 x}$. **1.49.** $\int \frac{dx}{a+bx}$.

1.50. $\int \frac{\sec^2 x}{a-b\tan x}\,dx$. **1.51.** $\int \frac{\cos\frac{x}{\sqrt{2}}}{2-3\sin\frac{x}{\sqrt{2}}}\,dx$.

1.52. $\int \cot x\,dx$. **1.53.** $\int 3^{4x}\,dx$.

1.54. $\int \cos(ax+b)\,dx$. **1.55.** $\int \sin(\ln x)\frac{dx}{x}$.

1.56. $\int \sin\sqrt{x}\cdot\frac{dx}{\sqrt{x}}$. **1.57.** $\int \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$.

1.58. $\int \frac{dx}{\sinh^2 3x}$. **1.59.** $\int \frac{x}{\sqrt[3]{x^2-1}}\,dx$. **1.60.** $\int x\cdot 5^{x^2}\,dx$.

1.61. $\int \frac{dx}{1-4x^2}$. **1.62.** $\int \frac{e^{-ax}}{1+e^{-2ax}}\,dx$. **1.63.** $\int \frac{dx}{\sqrt{5-3x^2}}$.

1.64. $\int \frac{dx}{\sqrt{9x^2-1}}$. **1.65.** $\int \frac{\sin x\,dx}{\sqrt{\cos^2 x+4}}$. **1.66.** $\int \frac{x^3\,dx}{x^8+1}$.

1.67. $\int \frac{x\,dx}{\sqrt{x^4+1}}$. **1.68.** $\int \frac{dx}{a^2+b^2x}$. **1.69.** $\int \frac{\sin ax}{\cos^3 ax}\,dx$.

1.70. $\int \cosh^2 x\sinh x\,dx$. **1.71.** $\int \frac{e^x}{(7-e^x)^2}\,dx$.

1.72. $\int \tan x\,dx$. **1.73.** $\int \coth 4x\,dx$. **1.74.** $\int \frac{a^{1/x}}{x^2}\,dx$.

1.75. $\int \frac{x\,dx}{\cosh^2(x^2+1)}$. **1.76.** $\int \frac{dx}{(a-b)x^2-(a+b)}\,(0<b<a)$.

1.77. $\int \frac{dx}{4x^2+7}$. **1.78.** $\int \frac{x\,dx}{4x^2+7}$.

1.79. $\int \frac{x^2\,dx}{\sqrt{x^6+1}}$. **1.80.** $\int \frac{a^x}{\sqrt{a^{2x}-1}}\,dx$.

In Problems 1.81 to 1.113 find the indicated indefinite integrals using various techniques.

1.81*. $\int \frac{x-1}{(x+2)^2}\,dx$. **1.82.** $\int \frac{x^2}{3+x^2}\,dx$.

1.83. $\int \frac{x^2-2x+3}{x^2-4}\,dx$. **1.84.** $\int \frac{x\,dx}{a^2x^4-b^2}$.

1.85. $\int \frac{x^3}{9-4x^8}\,dx$. **1.86.** $\int \frac{x^4+1}{x^5+5x-8}\,dx$.

1.87. $\int x^3 \sqrt{5x^4-3}\,dx$. **1.88.** $\int \left(3-\frac{x}{\sqrt{x^2+1}}\right)\frac{dx}{\sqrt{x^2+1}}$.

1.89. $\int \frac{x+1}{\sqrt{1-4x^2}}\,dx$.

1.90. $\int \frac{a^2+\sqrt{a^2+b^2x^2}}{a^2+b^2x^2}\,dx$.

1.91. $\int \frac{dx}{\sqrt[5]{a^x}}$. **1.92.** $\int e^x \sqrt[3]{4+e^x}\,dx$.

1.93. $\int \frac{e^x}{\sqrt{e^{2x}+4}}\,dx$. **1.94*.** $\int \frac{dx}{2^x+1}$.

1.95. $\int \frac{e^{\arcsin x}+x+1}{\sqrt{1-x^2}}\,dx$.

1.96. $\int \frac{xe^{\sqrt{x^2-1}}}{\sqrt{x^2-1}}\,dx$. **1.97.** $\int \sqrt{3-\cosh x}\sinh x\,dx$.

1.98. $\int \frac{dx}{x\sqrt{1-4\ln x}}$. **1.99.** $\frac{dx}{x\sqrt{1-4\ln^2 x}}$.

1.100*. $\int \sin^2 x\,dx$.

1.101*. $\int \cos^2 x\,dx$. **1.102.** $\int \frac{dx}{\sin\frac{x}{\sqrt{2}}}$.

1.103. $\int (\sin ax+\cos ax)^2\,dx$.

1.104. $\int \frac{x^2}{\cos(x^3)}\,dx$. **1.105.** $\int \frac{(1+\cos 2x)^2}{\cos 2x}\,dx$.

1.106. $\int \frac{\sin 2x}{\sqrt{3-\cos^2 x}}\,dx$. **1.107.** $\frac{\sin 2x}{\sqrt{\cos^4 x+3}}\,dx$.

1.108*. $\int \frac{dx}{\sin x\cos x}$. **1.109.** $\int \frac{dx}{\cot\sqrt{3x}}$

1.110. $\int \tanh ax\,dx$. **1.111.** $\int \tan^2(ax+b)\,dx$.

1.112. $\int x^2\cot^2(x^3-3)\,dx$.

1.113. $\int e^{\sec x}\tan x\sec x\,dx$.

(b) *The method of substitution.* Let it be required to compute the integral $\int f(x)\,dx$, where the function $f(x)$ is defined on a set X.

We introduce a new variable u by the formula

$$x = \varphi(u)\colon\ U \to X,$$

where the function $\varphi(u)$ is differentiable on some set U and maps U one-to-one onto X, i.e. has the inverse

$$u = \varphi^{-1}(x)\colon\ X \to U.$$

Substituting $x = \varphi(u)$ into the original integrand, we obtain

$$f(x)\,dx = f(\varphi(u))\,\varphi'(u)\,du = g(u)\,du.$$

The following equality holds true

$$\int f(x)\,dx = \int f(\varphi(u))\,\varphi'(u)\,du\Big|_{u=\varphi^{-1}(x)} = \int g(u)\,du\Big|_{u=\varphi^{-1}(x)},$$

i.e. the computation of the integral $\int f(x)\,dx$ is reduced to finding the integral $\int g(u)\,du$ (which may turn out to be simpler than the initial one) followed by the substitution $u = \varphi^{-1}(x)$.

Example 5. Compute the integral $\int \frac{1+x}{1+\sqrt{x}}\,dx$.

In the present case the domain of definition of the integrand $X = [0, +\infty)$. Let us make the substitution

$$x = \varphi(u) = u^2,\ u \in [0, +\infty).$$

Then $dx = 2u\,du$, $u = \varphi^{-1}(x) = \sqrt{x}$, whence

$$\begin{aligned}\int \frac{1+x}{1+\sqrt{x}}\,dx &= 2\int \frac{u^3+u}{u+1}\,du \\ &= 2\int (u^2-u+2)\,du - 4\int \frac{du}{u+1} \\ &= 2\left(\frac{1}{2}u^3 - \frac{1}{3}u^2 + 2u\right) - 4\ln(u+1) + C\Big|_{u=\sqrt{x}} \\ &= 2\left(\frac{1}{3}x^{3/2} - \frac{1}{2}x + 2x^{1/2}\right) - 4\ln(\sqrt{x}+1) + C.\end{aligned}$$

In Problems 1.114 to 1.127 find the indefinite integrals making the indicated substitutions.

1.114. $\int \frac{\sin x\,dx}{4+\cos^2 x}$. **1.115.** $\int \frac{dx}{x(1-\ln^2 x)}$.

1.116. $\int \frac{e^{2x}}{1+3e^{2x}}\,dx$. **1.117.** $\int x^2 e^{4x^3}\,dx$.

1.118. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. **1.119.** $\int \frac{x^3\,dx}{\sqrt{x-1}}$.

1.120. $\int \frac{dx}{1+\sqrt[3]{x+1}}$. **1.121.** $\int \sqrt{4x-1}\,dx$.

1.122. $\int \sqrt[3]{5+6x}\,dx$. **1.123.** $\int \sin(a+bx)\,dx$.

1.124. $\int \frac{dx}{x\sqrt{1-x^3}}$, $x=(1-t^2)^{1/3}$.

1.125. $\int \frac{dx}{x\sqrt{4-x^2}}$, $x=\frac{2}{t}$.

1.126. $\int \frac{dx}{x+\sqrt{x}}$, $x=t^2$.

1.127. $\int \frac{e^{2x}}{e^x+1}\,dx$, $x=\ln t$.

In Problems 1.128 to 1.133 find the indicated integrals using suitable substitutions.

1.128. $\int x(5x-1)^{19}\,dx$. **1.129.** $\int \frac{e^{3x}}{\sqrt{1-e^x}}\,dx$.

1.130. $\int \frac{x+2}{\sqrt{x+1}+1}\,dx$. **1.131.** $\int \frac{x}{(3-x)^7}\,dx$.

1.132. $\int \frac{dx}{\sqrt{3+e^x}}$. **1.133.** $\int \frac{dx}{x\sqrt{x^2+1}}$.

3. Method of Integration by Parts. If $u(x)$ and $v(x)$ are differentiable functions, then the following formula for integration by parts is valid

$$\int u\,dv = uv - \int v\,du. \tag{2}$$

This formula is used when the integrand $f(x)\,dx$ can be represented in the form $u\,dv$ so that, with a proper choice of expressing u and dv, the integral standing on the right of (2) may turn out to be simpler than the original integral. One should bear in mind that u must be supplied with such factors which get simplified during differentiation. For instance, if the integrand is a product of a polynomial by a trigonometric or exponential function, then the polynomial should be distributed to u, and the remaining expression to dv. Here formula (2) may be used repeatedly.

Example 6. Find $\int x^2 \cos x\, dx$.

We set $u = x^2$ and $dv = \cos x\, dx$. Then $du = 2x\, dx$ and $v = \int \cos x\, dx = \sin x$ (the constant C is assumed here to be equal to zero, i.e. we take as v one of the antiderivatives). By formula (2), we have

$$\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx.$$

We apply the formula for integration by parts once again to the integral standing on the right, equating u, as before, to the polynomial (i.e. to $2x$). We have: $u = 2x$, $dv = \sin x\, dx$. Hence,

$$du = 2dx \text{ and } v = \int \sin x\, dx = -\cos x.$$

Applying formula (2), we finally obtain:

$$\int x^2 \cos x\, dx = x^2 \sin x - (-2x \cos x - \int (-\cos x)\, 2dx)$$
$$= x^2 \sin x + 2x \cos x - 2 \sin x + C).$$

If a logarithmic or inverse trigonometric function is contained (as a factor) in the integrand, then it should be taken for u since these functions are simplified during differentiation.

Example 7. Find $\int \ln x\, dx$.

We set $u = \ln x$, $dv = dx$. Then $du = \frac{dx}{x}$ and $v = \int dx = x$. Substituting into formula (2), we find

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{dx}{x} x \ln x - x + C.$$

After repeated application of the formula for integration by parts, we sometimes arrive in the right-hand side at an expression containing the original integral, i.e. we obtain an equation with the sought-for integral as an unknown.

Example 8. Find $\int e^{ax} \sin bx\, dx$.

We set $u = e^{ax}$, $dv = \sin bx\, dx$. Then $du = ae^{ax}\, dx$, $v = -\frac{1}{b} \cos bx$. Substituting into (2), we have

$$\int e^{ax} \sin bx\, dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx\, dx.$$

Now, we set $u=e^{ax}$, $dv=\cos bx\,dx$. Then $du=ae^{ax}\,dx$, $v=\frac{1}{b}\sin bx$ and

$$\int e^{ax}\sin bx\,dx=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}\left(\frac{e^{ax}}{b}\sin bx-\frac{a}{b}\int e^{ax}\sin bx\,dx\right).$$

As the final result, we have obtained an equation with respect to the unknown integral $\int e^{ax}\sin bx\,dx$. Solving the equation, we find

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx=e^{ax}\,\frac{a\sin bx-b\cos bx}{b^2}+C_1,$$

or

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C.$$

In Problems 1.134 to 1.153 find the indicated integrals, applying the formula for integration by parts.

1.134. $\int \arccos x\,dx$. **1.135.** $\int x\cos x\,dx$.

1.136. $\int x\ln x\,dx$. **1.137.** $\int\frac{\ln x}{\sqrt[3]{x}}\,dx$.

1.138. $\int(x^2-x+1)\ln x\,dx$. **1.139.** $\int x^2\sin x\,dx$.

1.140. $\int x^2e^{-x}\,dx$. **1.141.** $\int x^3e^x\,dx$. **1.142*.** $\int x^3e^{x^2}\,dx$.

1.143. $\int\frac{\ln^2 x}{x^2}\,dx$. **1.144.** $\int x\arctan x\,dx$.

1.145. $\int\frac{x\sin x}{\cos^3 x}\,dx$. **1.146.** $\int e^{ax}\cos bx\,dx$.

1.147. $\int e^{\arccos x}\,dx$. **1.148.** $\int\ln\left(x+\sqrt{1+x^2}\right)dx$.

1.149. $\int x^3\ln x\,dx$. **1.150.** $\int x3^x\,dx$.

1.151. $\int(x^2-2x+3)\cos x\,dx$. **1.152.** $\int\frac{x\,dx}{\cos^2 x}$.

1.153. $\int\cos(\ln x)\,dx$.

In Problems 1.154 to 1.159 find the indicated integrals using various methods.

1.154*. $\int e^{\sqrt{x}}\,dx$. **1.155.** $\int x\,(\arctan x)^2\,dx$.

1.156. $\int \frac{\arcsin x}{x^2}\,dx$. **1.157.** $\int x\cot^2 x\,dx$.

1.158. $\int \frac{\cos^2 x}{e^x}\,dx$. **1.159*.** $\int \frac{x^2}{(x^2+1)^2}\,dx$.

1.160.** Derive the recurrence formula for integral $I_n = \int \frac{dx}{(x^2+a^2)^n}$. Find I_2 and I_3.

In Problems 1.161 to 1.167 find the indicated integrals.

1.161.** $\int \sqrt{x^2+a}\,dx$. **1.162**.** $\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$.

1.163. $\int x \arcsin x\,dx$. **1.164.** $\int \frac{\ln(\ln x)}{x}\,dx$.

1.165. $\int x^2 \arctan x\,dx$. **1.166.** $\frac{\arcsin\sqrt{x}}{\sqrt{x-1}}\,dx$.

1.167*. $\int \sqrt{a^2-x^2}\,dx$.

SEC. 6.2.
INTEGRATION OF THE BASIC CLASSES OF ELEMENTARY FUNCTIONS

1. Simplest Integrals Containing a Quadratic Trinomial. *Integrals of the form*

$$\int \frac{dx}{ax^2+bx+c} \text{ and } \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

are reduced to tabular integrals 10-14 (see the preceding section, subsection 1) by *separating a perfect trinomial square* from a quadratic trinomial.

Example 1. Find $\int \frac{dx}{\sqrt{1-4x-x^2}}$.

We have

$$\int \frac{dx}{\sqrt{1-4x-x^2}} = \int \frac{dx}{\sqrt{5-(x+2)^2}} = \arcsin\frac{x+2}{\sqrt{5}} + C.$$

Integrals of the form

$$\int \frac{mx+n}{ax^2+bx+c}\,dx \quad \text{and} \quad \int \frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$$

are reduced to integrals of the form

$$\int \frac{dx}{ax^2+bx+c} \quad \text{and} \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

by separating in the numerator the derivative $2a + b$ of a quadratic trinomial.

Example 2. Find $\int \frac{x-1}{3x^2+2x+1}\,dx$.

Since $(3x^2+2x+1)' = 6x+2$, $x-1 = \frac{1}{6}(6x+2) - \frac{4}{3}$, we get

$$\int \frac{x-1}{3x^2+2x+1}\,dx = \frac{1}{6}\int \frac{6x+2}{3x^2+2x+1}\,dx - \frac{4}{3}\int \frac{dx}{3\left(x^2+\frac{2}{3}x+\frac{1}{3}\right)}$$

$$= \frac{1}{6}\ln(3x^2+2x+1) - \frac{4}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}}$$

$$= \frac{1}{6}\ln(3x^2+2x+1) - \frac{2\sqrt{2}}{3}\arctan\frac{3x+1}{\sqrt{2}} + C.$$

Integrals of the form

$$\int \frac{dx}{(mx+n)^r\sqrt{ax^2+bx+c}} \qquad (r=1,\ 2)$$

are reduced to the above considered integrals with the aid of the substitution $mx+n = \frac{1}{t}$.

Example 3. Find $\int \frac{dx}{x\sqrt{x^2-2x-1}}$.

We set $x = \frac{1}{t}$. Then $dx = -\frac{dt}{t^2}$,

$$\sqrt{x^2-2x-1} = \left|\sqrt{\frac{1}{t^2}-\frac{2}{t}-1}\right| = \frac{\sqrt{1-2t-t^2}}{t}$$

and

$$\int \frac{dx}{x\sqrt{x^2-2x-1}} = -\int \frac{dt}{t^2 \cdot \frac{1}{t} \cdot \frac{\sqrt{1-2t-t^2}}{t}}$$

$$= -\int \frac{dt}{\sqrt{2-(t+1)^2}} = -\arcsin\frac{t+1}{\sqrt{2}} + C$$

$$= -\arcsin\frac{\frac{1}{x}+1}{\sqrt{2}} + C = -\arcsin\frac{x+1}{x\sqrt{2}} + C.$$

In Problems 2.1 to 2.18 find the indicated integrals.

2.1. $\int \frac{dx}{x^2+4x-5}$. 2.2. $\int \frac{dx}{2x^2-4x+5}$.

2.3. $\int \frac{dx}{\sqrt{8x-x^2}}$. 2.4. $\int \frac{dx}{\sqrt{x^2+px+q}}$.

2.5. $\int \frac{dx}{x^2-6x}$. 2.6. $\int \frac{dx}{\sqrt{4-6x-3x^2}}$.

2.7. $\int \frac{x\,dx}{x^2-5x+4}$. 2.8. $\int \frac{dx}{x^2-3x+3}$.

2.9. $\int \frac{x+4}{\sqrt{2-x-x^2}}\,dx$. 2.10. $\int \frac{x-3}{\sqrt{x^2-6x+1}}\,dx$.

2.11. $\int \frac{x\,dx}{x^4+6x^2+13}$. 2.12. $\int \frac{3^x\,dx}{3^{2x}-4\cdot 3^x+3}$.

2.13. $\int \frac{4x-3}{x^2-2x+6}\,dx$. 2.14. $\int \frac{x}{\sqrt{x^2+x+1}}\,dx$.

2.15. $\int \frac{dx}{x\sqrt{x^2+8x+1}}$. 2.16. $\int \frac{dx}{(x-1)\sqrt{6x-x^2-5}}$.

2.17. $\int \frac{dx}{x^2\sqrt{1-x+2x^2}}$. 2.18. $\int \frac{dx}{(x+2)^2\sqrt{x^2+5}}$.

2. Integration of Rational Fractions. Fractions of the form $\frac{A}{x-\alpha}$, $\frac{A}{(x-\alpha)^k}$, $\frac{Ax+B}{x^2+px+q}$, $\frac{Ax+B}{(x^2+px+q)^k}$, where $k=2, 3, \ldots$; A, B, α, p, and q are constants, and $p^2 - 4q < 0$, are called *partial*.

The integrals of the partial fractions of the first two types are found in an elementary way, the integration of the partial fraction of the third type was considered in Example 2.

The integration of the fraction of the fourth type after separation in the numerator of the derivative of the quadratic trinomial standing

in the denominator and separation of a perfect square in this trinomial is reduced to computing the integrals

$$\int (x^2+px+q)^{-k}\,d\,(x^2+px+q) = -\frac{1}{(k-1)\,(x^2+px+q)^{k-1}}\,,$$

and

$$\int \frac{du}{(u^2+a^2)^k}\,.$$

The last integral can be computed by a recurrence formula (see Problem 1.160).

In the general case, the integration of an arbitrary rational fraction $\frac{P_m(x)}{Q_n(x)} = \frac{a_m x^m + \ldots + a_1 x + a_0}{b_n x^n + \ldots + b_1 x + b_0}$ with real coefficients is performed in the following way.

(1) If $m \geqslant n$, i.e. if the original fraction $\frac{P_m(x)}{Q_n(x)}$ is *improper*, then first it is necessary to take out the *integral part* of this fraction, i.e. to represent it in the form

$$\frac{P_m(x)}{Q_n(x)} = M_{m-n}(x) + \frac{R_r(x)}{Q_n(x)}\,, \tag{1}$$

where $M_{m-n}(x)$ and $R_r(x)$ are polynomials of degrees $m-n \geqslant 0$ and r, respectively, and $r < n$, i.e. the fraction $\frac{R_r(x)}{Q_n(x)}$ is *proper*.

The integral part of an improper fraction $\frac{P_m(x)}{Q_n(x)}$ is taken out by dividing the numerator by the denominator.

Example 4. Take out the integral part of the fraction

$$\frac{P_m(x)}{Q_n(x)} = \frac{(x^2+1)^3}{x\,(x^2-2x+1)}\,.$$

The given fraction is improper, since $m = 6 > n = 3$. To take out the integral part, we write both the numerator and denominator in the canonical form:

$$(x^2+1) = x^6 + 3x^4 + 3x^2 + 1,$$
$$x\,(x^2-2x+1) = x^3 - 2x^2 + x,$$

and then, dividing the first polynomial by the second, we obtain $x^3 + 2x^2 + 6x + 10$ as a partial quotient and $17x^2 - 10x + 1$ as a remainder. Consequently,

$$\frac{(x^2+1)}{x\,(x^2-2x+1)} = x^3 + 2x^2 + 6x + 10 + \frac{17x^2-10x+1}{x^3-2x^2+x}\,,$$

the integral part thus having been taken out.

(2) As is seen from formula (1), the operation of separating the integral part reduces the integration of an arbitrary rational fraction to integration of a polynomial and a proper rational fraction.

Integration of the proper rational fraction $\frac{P_m(x)}{Q_n(x)}$, $m < n$, is performed by expanding the fraction into a sum of partial fractions of the above indicated four types which are then integrated.

This expansion is carried out in the following way. Let the denominator $Q_n(x) = a_n x^n + \ldots + a_1 x + a_0$ have real roots $\alpha_1, \ldots, \alpha_l$ of respective multiplicities $s_1, \ldots, s_l$ and complex conjugate pairs of roots $\beta_1, \overline{\beta}_1, \ldots, \beta_k, \overline{\beta}_k$ of multiplicities $t_1, \ldots, t_k$, respectively $(s_1 + \ldots + s_l + 2t_1 + \ldots + 2t_k = n)$, i.e. the following expansion is valid:

$$Q_n(x) = a_n (x - \alpha_1)^{s_1} \ldots (x - \alpha_l)^{s_l} (x^2 + p_1 x + q_1)^{t_1} \ldots \ldots (x^2 p_k x + q_k)^{t_k},$$

where

$$x^2 + p_\nu x + q_\nu = (x - \beta_\nu)(x - \overline{\beta}_\nu), \quad \nu = 1, \ldots, k.$$

Then the expansion of the fraction $\frac{P_m(x)}{Q_n(x)}$ into a sum of partial fractions has the form

$$\frac{P_m(x)}{Q_n(x)} = \frac{A_1^{(1)}}{x-\alpha_1} + \ldots + \frac{A_{s_1}^{(1)}}{(x-\alpha_1)^{s_1}} + \ldots + \frac{A_1^{(l)}}{x-\alpha_l} + \ldots + \frac{A_{s_l}^{(l)}}{(x-\alpha_l)^{s_l}} + \frac{B_1^{(1)}x + C_1^{(1)}}{x^2 + p_1 x + q_1} + \ldots + \frac{B_{t_1}^{(1)}x + C_{t_1}^{(1)}}{(x^2 + p_1 x + q_1)^{t_1}} + \ldots + \frac{B_1^{(k)}x + C_1^{(k)}}{x^2 + p_k x + q_k} + \ldots + \frac{B_{t_k}^{(k)}x + C_{t_k}^{(k)}}{(x^2 + p_k x + q_k)^{t_k}}. \quad (2)$$

The coefficients A_i, B_i, and C_i in this expansion are determined by equating the coefficients of like powers of x in the polynomial $P_m(x)$ and the polynomial obtained in the numerator of the right-hand member of (2) on reducing it to a common denominator (method of undetermined coefficients). These coefficients can also be determined by setting x in equality (2) or its equivalent equal to suitably chosen numbers (in the first place to the values of real roots of the denominator $Q_n(x)$).

Example 5. Find $\int \frac{x^2+4x+4}{x(x-1)^2}\,dx$.

The integrand $\frac{x^2+4x+4}{x(x-1)^2}$ is a proper fraction, its expansion into a sum of partial fractions has the form

$$\frac{x^2+4x+4}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Reducing the right-hand side to a common denominator, we obtain

$$x^2 + 4x + 4 = A(x-1)^2 + Bx(x-1) + Cx \qquad (3)$$

(identical equality of the numerators), whence, equating the coefficients of like powers of x, we have

$$A + B = 1, \quad -2A - B + C = 4, \quad A = 4,$$

which yields

$$B = -3, \quad C = 9.$$

Hence,

$$\int \frac{x^2+4x+4}{x(x-1)^2}\,dx = \int \left(\frac{4}{x} - \frac{3}{x-1} + \frac{9}{(x-1)^2}\right) dx$$

$$= 4\ln|x| - 3\ln|x-1| - \frac{9}{x-1} + C.$$

We could determine the coefficients A, B, C by putting in identity (3) $x = 0$, $x = 1$ and, in addition, $x = -1$. Then for $x = 0$ we find $A = 4$, for $x = 1$ we obtain $C = 9$, and for $x = -1$ we have $4A + 2B - C = 1$, that is, $B = -3$.

When solving this example, it would be best of all to combine the 1st and 2nd methods, i.e. to find $A = 4$ for $x = 0$, $C = 9$ for $x = 1$, and to determine B from the condition of equality of the coefficients of x^2 in (3), i.e. from the equality $A + B = 1$.

Example 6. Find $\int \frac{dx}{x(x^2+1)^2}$.

The integrand $\frac{1}{x(x^2+1)^2}$ is a proper fraction, its expansion into a sum of partial fractions has the form

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

We have

$$1 = A(x^2+1)^2 + Bx^2(x^2+1) + Cx(x^2+1) + Dx^2 + Ex.$$

Setting $x = 0$, we find $A = 1$. Equating the coefficients of like powers of x, we get $0 = A + B$, $0 = C$, $0 = 2A + B + D$, $0 = C + E$, that is,

$$B = -1, \ C = 0, \ D = -1 \text{ and } E = 0.$$

Consequently,

$$\int \frac{dx}{x(x^2+1)^2} = \int \left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right) dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.$$

Note that we can also expand $\frac{1}{x(x^2+1)^2}$ into partial fractions without applying the method of undetermined coefficients, namely,

$$\frac{1}{x(x^2+1)^2}=\frac{(1+x^2)-x^2}{x(x^2+1)^2}=\frac{1}{x(x^2+1)}-\frac{x}{(x^2+1)^2}$$
$$=\frac{(1+x^2)-x^2}{x(x^2+1)}-\frac{x}{(x^2+1)^2}=\frac{1}{x}-\frac{x}{x^2+1}-\frac{x}{(x^2+1)^2}.$$

In Problems 2.19 to 2.34 find the indicated integrals.

2.19. $\int \frac{dx}{(x-3)(x+4)}$. **2.20.** $\int \frac{2x^2-1}{x^3-5x^2+6x}\,dx$.

2.21. $\int \frac{x^3+2}{x^3-4x}\,dx$. **2.22.** $\int \frac{x^4+3x^3+3x^2-5}{x^3+3x^2+3x+1}\,dx$.

2.23. $\int \frac{3x^2+2x-1}{(x-1)^2(x+2)}\,dx$. **2.24.** $\int \frac{2x-5}{(x^2-5x+4)^3}\,dx$.

2.25. $\int \frac{dx}{x(x^2+2)}$. **2.26.** $\int \frac{dx}{x^4+1}$.

2.27*. $\int \frac{(x-1)\,dx}{(x^2+1)^3}$. **2.28*.** $\int \frac{x\,dx}{(x-1)(x^2+x+1)^2}$.

2.29. $\int \frac{dx}{(x-a)(x-b)}$. **2.30.** $\int \frac{x^2-x+4}{(x+1)(x-2)(x-3)}\,dx$.

2.31. $\int \frac{dx}{x^3+8}$. **2.32.** $\int \frac{5x-13}{(x^2-5x+6)^2}\,dx$.

2.33. $\int \frac{x^4+1}{x^4-1}\,dx$. **2.34.** $\int \frac{dx}{x^4+2x^2+1}$.

In Problems 2.35 to 2.42 find the indicated integrals without applying the method of undetermined coefficients.

2.35*. $\int \frac{dx}{x^4+a^2x^2}$. **2.36*.** $\int \frac{dx}{x^4-a^4}$.

2.37. $\int \frac{dx}{x^4-4x^2+3}$. **2.38*.** $\int \frac{dx}{x(x^6+1)^2}$.

2.39. $\int \frac{dx}{x^7+x^5}$. **2.40*.** $\int \frac{x^7}{(x^4+1)(x^4-2)}\,dx$.

2.41. $\int \frac{x^2-x}{(x+1)^9}\,dx$. **2.42.** $\int \frac{x^5+x^2}{x^9+x^3-2}\,dx$.

3. Integration of Trigonometric and Hyperbolic Functions.

(a) Integrals of the form $\int \sin^m x \cos^n x\,dx$.

If m or n is an odd positive integer, then, separating one factor from the odd power and expressing the remaining even power in terms of an additional function with the aid of the formula $\sin^2 x + \cos^2 x = 1$, we arrive at a tabular integral.

Example 7. Find $\int \frac{\sin^3 x}{\sqrt[4]{\cos x}}\,dx$.

We have:

$$\int \frac{\sin^3 x}{\sqrt[4]{\cos x}}\,dx = \int \frac{\sin^2 x}{\sqrt[4]{\cos x}} \sin x\,dx = -\int \frac{1-\cos^2 x}{\sqrt[4]{\cos x}}\,d\cos x$$

$$= -\int \frac{d\cos x}{\sqrt[4]{\cos x}} + \int \frac{\cos^2 x}{\sqrt[4]{\cos x}}\,d\cos x$$

$$= -\frac{4}{3}\sqrt[4]{\cos^3 x} + \frac{4}{11}\sqrt[4]{\cos^{11} x} + C.$$

And if both m and n are even nonnegative numbers, then powers are reduced by passing to a double argument (angle) with the aid of the following trigonometric formulas:

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}, \quad \sin x \cos x = \frac{1}{2}\sin 2x.$$

Example 8. Find $\int \sin^2 x \cos^4 x\,dx$.

We have:

$$\int \sin^2 x \cos^4 x\,dx = \int (\sin x \cos x)^2 \cos^2 x\,dx$$

$$= \int \frac{\sin^2 2x}{4} \cdot \frac{1+\cos 2x}{2}\,dx = \frac{1}{8}\int \sin^2 2x\,dx$$

$$+ \frac{1}{8}\int \sin^2 2x \cdot \cos 2x\,dx = \frac{1}{8}\int \frac{1-\cos 4x}{2}\,dx$$

$$+ \frac{1}{16}\int \sin^2 2x\,d\sin 2x = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

If $m + n = -2k$, $k \in \mathbb{N}$, i.e. $m + n$ is an even negative integer, then it is advisable to use the substitutions $\tan x = t$ and $\cot x = t$.

Example 9. Find $\int \sin^{1/3} x \cos^{-13/3} x\,dx$.

Since $\frac{1}{3} - \frac{13}{3} = -4$, the computation of the given integral is reduced to integrating the powers of tangent:

$$\int \sin^{1/3} x \cos^{-13/3} x\,dx = \int \tan^{1/3} x \cdot \frac{dx}{\cos^4 x}$$

$$= \int \tan^{1/3} x\,(1+\tan^2 x)\,\frac{dx}{\cos^2 x} = \int \tan^{1/3} x\,d\tan x$$

$$+ \int \tan^{7/3} x\,d\tan x = \frac{3}{4}\tan^{4/3} x + \frac{3}{10}\tan^{10/3} x + C.$$

To compute integrals of the form $\int \tan^m x\,dx$, $\int \cot^m x\,dx$, where $m = 2, 3, \ldots$, the following trigonometric formulas are used:

$$\tan^2 x = \sec^2 x - 1,\quad \cot^2 x = \csc^2 x - 1.$$

Example 10. Compute $\int \cot^4 x\,dx$.

We have:

$$\int \cot^4 x\,dx = \int \cot^2 x\,(\csc^2 x - 1)\,dx$$

$$= -\int \cot^2 x\,d\cot x - \int (\csc^2 x - 1)\,dx = -\frac{\cot^3 x}{3} + \cot x + x + C.$$

In the general case, integrals of the form $\int \sin^m x\,\cos^n x\,dx$, where m and n are integers, are computed with the aid of recurrence formulas which are derived by integration by parts.

Example 11. Derive the recurrence formula for $\int \frac{dx}{\cos^{2k+1} x}$ and find $\int \frac{dx}{\cos^3 x}$ with the aid of the derived formula.

We have:

$$I_{2k+1} = \int \frac{dx}{\cos^{2k+1} x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^{2k+1} x}\,dx$$

$$= \int \frac{\sin^2 x}{\cos^{2k+1} x}\,dx + \int \frac{dx}{\cos^{2k-1} x} = \int \sin x\,\frac{\sin x}{\cos^{2k+1} x}\,dx + I_{2k-1}.$$

We set $u = \sin x$, $dv = \frac{\sin x}{\cos^{2k+1} x}\,dx$. Then $du = \cos x\,dx$, $v = \frac{1}{2k\cos^{2k} x}$, and, integrating by parts, we obtain

$$I_{2k+1} = \frac{\sin x}{2k\cos^{2k} x} - \frac{1}{2k}\int \frac{dx}{\cos^{2k-1} x} + I_{2k-1},$$

or

$$I_{2k+1} = \frac{\sin x}{2k\cos^{2k} x} + \left(1 - \frac{1}{2k}\right) I_{2k-1}$$

(recurrence formula).

In particular, for $k=1$ we have

$$I_3 = \int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\int \frac{dx}{\cos x}$$

$$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln|\tan x + \sec x| + C. \qquad (58)$$

In Problems 2.43 to 2.64 find the indicated integrals.

2.43. $\int \sin^3 x\,dx$. **2.44.** $\int \frac{\sin^3 x}{\cos^8 x}\,dx$.

2.45. $\int \cos^7 x\,dx$. **2.46.** $\int \cos^4 \frac{x}{2}\,dx$.

2.47. $\int \sin^2 x \cos^2 x\,dx$. **2.48.** $\int \cos^2 x \sin^4 x\,dx$.

2.49. $\int \frac{dx}{\sin^6 x}$. **2.50.** $\int \frac{\sin^2 x}{\cos^6 x}\,dx$.

2.51. $\int \frac{dx}{\sin^3 x \cos^5 x}$. **2.52.** $\frac{dx}{\sin^4 x \cos^2 x}$.

2.53. $\int \frac{\cos\left(x+\frac{\pi}{4}\right)}{\sin x \cos x}\,dx$. **2.54.** $\frac{dx}{\cos^5 x}$.

2.55. $\int \tan^3 x\,dx$. **2.56.** $\int \left(\cot^3 \frac{x}{2} + \cot^4 \frac{x}{2}\right)dx$.

2.57. $\int \frac{dx}{\sqrt{\cos x \sin^3 x}}$. **2.58.** $\int \cos^5 x\,dx$.

2.59. $\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$. **2.60.** $\int \sin^6 2x\,dx$.

2.61. $\int \frac{dx}{\cos^4 x}$. **2.62.** $\frac{dx}{\cos \frac{x}{3} \sin^3 \frac{x}{3}}$.

2.63. $\int \frac{dx}{\sin^3 x}$. **2.64.** $\int \cos x \cos^2 2x\,dx$.

(b) For integrating the products of sines and cosines of different arguments the following trigonometric formulas are used:

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)),$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)),$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)).$$

Example 12. Find $\int \cos 9x \cos 5x\, dx$.

We have

$$\int \cos 9x \cos 5x\, dx = \frac{1}{2}\int (\cos 4x + \cos 14x)\, dx$$

$$= \frac{1}{8}\sin 4x + \frac{1}{28}\sin 14x + C.$$

In Problems 2.65 to 2.70 find the indicated integrals.

2.65. $\int \sin 3x \cos 5x\, dx$. **2.66.** $\int \sin 10x \sin 15x\, dx$.

2.67. $\int \cos \frac{x}{2} \cos \frac{x}{3}\, dx$. **2.68.** $\int \sin \frac{x}{3} \cos \frac{2x}{3}\, dx$.

2.69. $\int \cos x \cos^2 3x\, dx$. **2.70.** $\int \sin x \sin 2x \sin 3x\, dx$.

(c) Integrals of the form

$$\int R(\sin x, \cos x)\, dx,$$

where $R(u, v)$ is a rational function of two variables, are reduced to integrals of a rational function of a new argument t by making the substitution $\tan \frac{x}{2} = t$. In this case the following formulas are used:

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2\,dt}{1+t^2}.$$

Example 13. Find $\int \frac{dx}{4\cos x + 3\sin x + 5}$.

We set $\tan \frac{x}{2} = t$. Then

$$\int \frac{dx}{4\cos x + 3\sin x + 5} = 2\int \frac{dt}{\left(4\cdot\frac{1-t^2}{2+t^2} + 3\cdot\frac{2t}{1+t^2} + 5\right)(1+t^2)}$$

$$= 2\int \frac{dt}{t^2+6t+9} = 2\int \frac{dt}{(t+3)^2} = -\frac{2}{t+3} + C = -\frac{2}{\tan\frac{x}{2}+3} + C.$$

If the integrand contains only even powers of $\sin x$ and $\cos x$, then it is more convenient to use the substitution $\tan x = t$.

Example 14. Find $\int \frac{dx}{1-5\sin^2 x}$.

Dividing both the numerator and denominator by $\cos^2 x$ and making the substitution $\tan x = t$, we obtain

$$\int \frac{dx}{1-5\sin^2 x} = \int \frac{d\tan x}{1+\tan^2 x-5\tan^2 x} = \int \frac{dt}{1-4t^2}$$
$$= \frac{1}{4}\ln\left|\frac{1+2t}{1-2t}\right| + C = \frac{1}{4}\ln\frac{1+2\tan x}{1-2\tan x} + C.$$

In Problems 2.71 to 2.80 find the indicated integrals.

2.71. $\int \frac{dx}{3\cos x+2}$ **2.72.** $\int \frac{dx}{3-2\sin x+\cos x}$.

2.73*. $\int \frac{\sin x}{1+\sin x}dx$. **2.74.** $\int \frac{dx}{4\sin^2 x-7\cos^2 x}$.

2.75. $\int \frac{\sin x}{\cos^2 x-2\cos x+5}dx$. **2.76.** $\int \frac{\sin 2x}{1+4\cos^2 x}dx$.

2.77. $\int \frac{dx}{2-\sin x}$. **2.78*.** $\int \frac{dx}{(\sin x+4)(\sin x-1)}$.

2.79. $\int \frac{1+\cot x}{1-\cot x}dx$.

2.80. $\int \frac{dx}{\sin^2 x+8\sin x\cos x+12\cos^2 x}$.

(d) Integration of hyperbolic functions is performed analogously to integration of trigonometric functions, the following formulas being used:

$$\cosh^2 x - \sinh^2 x = 1, \quad \sinh x \cosh x = \frac{1}{2}\sinh 2x,$$
$$\cosh^2 x = \frac{1}{2}(\cosh 2x+1), \quad \sinh^2 x = \frac{1}{2}(\cosh 2x-1),$$
$$1-\tanh^2 x = \operatorname{sech}^2 x, \quad 1-\coth^2 x = \operatorname{csch}^2 x.$$

In Problems 2.81 to 2.90 find the indicated integrals.

2.81. $\int \cosh^3 3x\,dx$. **2.82.** $\sinh^3 2x\,dx$.

2.83. $\int \sinh^2 x\cosh^2 x\,dx$. **2.84.** $\int \cosh^4 x\,dx$.

2.85. $\int \frac{dx}{\sinh^2 x\cosh^2 x}dx$. **2.86*.** $\int \frac{dx}{\sinh^2 x-4\cosh^2 x}$.

2.87*. $\int \frac{dx}{\cosh x-1}$. **2.88.** $\int \sqrt{\cosh x+1}\,dx$.

2.89. $\int \coth^3 x\,dx$. **2.90.** $\int \tanh^4 x\,dx$.

4. Integration of Some Irrational Functions.
(a) Integrals of the form

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_2}{n_2}}, \ldots\right) dx,$$

where $R(x, y, z, \ldots)$ is a rational function of its arguments, m_1, n_1, m_2, n_2, ..., are whole numbers computed with the aid of the substitution $\frac{ax+b}{cx+d} = t^s$, where s is the common denominator of the fractions $\frac{m_1}{n_1}, \frac{m_2}{n_2}, \ldots$

Example 15. Find $\int \frac{dx}{(\sqrt[4]{x+3}-1)\sqrt{x+3}}$.

We make the substitution $x+3=t^4$. Then $dx=4t^3\,dt$ and, consequently,

$$\int \frac{dx}{(\sqrt[4]{x+3}-1)\sqrt{x+3}} = 4\int \frac{t^3\,dt}{(t-1)t^2} = 4\int \frac{t\,dt}{t-1}$$
$$= 4\int \frac{(t-1)+1}{t-1}\,dt = 4(t+\ln|t-1|)+C$$
$$= 4(\sqrt[4]{x+3}+\ln|\sqrt[4]{x+3}-1|)+C.$$

In Problems 2.91 to 2.98 find the indicated integrals.

2.91. $\int \frac{dx}{(5+x)\sqrt{1+x}}$. **2.92.** $\int \frac{x\,dx}{\sqrt[3]{2x-3}}$.

2.93. $\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}}$. **2.94.** $\int \frac{\sqrt[6]{x+a}-1}{(x+a)(1+\sqrt[3]{x+a})}\,dx$.

2.95. $\int \sqrt[3]{\frac{x+1}{x-1}}\,\frac{dx}{(x-1)^3}$. **2.96.** $\int \frac{dx}{(x-1)\sqrt[4]{x^3}}$.

2.97. $\frac{dx}{(\sqrt[3]{x}+4)\sqrt{x}}$. **2.98.** $\int \frac{1}{x}\sqrt{\frac{x-1}{x+1}}\,dx$.

(b) Integrals of the form

$$\int R\,(x,\ \sqrt{ax^2+bx+c})\,dx,$$

where R is a rational function of two arguments are computed with the aid of trigonometric substitutions in the following way. By separating a perfect square in the quadratic trinomial and subsequent change of the variable $u=x+\frac{b}{2a}$, the original integral is reduced to an integral of one of the following three types:

(1) $\int R\,(u,\ \sqrt{l^2-u^2})\,du,$

(2) $\int R\,(u,\ \sqrt{l^2+u^2})\,du,$

(3) $\int R\,(u,\ \sqrt{u^2-l^2})\,du.$

The above integrals are reduced to integrals of the form $\int R\ (\sin\ t,\ \cos\ t)\,dt$ or $\int R\,(\sinh\ t,\ \cosh\ t)\,dt$ with the aid of a trigonometric or hyperbolic substitution:

(1) $u = l \sin t$ or $u = l \tanh t$,

(2) $u = l \tan t$ or $u = l \sinh t$,

(3) $u = l \sec t$ or $u = l \cosh t$,

respectively.

Example 16. Find $\int \frac{dx}{\sqrt{(x^2+4x+7)^3}}$.

Separating a perfect square in the quadratic trinomial, we have

$$\int \frac{dx}{\sqrt{(x+4x+7)^3}}=\int \frac{du}{\sqrt{(u^2+3)^3}},\ \text{where } u=x+2.$$

Making now the substitution $u=\sqrt{3}\tan t$, $du=\frac{\sqrt{3}}{\cos^2 t}\,dt$, $\sqrt{u^2+3}=$ $=\sqrt{3}\sec t$, we obtain:

$$\int \frac{dx}{\sqrt{(x^2+4x+3)^3}}=\int \frac{\sqrt{3}}{\cos^2 t\ \sqrt{3^3}\sec^3 t}\,dt$$

$$=\frac{1}{3}\int \cos t\,dt=\frac{1}{3}\sin t+C=\frac{1}{3}\,\frac{u}{\sqrt{u^2+3}}$$

$$=\frac{1}{3}\,\frac{x+2}{\sqrt{x^2+4x+7}}+C.$$

Example 17. Find $\int \sqrt{x^2-a^2}\,dx$.

We make the substitution $x=a\cosh t$. Then $dx=a\sinh t\,dt$, $\sqrt{x^2-a^2}=a\sinh t$ and further

$$\int \sqrt{x^2-a^2}\,dx=a^2\int \sinh^2 t\,dt=\frac{a^2}{2}\int(\cosh 2t-1)\,dt$$

$$=\frac{a^2}{2}\left(\frac{\sinh 2t}{2}-t\right)+C=\frac{a^2}{2}(\sinh t\cosh t-t)+C$$

$$=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+C.$$

In Problems 2.99 to 2.112 find the indicated integrals.

2.99. $\int \frac{dx}{(x^2-3)\sqrt{4-x^2}}$. **2.100.** $\int \frac{dx}{(x^2+1)(x+\sqrt{x^2+1})}$.

2.101. $\int \frac{\sqrt{x^2-1}}{x}\,dx$. **2.102.** $\int \frac{x^3}{\sqrt{a^2-x^2}}\,dx$.

2.103. $\int \sqrt{1-2x-x^2}\,dx$. **2.104.** $\int \sqrt{(3-2x-x^2)^3}\,dx$.

2.105. $\int \frac{x^2\,dx}{\sqrt{(x^2+1)^5}}$. **2.106.** $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$.

2.107. $\int \sqrt{x^2-2x+10}\,dx$. **2.108.** $\int \sqrt{4x-x^2}\,dx$.

2.109. $\int \frac{\sqrt{x^2+5}}{x^2}\,dx$. **2.110.** $\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$.

2.111. $\int \frac{dx}{\sqrt{(x^2+9)^3}}$. **2.112.** $\int \sqrt{(x^2-1)^3}\,dx$.

SEC. 6.3.
MIXED PROBLEMS ON INTEGRATION

In Problems 3.1 to 3.50 find the indicated integrals.

3.1. $\int \frac{x+3}{x^2+2x+4}\,dx$. **3.2.** $\int \frac{x^3}{x^2-x-1}\,dx$.

3.3. $\int \frac{dx}{(x-2)^2(x+3)}$. **3.4.** $\int \frac{dx}{(x^3-1)^2}$.

3.5. $\int \frac{dx}{x^5 (x^4+1)^2}$. **3.6.** $\int \frac{x\,dx}{\sqrt{x^2+x+2}}$.

3.7. $\int \frac{\ln x\,dx}{x\sqrt{6+4\ln x-\ln^2 x}}$. **3.8.** $\int \frac{dx}{x\sqrt{x^2+8x+4}}$.

3.9. $\int x\sqrt{x^2-4}\,dx$. **3.10.** $\int x\sqrt{x^2+4x-5}\,dx$.

3.11. $\int \sqrt{x^2+4x+5}\,dx$. **3.12.** $\int \frac{dx}{(x^2+9)\sqrt{16-x^2}}$.

3.13. $\int \frac{x\,dx}{\sqrt{x^4+16}}$. **3.14.** $\int \frac{dx}{\sqrt{(x^2+4)^3}}$.

3.15. $\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}+1}$. **3.16.** $\int \frac{1}{(1+x)^2}\sqrt{\frac{1+x}{1-x}}\,dx$.

3.17. $\int \frac{\sin x}{1-\sin x}\,dx$. **3.18.** $\int \frac{x\,dx}{1+\cos x}$.

3.19. $\int \frac{\cos x}{(1-\sin x)^4}\,dx$. **3.20.** $\int \frac{dx}{2+\cos x}$.

3.21. $\int \frac{dx}{3-4\sin^2 x}$. **3.22.** $\int \frac{2-\sqrt[3]{\tan x}}{\cos^2 x}\,dx$.

3.23. $\int \frac{\sec x \tan x}{\sqrt{5-\sec^2 x}}\,dx$. **3.24.** $\int \frac{\cos^3 x}{\sin x+5}\,dx$.

3.25. $\int \frac{dx}{\sin x \cos^5 x}$. **3.26.** $\int \frac{dx}{\cos^6 x}$.

3.27. $\int \frac{dx}{\sin^5 x}$. **3.28.** $\int x \sin x \cos 2x\,dx$.

3.29. $\int \frac{dx}{\sinh x \cosh x}$. **3.30.** $\int \frac{dx}{\sinh^2 x \cosh^2 x}$.

3.31. $\int \tanh^5 x\,dx$. **3.32.** $\int \frac{\cosh \sqrt{1+x}}{\sqrt{1+x}}\,dx$.

3.33. $\int \frac{x\,dx}{\cosh^2 x}$. **3.34.** $\int \sin^2 (\ln x)\,dx$.

3.35. $\int xe^{2x}\,dx$. **3.36.** $\int xe^{-x^2}\,dx$.

3.37. $\int \frac{e^x\,dx}{e^{2x}+4e^x-5}$. **3.38.** $\int \frac{(a^x-b^x)^2}{a^x b^x}\,dx$.

3.39. $\int e^{\arcsin x}\,dx$. **3.40.** $\int \sqrt{e^x-1}\,dx$.

3.41. $\int \frac{\arcsin x}{x^2}\,\frac{1+x^2}{\sqrt{1-x^2}}\,dx$. **3.42.** $\int \frac{\arcsin e^x}{e^x}\,dx$.

3.43. $\int \frac{x^4 \arctan x}{1+x^2}\,dx$.

3.44. $\int x\,(1+x^2)\arctan x\,dx$.

3.45. $\int \frac{\ln(1+x+x^2)}{(1+x)^2}\,dx$. **3.46.** $\int x \ln(4+x^4)\,dx$.

3.47. $\int x\sqrt{x^2+1}\ln\sqrt{x^2-1}\,dx$.

3.48. $\int \frac{x}{\sqrt{1-x^2}}\ln\frac{x}{\sqrt{1-x^2}}\,dx$.

3.49. $\int x^x(1+\ln x)\,dx$. **3.50.** $\int \frac{\arctan e^{x/2}}{e^{x/2}(1+e^x)}\,dx$.

SEC. 6.4.

THE DEFINITE INTEGRAL AND METHODS OF ITS COMPUTATION

1 The Definite Integral As the Limit of Integral Sum. If a function $f(x)$ is defined on the interval $a \leqslant x \leqslant b$ and $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ is an arbitrary division of this interval into n parts (Fig. 56), then the *integral sum* of the functions $f(x)$ on $[a, b]$ is defined as a sum of the form

$$S_n = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k,$$

where $x_{k-1} \leqslant \xi_k \leqslant x_k$, $\Delta x_k = x_k - x_{k-1}$, $k = 1, 2, 3, \ldots, n$. Geometrically, S_n is an algebraic sum of areas of the rectangles having bases Δx_k and altitudes $f(\xi_k)$.

If a function $f(x)$, defined on an interval $[a, b]$, is such that there exists a finite limit of the sequence of integral sums S_n on condition that the greatest of the differences Δx_k tends to zero and this limit depends neither on the way in which the interval $[a, b]$ is divided into the subintervals $[x_{k-1}, x_k]$, nor on the choice of the points ξ_k on these subintervals, then the function $f(x)$ is said to be *integrable* on the interval $[a, b]$ and the limit is called the *definite integral* of the

function $f(x)$ in the limits from a to b and is denoted by the symbol $\int_a^b f(x)\,dx$. Thus,

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k. \tag{1}$$

A function continuous on an interval $[a, b]$ is integrable on this interval.

Geometrically, the definite integral (1) represents an algebraic sum of areas of the figures bounded by the graph of the function $y = f(x)$,

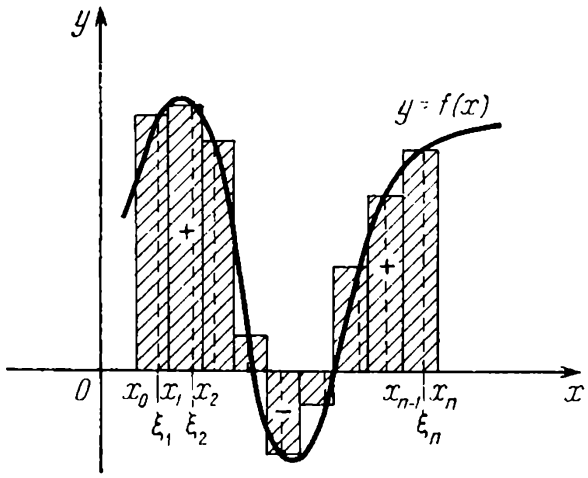

Fig. 56.

the x-axis and the straight lines $x = a$ and $x = b$, where the areas above the x-axis enter into this sum with the plus sign, and those below the x-axis with the minus sign.

Example 1. Compute $\int_1^2 x^2\,dx$, regarding the definite integral as the limit of integral sums.

First method. We divide the interval of integration [1, 2] into n equal parts of length $\Delta x = \frac{1}{n}$. The division points are:

$$x_0 = 1,\quad x_1 = 1 + \frac{1}{n},\quad x_2 = 1 + \frac{2}{n},\quad \ldots,\quad x_{n-1} = 1 + \frac{n-1}{n},\quad x_n = 2.$$

We choose, for instance, the left end points of the obtained subintervals as points ξ_k. Then

$$f(x_0)=1, \quad f(x_1)=\left(1+\frac{1}{n}\right)^2, \quad f(x_2)=\left(1+\frac{2}{n}\right)^2, \quad \ldots$$

$$\ldots, \quad f(x_{n-1})=\left(1+\frac{n-1}{n}\right)^2.$$

Consequently,

$$S_n=\frac{1}{n}\left(1+\left(1+\frac{1}{n}\right)^2+\left(1+\frac{2}{n}\right)^2+\ldots+\left(1+\frac{n-1}{n}\right)^2\right)$$

$$=\frac{1}{n^3}\left(n^2+(n+1)^2+(n+2)^2+\ldots+(2n-1)^2\right)$$

$$=\frac{1}{n^3}\left(\sum_{k=1}^{2n-1}k^2-\sum_{k=1}^{n-1}k^2\right).$$

Applying the formula for the sum of squares of integers

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6},$$

we find

$$S_n=\frac{1}{n^3}\left(\frac{(2n-1)\,2n\,(4n-1)}{6}-\frac{(n-1)\,n\,(2n-1)}{6}\right)$$

$$=\frac{14n^2-9n+1}{6n^2},$$

whence

$$\int_1^2 x^2\,dx=\lim_{n\to\infty}\frac{14n^2-9n+1}{6n^2}=\frac{7}{3}.$$

Second method. We subdivide the interval [1, 2] into parts so that the abscissas of the division points form a geometric progression:

$$x_0=1,\ x_1=q,\ x_2=q^2,\ \ldots,\ x_{n-1}=q^{n-1},\ x_n=q^n=2,$$

where $q=2^{1/n}$, and take the left end point of the kth subinterval as a point ξ_k. Then

$$f(x_0)=1,\ f(x_1)=q^2,\ f(x_2)=q^4,\ \ldots,\ f(x_{n-1})=q^{2(n-1)},$$

$$\Delta x_1=q-1,\ \Delta x_2=q^2-q=q(q-1),$$

$$\Delta x_3=q^2(q-1),\ \ldots,\ \Delta x_n=q^{n-1}(q-1),$$

$$S_n = 1\cdot(q-1) + q^3(q-1) + q^6(q-1) + \ldots + q^{3(n-1)}(q-1)$$
$$+ (q-1)(1+q^3+q^6+\ldots+q^{3(n-1)}) = (q-1)\frac{q^{3n}-1}{q^3-1}$$
$$= \frac{q^{3n}-1}{q^2+q+1} = \frac{2^3-1}{2^{2/n}+2^{1/n}+1} = \frac{7}{2^{2/n}+2^{1/n}+1}.$$

Consequently,

$$\int_1^2 x^2\,dx = \lim_{n\to\infty}\frac{7}{2^{2/n}+2^{1/n}+1} = \frac{7}{3}.$$

In Problems 4.1 to 4.4 compute the indicated integrals regarding them as the limits of the corresponding integral sums.

4.1*. $\int_0^5 (1+x)\,dx$. **4.2*.** $\int_0^{\pi/2} \cos x\,dx$.

4.3*. $\int_0^{10} e^x\,dx$. **4.4*.** $\int_1^3 \frac{dx}{x^2}$.

2. Computing Simplest Integrals with the Aid of the Newton-Leibnitz Formula. If $F(x)$ is one of the antiderivatives of a function $f(x)$, continuous on $[a, b]$, then the following *Newton-Leibnitz formula* holds true:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Example 2. Compute $\int_e^{e^2} \frac{dx}{x\ln x}$.

We have

$$\int_e^{e^2}\frac{dx}{x\ln x} = \int_e^{e^2}\frac{d(\ln x)}{\ln x} = \ln|\ln x|\Big|_e^{e^2}$$
$$= \ln(\ln e^2) - \ln(\ln e) = \ln 2 \approx 0.69.$$

In Problems 4.5 to 4.33, using the Newton-Leibnitz formula, compute the indicated integrals.

4.5. $\int_{-1}^2 x^3\,dx$. **4.6.** $\int_1^8 \frac{dx}{\sqrt[3]{x}}$.

4.7. $\int_1^2 (3x^2-2x+1)\,dx$. **4.8.** $\int_0^1 (\sqrt{x}+\sqrt[3]{x^2})\,dx$.

4.9. $\int_1^8 \frac{2+5\sqrt[3]{x}}{x^3}\,dx$. **4.10.** $\int_2^9 \sqrt[3]{x-1}\,dx$.

4.11. $\int_{\pi/2}^{\pi} \sin x\,dx$. **4.12.** $\int_{-\pi/4}^{0} \frac{dx}{\cos^2 x}$.

4.13. $\int_1^2 e^x\,dx$. **4.14.** $\int_0^3 2^x\,dx$.

4.15. $\int_2^5 \frac{dx}{x}$. **4.16.** $\int_1^2 \frac{dx}{2x-1}$.

4.17. $\int_0^1 \frac{x^2\,dx}{1+x^6}$. **4.18.** $\int_0^{\pi/4} \sin^2\varphi\,d\varphi$.

4.19. $\int_{\pi/6}^{\pi/3} \tan^4 x\,dx$. **4.20.** $\int_0^2 \sinh^3 x\,dx$.

4.21. $\int_0^1 \frac{dx}{4x^2+4x+5}$. **4.22.** $\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}}$.

4.23. $\int_3^4 \frac{x^2+3}{x-2}\,dx$. **4.24.** $\int_{-2}^{-1} \frac{x+1}{x^3-x^2}\,dx$.

4.25. $\int_1^2 \frac{e^{1/x^2}}{x^3}\,dx$. **4.26.** $\int_1^e \frac{\cos(\ln x)}{x}\,dx$.

4.27. $\int_1^e \frac{dx}{x(1+\ln^2 x)}$. **4.28.** $\int_0^{\pi/2} \cos^3\alpha\,d\alpha$.

4.29. $\int_0^{1/3} \cosh^2 3x\,dx$. **4.30.** $\int_2^3 \frac{dy}{y^2-2y-8}$.

4.31. $\int\limits_{3/4}^{2} \frac{dx}{\sqrt{2+3x-2x^2}}$. **4.32.** $\int\limits_{0}^{2} \frac{2x-1}{2x+1}\,dx$.

4.33. $\int\limits_{0}^{1} \frac{x^2+3x}{(x+1)(x^2+1)}\,dx$.

In Problems 4.34 to 4.36 find the limits of the indicated sums with the aid of definite integrals.

4.34.** $\lim\limits_{n\to\infty} \left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+n^2} \right)$.

4.35. $\lim\limits_{n\to\infty} \frac{\pi}{2n} \left(1 + \cos\frac{\pi}{2n} + \cos 2\frac{\pi}{2n} + \ldots \right.$

$\left. \ldots + \cos(n-1)\frac{\pi}{2n} \right)$.

4.36. $\lim\limits_{n\to\infty} \frac{1}{n} \left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \ldots \right.$

$\left. \ldots + \sqrt{1+\frac{n}{n}} \right)$.

In Problems 4.37 to 4.44 compute the areas of the figures bounded by the indicated lines.

4.37. $y = \frac{1}{2}x^2$, $y = 0$, $x = 2$, $x = 3$.

4.38. $y = \sqrt[3]{x}$, $y = 0$, $x = 1$, $x = 8$.

4.39. $y = 6 - x - 2x^2$, $y = x + 2$.

4.40. $y = \frac{x^2}{4}$, $y = 2\sqrt{x}$.

4.41. $y = \cos x$, $y = 0$, $x = -\frac{\pi}{2}$, $x = -\frac{\pi}{4}$.

4.42. $y = e^{-x}$, $y = 0$, $x = 1$, $x = 2$.

4.43. $y = \frac{2}{x}$, $y = 0$, $x = 2$, $x = 3$.

4.44. $y = \frac{3}{x}$, $x + y = 4$.

3. Properties of the Definite Integral.

(1) If $f(x) \geqslant 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \geqslant 0$.

(2) If $f(x) \leqslant g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$$

(3) $\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx.$

(4) If $f(x)$ is continuous on $[a, b]$, m is the least and M is the greatest value of $f(x)$ on $[a, b]$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

(the *theorem on estimation of the definite integral*).

Example 3. Estimate the integral

$$I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}.$$

We have: $1 \leqslant 1+x^4 \leqslant 2$ for $0 \leqslant x \leqslant 1$;

$$\frac{1}{\sqrt{2}} \leqslant \frac{1}{\sqrt{1+x^4}} \leqslant 1,$$

i. e. $m = \frac{1}{\sqrt{2}}$, $M = 1$, $b-a = 1$. Consequently, $\frac{1}{\sqrt{2}} \leqslant I \leqslant 1$.

(5) If $f(x)$ is continuous, and $g(x)$ is integrable on $[a, b]$, $g(x) \geqslant 0$, m and M are the least and the greatest values (respectively) of $f(x)$ on $[a, b]$, then

$$m\int_a^b g(x)\,dx = \int_a^b f(x)\,g(x)\,dx \leqslant M\int_a^b g(x)\,dx$$

(the *generalized theorem on estimation of the definite integral*).

(6) If $f(x)$ is continuous on $[a, b]$, then there exists a point $c \in (a, b)$ such that the following equality is valid:

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

(the *mean-value theorem*).

The number

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

is called the *mean value* of the function $f(x)$ on the interval $[a, b]$.

(7) If $f(x)$ is continuous, and $g(x)$ is integrable on $[a, b]$ and $g(x) \geqslant 0$, then there exists a point $c \in (a, b)$ such that the following equality holds:

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

(the *generalized mean-value theorem*).

(8) If $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$, then

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leqslant \sqrt{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}$$

(the *Cauchy-Buniakovski inequality*).

(9) Integration of *even* and *odd* functions over symmetric limits. If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$. And if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$.

(10) If a function $f(x)$ is continuous on the interval $[a, b]$, then the *integral with a variable upper limit*

$$\Phi(x) = \int_a^x f(t)\,dt$$

is an antiderivative for the function $f(x)$, i.e.

$$\Phi'(x)=\left(\int_a^x f(t)\,dt\right)'=f(x),\quad x\in[a,\ b].$$

(11) If the functions $\varphi(x)$ and $\psi(x)$ are differentiable at a point $x\in(a,\ b)$ and $f(t)$ is continuous for $\varphi(a)\leqslant t\leqslant\psi(b)$, then

$$\left(\int_{\varphi(x)}^{\psi(x)} f(t)\,dt\right)=f(\psi(x))\,\psi'(x)-f(\varphi(x))\,\varphi'(x).$$

Example 4. $I(x)=\int_0^{x^2} e^{-t^2}\,dt$. Find $I'(x)$.

Making use of property (11) and taking into account that $\varphi(x)=0$, i.e. $\varphi'(x)=0$, we have

$$L'(x)=e^{-(x^2)^2}\cdot(x^2)'=2xe^{-x^4}.$$

4.45. Without computing the indicated integrals, determine their signs:

(a)* $\int_{-2}^{1}\sqrt[3]{x}\,dx$; (b) $\int_{-1}^{1}x^3e^x\,dx$; (c) $\int_{1/3}^{1}x\ln x\,dx$.

4.46. Without evaluating the indicated integrals, find out which of the two integrals is greater:

(a) $\int_1^2\frac{dx}{\sqrt{1+x^2}}$ or $\int_1^2\frac{dx}{x}$; (b) $\int_1^2\frac{dx}{x^2}$ or $\int_1^2\frac{dx}{x^3}$;

(c) $\int_0^1 e^{-x}\cos^2 x\,dx$ or $\int_0^1 e^{-x^2}\cos^2 x\,dx$.

4.47. Find the mean value of the function on the given intervals:

(a) x^3, $0\leqslant x\leqslant 1$; (c) $\cos x$, $0\leqslant x\leqslant\pi/2$;

(b) $\sqrt[3]{x}$, $0\leqslant x\leqslant 1$; (d) $\cos^3 x$, $0\leqslant x\leqslant\pi/2$.

4.48. The strength of an alternating current changes according to the law $I = I_0 \sin\left(\frac{2\pi}{T}t + \varphi\right)$, where T is the cycle. Find the mean value of the strength of the current during half-cycle.

4.49. Estimate the integral $\int\limits_{-1}^{1} \sqrt{8 + x^3}\, dx$.

4.50. Estimate the integral $\int\limits_{0}^{2\pi} \frac{dx}{\sqrt{5+2\sin x}}$.

4.51. Estimate the integral $\int\limits_{0}^{1} \sqrt{(1+x)(1+x^3)}\, dx$ using:

(a) the generalized theorem of estimation of the definite integral;
(b) the Cauchy-Buniakovski inequality.

4.52. Estimate the integral $\int\limits_{0}^{1} \sqrt{(4+x^3)\,x}\, dx$ using:

(a) the generalized theorem on estimation of the definite integral;
(b) the Cauchy-Buniakovski inequality.

4.53. Find (a) $\frac{dI}{d\beta}$, (b) $\frac{dI}{d\alpha}$ if

$$I = \int\limits_{\alpha}^{\beta} \frac{e^x}{x}\, dx \qquad (0 < \alpha < \beta).$$

4.54. Find the points of extremum of the function

$$\Phi(x) = \int\limits_{a}^{x} \frac{\cos t}{t}\, dt \qquad \left(x > 0,\ 0 < a < \frac{\pi}{2}\right).$$

In Problems 4.55 to 4.58 find the derivatives of the given functions.

4.55. $\Phi(x) = \int_0^x \frac{\sin t}{t}\, dt$. 4.56. $\Phi(x) = \int_{1/x}^{\sqrt{x}} \sin(t^2)\, dt$.

4.57. $\Phi(x) = \int_x^0 \frac{dt}{\sqrt{1+t^3}}$. 4.58. $\Phi(x) = \int_{x^2}^{x^3} \frac{dt}{\ln t}$ $(x>0)$.

4.59. Prove that

$$\int_{-3}^{3} \frac{x^2 \sin x}{1+x^4}\, dx = 0.$$

4. Change of Variable in Definite Integral. If the function $f(x)$ is continuous on the interval $[a, b]$, and the function $x = \varphi(t)$ is continuously differentiable on the interval $[t_1, t_2]$, and $a = \varphi(t_1)$, $b = \varphi(t_2)$, then

$$\int_a^b f(x)\, dx = \int_{t_1}^{t_2} f(\varphi(t))\, \varphi'(t)\, dt.$$

Example 5. Compute $\int_{\sqrt{2}/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\, dx$.

Let us apply the substitution $x = \sin t$. Then $dx = \cos t\, dt$, $= \arcsin x$, $t_1 = \arcsin \frac{\sqrt{2}}{2} = \frac{\pi}{4}$ and $t_2 = \arcsin 1 = \frac{\pi}{2}$. Consequently,

$$\int_{\sqrt{2}/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\, dx = \int_{\pi/4}^{\pi/2} \frac{\sqrt{1-\sin^2 t}}{\sin^2 t} \cos t\, dt$$

$$= \int_{\pi/4}^{\pi/2} \frac{\cos^2 t}{\sin^2 t}\, dt = \int_{\pi/4}^{\pi/2} \frac{1-\sin^2 t}{\sin^2 t}\, dt$$

$$= (-\cot t - t)\Big|_{\pi/4}^{\pi/2} = -\frac{\pi}{2} + 1 + \frac{\pi}{4} = 1 - \frac{\pi}{4}.$$

4.60. Is it possible to evaluate the integral $\int_0^2 x \sqrt[3]{1-x^2}\,dx$ with the aid of the substitution $x = \sin t$?

In Problems 4.61 to 4.66 evaluate the given integrals with the aid of the indicated substitutions.

4.61. $\int_1^6 \frac{dx}{1+\sqrt{3x-2}}$, $3x-2=t^2$.

4.62. $\int_{\ln 3}^{\ln 8} \frac{dx}{\sqrt{e^x+1}}$, $e^x+1=t^2$.

4.63. $\int_0^{\sinh 1} \sqrt{x^2+1}\,dx$, $x=\sinh t$.

4.64. $\int_0^{\pi/2} \frac{dx}{3+2\cos x}$, $\tan\frac{x}{2}=t$.

4.65. $\int_0^{\pi/4} \frac{dx}{1+2\sin^2 x}$, $\tan x = t$.

4.66. $\int_{-1}^{1} \sqrt{3-2x-x^2}\,dx$, $x+1=2\sin t$.

In Problems 4.67 to 4.76 evaluate the indicated integrals by changing the variable.

4.67. $\int_{2/\sqrt{3}}^{2} \frac{dx}{x\sqrt{x^2-1}}$.

4.68. $\int_{-2}^{0} \frac{dx}{\sqrt{x+3}+\sqrt{(x+3)^3}}$.

4.69. $\int_2^{4/\sqrt{3}} \frac{\sqrt{x^2-4}}{x}\,dx$.

4.70. $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{\sqrt{(1+x^2)^3}}$.

4.71. $\int_{-2}^{2} \frac{dx}{(4+x^2)^2}$.

4.72. $\int_1^5 \frac{dx}{x+\sqrt{2x-1}}$.

4.73. $\int\limits_{1/4}^{1} \frac{dx}{x\sqrt{1+4x^2}}$. **4.74.** $\int\limits_{-1}^{1} \frac{x\,dx}{\sqrt{5-4x}}$.

4.75. $\int\limits_{\ln 2}^{\ln 6} \frac{e^x\sqrt{e^x-2}}{e^x+2}\,dx$. **4.76.** $\int\limits_{0}^{3} x^2\sqrt{9-x^2}\,dx$.

4.77. Show that $\int\limits_{e}^{e^2} \frac{dx}{\ln x} = \int\limits_{1}^{2} \frac{e^x}{x}\,dx$.

4.78. Show that $\int\limits_{1/\sqrt{2}}^{1} \frac{dx}{\arcsin x} = \int\limits_{\pi/4}^{\pi/2} \frac{\cos x}{x}\,dx$.

4.79. Make sure that $\int\limits_{-2}^{2} \frac{3x^7-2x^5+x^3-3}{x^4+3x^2+1}\,dx = 0$.

5. **Integration by Parts.** If two functions $u = u(x)$, $v = v(x)$ and their derivatives $u'(x)$ and $v'(x)$ are continuous on the interval $[a, b]$, then

$$\int\limits_{a}^{b} u\,dv = uv\Big|_a^b - \int\limits_{a}^{b} v\,du$$

(the *formula for integration by parts*).

Example 6. Compute $\int\limits_{1}^{e} \ln x\,dx$.

Let us set $u = \ln x$, $dv = dx$, then $du = \frac{dx}{x}$, $v = x$. We have

$$\int\limits_{1}^{e} \ln x\,dx = x\ln x\Big|_1^e - \int\limits_{1}^{e} x\cdot\frac{dx}{x} = e - x\Big|_1^e = e - e + 1 = 1.$$

In Problems 4.80 to 4.89 evaluate the given integrals using the method of integration by parts.

4.80. $\int\limits_{0}^{1} xe^x\,dx$. **4.81.** $\int\limits_{0}^{1} \frac{\arcsin x}{\sqrt{1+x}}\,dx$.

4.82. $\int\limits_{\pi/6}^{\pi/3} \frac{x\,dx}{\cos^2 x}$. **4.83.** $\int\limits_{1}^{e} \ln^2 x\,dx$.

4.84. $\int\limits_{0}^{\pi/4} e^{3x} \sin 4x\,dx$. **4.85.** $\int\limits_{2}^{2\sqrt{3}} \frac{\sqrt{x^2+4}}{x^2}\,dx$.

4.86. $\int\limits_{1}^{e} x \ln x\,dx$. **4.87.** $\int\limits_{0}^{1} x \arctan x\,dx$.

4.88. $\int\limits_{0}^{\pi/4} x^2 \cos 2x\,dx$. **4.89.** $\int\limits_{0}^{\pi/2} e^x \cos x\,dx$.

4.90. Show that for the integral

$$I_n = \int\limits_{0}^{\pi/2} \sin^n x\,dx = \int\limits_{0}^{\pi/2} \cos^n x\,dx, \quad n \in \mathbb{N},$$

the recurrence formula $I_n = \frac{n-1}{n} I_{n-2}$ is valid. Evaluate I_7 and I_8.

4.91. Show that for the integral

$$I_n = \int\limits_{0}^{1} x^n e^{-x}\,dx, \quad n \in \mathbb{N},$$

the recurrence formula $I_n = -\frac{1}{e} + nI_{n-1}$ is valid. Evaluate I_4.

SEC. 6.5.
IMPROPER INTEGRALS

1. Integrals with Infinite Limits. If a function $f(x)$ is continuous for $a \leqslant x < +\infty$, then, by definition,

$$\int\limits_{a}^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int\limits_{a}^{b} f(x)\,dx. \tag{1}$$

If there exists a finite limit in the right-hand side of formula (1), then the improper integral is said to be *convergent*, and if this limit does not exist, then it is said to be *divergent*.

Geometrically, improper integral (1) for $f(x) > 0$ is the area of the figure bounded by the graph of the function $y = f(x)$, the straight line $x = a$, and the x-axis (asymptote).

The integrals $\int\limits_{-\infty}^{b} f(x)\,dx$ and

$$\int\limits_{-\infty}^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int\limits_{c}^{b} f(x)\,dx + \lim_{a \to -\infty} \int\limits_{a}^{c} f(x)\,dx \qquad (2)$$

are determined in a similar way.

We shall consider the *tests for convergence* only for integrals of form (1).

(1) If $F(x)$ is an antiderivative for $f(x)$ and there exists a finite limit $\lim\limits_{x \to +\infty} F(x) = F(+\infty)$, then integral (1) converges and is equal to

$$\int\limits_{a}^{+\infty} f(x)\,dx = F(+\infty) - F(a);$$

and if $\lim\limits_{x \to +\infty} F(x)$ does not exist, then integral (1) is divergent.

(2) Let for $a \leqslant x < +\infty$ $0 \leqslant f(x) \leqslant g(x)$. If $\int\limits_{a}^{+\infty} g(x)\,dx$ converges, then $\int\limits_{a}^{+\infty} f(x)\,dx$ converges as well, and $\int\limits_{a}^{+\infty} f(x)\,dx \leqslant \int\limits_{a}^{+\infty} g(x)\,dx$.

If $\int\limits_{a}^{+\infty} f(x)\,dx$ diverges, then $\int\limits_{a}^{+\infty} g(x)\,dx$ also diverges (*comparison tests*).

(3) If for $a \leqslant x < +\infty$ $f(x) > 0$, $g(x) > 0$ and there exists a finite limit $\lim\limits_{x \to +\infty} \dfrac{f(x)}{g(x)} \neq 0$, then the integrals $\int\limits_{a}^{+\infty} f(x)\,dx$ and $\int\limits_{a}^{+\infty} g(x)\,dx$ either converge or diverge simultaneously (*limiting comparison test*).

(4) If $\int\limits_{a}^{+\infty} |f(x)|\,dx$ converges, then $\int\limits_{a}^{+\infty} f(x)\,dx$ converges as well (in this case the latter integral is called *absolutely convergent*).

(5) If for $x \to +\infty$ the function $f(x) > 0$ is an infinitesimal of order α as compared with $1/x$, then the integral $\int_a^{+\infty} f(x)\,dx$ is convergent for $\alpha > 1$ and divergent for $\alpha \leqslant 1$.

Example 1. Evaluate $\int_0^{+\infty} e^{-3x}\,dx$.

We have:

$$\int_0^{+\infty} e^{-3x}\,dx = \lim_{b\to+\infty} \int_0^b e^{-3x}\,dx = \lim_{b\to+\infty} \left(-\frac{1}{3}\,e^{-3x}\,\Big|_0^b\right)$$

$$= \frac{1}{3} \lim_{b\to+\infty} (1 - e^{-3b}) = \frac{1}{3}.$$

Example 2. Investigate the integral $\int_1^{+\infty} \frac{x+1}{\sqrt{x^3}}\,dx$ for convergence.

We have

$$\frac{x+1}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}.$$

The given integral is divergent, since $\int_1^{+\infty} \frac{dx}{\sqrt{x}}$ is divergent.

In Problems 5.1 to 5.9 evaluate the given improper integral (or prove that it is divergent).

5.1. $\int_e^{+\infty} \frac{dx}{x \ln^3 x}$.

5.2. $\int_e^{+\infty} \frac{dx}{x\sqrt{\ln x}}$.

5.3. $\int_{-\infty}^{+\infty} \frac{dx}{x^2+6x+11}$.

5.4. $\int_0^{+\infty} e^{-2x} \cos x\,dx$.

5.5. $\int_0^{+\infty} \frac{x\,dx}{x^2+4}$.

5.6. $\int_1^{+\infty} \frac{1+2x}{x^2(1+x)}\,dx$.

5.7. $\int_2^{+\infty} \frac{x\,dx}{\sqrt{(x^2+5)^3}}$. 5.8. $\int_0^{+\infty} xe^{-x^2}\,dx$.

5.9. $\int_0^{+\infty} x\cos x\,dx$.

In Problems 5.10 to 5.21 investigate the given integrals for convergence.

5.10. $\int_0^{\infty} \frac{x\,dx}{x^3+4}$. 5.11. $\int_1^{\infty} \frac{x^2+1}{x^3}\,dx$.

5.12. $\int_0^{\infty} \frac{dx}{\sqrt{1+x^3}}$. 5.13. $\int_2^{\infty} \frac{dx}{\sqrt[3]{x^3-1}}$.

5.14. $\int_0^{\infty} e^{-x}\,dx$. 5.15. $\int_1^{+\infty} \frac{dx}{3+2x^2+5x^4}$.

5.16. $\int_1^{+\infty} \frac{\sqrt{x^3}+\sqrt{x^2+1}}{x^3+3x+1}\,dx$. 5.17. $\int_1^{+\infty} \frac{3x^2+\sqrt{(x+1)^3}}{2x^3+\sqrt[3]{x^5}+1}\,dx$.

5.18. $\int_1^{+\infty} \frac{3+\sin x}{\sqrt[3]{x}}\,dx$. 5.19. $\int_1^{+\infty} \frac{dx}{\sqrt{x(x+1)(x+2)}}$.

5.20. $\int_1^{+\infty} \frac{\sin\frac{1}{x}}{2+x\sqrt{x}}\,dx$. 5.21. $\int_1^{+\infty} \frac{dx}{\sqrt{x}+\cos^2 x}$.

2. Integrals of Unbounded Functions. If a function $f(x)$ is continuous for $a \leqslant x < b$ and $f(b) = \infty$, then, by definition,

$$\int_a^b f(x)\,dx = \lim_{\gamma \to 0} \int_a^{b-\gamma} f(x)\,dx. \tag{3}$$

If a finite limit exists in the right-hand side of formula (3), then we say that the improper integral is *convergent*. If this limit does not exist, then the equality becomes meaningless, and the improper integral written on the left is said to *be divergent*.

Geometrically, if $f(x) > 0$ improper integral (3) is the area of the figure bounded by the graph of the function $y = f(x)$, the straight line $x = a$, and the vertical asymptote $x = b$.

If $f(a) = \infty$, the improper integral is defined in a similar way.

If $c \in (a, b)$ is a point of discontinuity and $f(c) = \infty$, then

$$\int_a^b f(x)\,dx = \lim_{\gamma_1 \to 0} \int_a^{c-\gamma_1} f(x)\,dx + \lim_{\gamma_2 \to 0} \int_{c+\gamma_2}^b f(x)\,dx. \tag{4}$$

If the antiderivative $F(x)$ of the function $f(x)$ is continuous for $a \leqslant x \leqslant b$, then the Newton-Leibnitz formula $\int_a^b f(x)\,dx = F(b) - F(a)$ is applicable to integrals (3) and (4).

The tests for convergence of improper integrals of unbounded functions are analogous to those from subsection 1.

As a standard for comparison, we usually take the integral

$$\int_a^b \frac{dx}{(b-x)^\alpha} \quad (\alpha > 0), \tag{5}$$

which is convergent for $\alpha < 1$ and divergent for $\alpha \geqslant 1$.

Example 3. Investigate the integral $\int_1^2 \frac{dx}{\ln x}$ for convergence.

As $x \to 1$ $\frac{1}{\ln x} \sim \frac{1}{x-1}$ (equivalent infinites), since

$$\lim_{x \to 1} \frac{\frac{1}{\ln x}}{\frac{1}{x-1}} = \lim_{x \to 1} \frac{x-1}{\ln x} = \lim_{x \to 1} \frac{1}{1/x} = 1.$$

The integral $\int_1^2 \frac{dx}{x-1}$ diverges as an integral of type (5) for $\alpha = 1$.

Hence, $\int_1^2 \frac{dx}{\ln x}$ diverges as well.

In Problems 5.22 to 5.30 evaluate the given integral (or prove that it is divergent).

5.22. $\int_0^1 \frac{dx}{x^2 + x^4}$. 5.23. $\int_0^2 \frac{x\,dx}{(x^2-1)^{4/5}}$.

5.24. $\int_1^e \frac{dx}{x \ln^3 x}$.

5.25. $\int_2^4 \frac{dx}{\sqrt{6x-x^2-8}}$.

5.26. $\int_{1/3}^{2/3} \frac{dx}{x\sqrt{9x^2-1}}$.

5.27. $\int_0^2 \frac{x^3\,dx}{\sqrt{4-x^2}}$.

5.28. $\int_1^{e^2} \frac{dx}{x\sqrt{\ln x}}$.

5.29. $\int_0^{\sqrt{2/\pi}} \cos\frac{1}{x^2}\cdot\frac{dx}{x^3}$.

5.30. $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$.

In Problems 5.31 to 5.40 investigate the given integrals for convergence.

5.31. $\int_0^1 \frac{\sqrt{x}}{\sqrt[4]{(1-x)^3}}\,dx$.

5.32. $\int_0^1 \frac{dx}{\sqrt{x}-1}$.

5.33. $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$.

5.34. $\int_0^1 \frac{1}{x^3} e^{\frac{1}{x}}\,dx$.

5.35. $\int_{-1}^0 \frac{1}{x^3} e^{\frac{1}{x}}\,dx$.

5.36. $\int_0^1 \frac{\cos\frac{1}{x}}{\sqrt[3]{x}}\,dx$.

5.37. $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$.

5.38. $\int_0^1 \frac{dx}{\tan x - x}$.

5.39. $\int_0^1 \frac{\ln(1+\sqrt[3]{x^2})}{e^x-1}\,dx$.

5.40. $\int_0^1 \frac{dx}{e^x-\cos x}$.

5.41. Prove that for $\alpha > 0$ Euler's integral $\Gamma(\alpha) = \int_0^{+\infty} e^{-x}x^{\alpha-1}\,dx$ defining the *gamma function* $\Gamma(\alpha)$ is convergent, and establish the following relationships:

(a) if $\alpha = n$ is an integer, then $\Gamma(n+1) = n!$;

(b) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for any $\alpha > 0$;

(c) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$;

(d) $\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$;

(e) $\Gamma\left(n+\frac{1}{2}\right)=1\cdot 3\cdot 5\ldots(2n-1)\frac{\sqrt{\pi}}{2^n}$, n is an integer.

SEC. 6.6.
GEOMETRICAL APPLICATIONS OF THE DEFINITE INTEGRAL

1. Area of a Plane Figure. The area of the figure bounded by the graph of a continuous function $y=f(x)$ ($f(x)\geqslant 0$), two straight lines $x=a$ and $x=b$, and the x-axis, or the area of a *curvilinear trapezoid* bounded by an arc of the graph of the function $y=f(x)$, $a\leqslant x\leqslant b$ (Fig. 57), is computed by the formula

$$S=\int_a^b f(x)\,dx. \qquad (1)$$

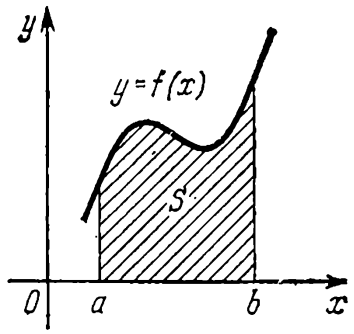

Fig. 57.

The area of the figure bounded by the graphs of two continuous functions $y=f_1(x)$ and $y=f_2(x)$, $f_1(x)\leqslant f_2(x)$, and two straight lines $x=a$, $x=b$ (Fig. 58), is determined by the formula

$$S=\int_a^b (f_2(x)-f_1(x))\,dx. \qquad (2)$$

Simplest problems on application of formulas (1) and (2) were given in Sec. 6.4 (Problems 4.37 to 4.44).

Example 1. Find the area of the figure lying in the right-hand half-plane and bounded by the circle $x^2+y^2=8$ and the parabola $y^2=2x$.

Let us first find the points of intersection of the given curves (Fig. 59) by solving the system of equations

$$x^2+y^2=8,$$
$$y^2=2x.$$

We obtain the points (2, 2) and (2, —2). Making use of the symmetry about the x-axis, we find the desired area S as the doubled sum of the

areas of the curvilinear trapezoids bounded, respectively, by the arcs of parabola $y = \sqrt{2x}$, $0 \leqslant x \leqslant 2$, and the circle $y = \sqrt{8 - x^2}$,

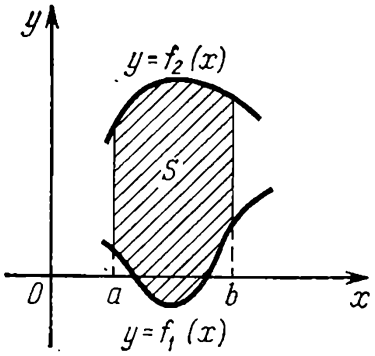

Fig. 58.

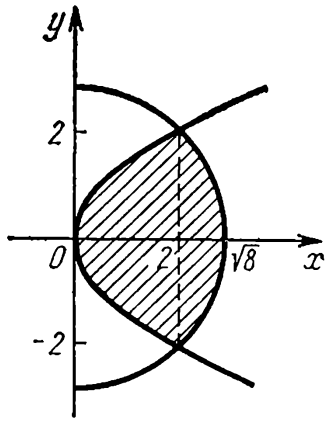

Fig. 59.

$2 \leqslant x \leqslant \sqrt{8}$:

$$S = 2\left(\int_0^2 \sqrt{2x}\,dx + \int_2^{\sqrt{8}} \sqrt{8-x^2}\,dx\right)$$

$$= 2\left(\sqrt{2}\cdot\frac{2}{3}x^{3/2}\Big|_0^2 + \left(\frac{x}{2}\sqrt{8-x^2} + 4\arcsin\frac{x}{\sqrt{8}}\right)\Big|_2^{\sqrt{8}}\right)$$

$$= 2\left(\frac{8}{3} + 2\pi - 2 - \pi\right) = 2\pi + \frac{4}{3}.$$

It is sometimes convenient to use formulas, analogous to (1) and (2) but with respect to y (regarding x as a function of y), in particular,

$$S = \int_c^d (f_2(y) - f_1(y))\,dy. \tag{3}$$

Example 2. Find the area of the figure bounded by the parabola $(y-2)^2 = x - 1$, the tangent to it at the point with the ordinate $y_0 = 3$, and the x-axis (Fig. 60).

The shape of the given figure does not enable us to apply directly formula (1) or (2). But if we consider the figure with respect to the y-axis, then formula (3) can be utilized. So, let y be an independent variable. Then the equation of the parabola will be written in the form $x = y^2 - 4y + 5$. Now, we find the equation of the tangent to the

parabola. It has the form: $x - x_0 = x_0' (y - y_0)$. Since $x_y' = 2(y - 2)$, we have $x_0' = x'|_{y=3} = 2$. Finding, further, the abscissa of the point of tangency $x_0 = 2$, we get the equation of the tangent:

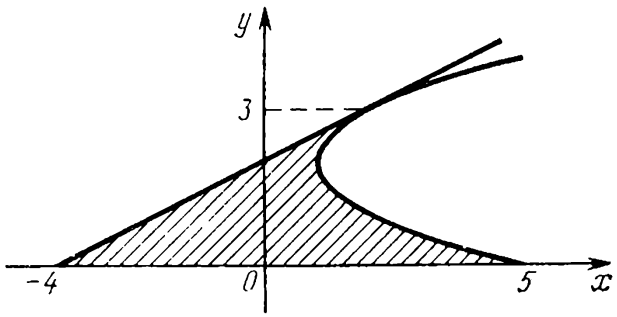

Fig. 60.

$$x - 2 = 2(y - 3), \text{ or } x = 2y - 4.$$

Setting in (3) $f_1(y) = 2y - 4$, $f_2(y) = y^2 - 4y + 5$, we have:

$$S = \int_0^3 ((y^2 - 4y + 5) - (2y - 4))\, dy = \int_0^3 (y^2 - 6y + 9)\, dy$$

$$= \int_0^3 (y - 3)^2\, dy = \frac{1}{3}(y - 3)^3 \Big|_0^3 = 9.$$

Note that the use of formulas (1) and (2) for solving Example 2 would require the computation of the sum of three integrals:

$$S = \int_{-4}^{1} \left(\frac{1}{2}x + 2\right) dx + \int_1^2 \left(\left(\frac{1}{2}x + 2\right) - (2 + \sqrt{x-1})\right) dx$$

$$+ \int_1^5 (2 - \sqrt{x-1})\, dx.$$

Example 3. Find the area of the figure bounded by the curve $y = 1/x^2$, the x-axis, and the straight line $x = 1$, and lying on the right of this line (Fig. 61).

The sought-for area is expressed by the improper integral

$$S = \int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = 1.$$

If a figure is bounded by a curve, having parametric equations $x = x(t)$, $y = y(t)$, straight lines $x = a$, $x = b$, and the x-axis, then

its area is computed by the formula

$$S = \int_{t_1}^{t_2} y(t)\, x'(t)\, dt = \int_{t_1}^{t_2} y(t)\, dx(t), \tag{4}$$

where the limits of integration are found from the equations $a = x(t_1)$, $b = x(t_2)$ ($y(t) \geqslant 0$ on the interval $[t_1, t_2]$).

Formula (4) is also applicable for calculating the area of a figure bounded by a closed curve (in this case the change of the parameter t from t_1 to t_2 must correspond to tracing the contour clockwise).

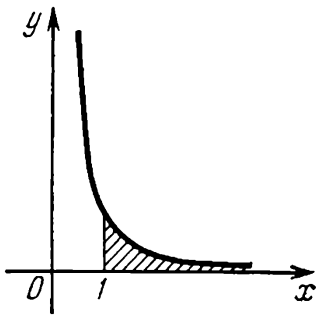

Fig. 61.

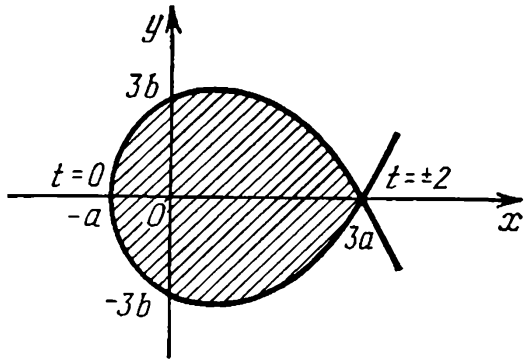

Fig. 62.

Example 4. Find the area of the loop made by the curve

$$x = a(t^2 - 1),\ y = b(4t - t^3) \quad (a > 0,\ b > 0).$$

Let us first find the points of intersection of the given curve and the coordinate axes. We have: $x = 0$ for $t = \pm 1$; $y = 0$ for $t = 0$ $t = \pm 2$. Consequently, we obtain the following points: $(0, 3b)$ for $t = 1$; $(0, -3b)$ for $t = -1$; $(-a, 0)$ for $t = 0$; $(3a, 0)$ for $t = \pm 2$. The point $(3a, 0)$ is a point of self-intersection of the curve. For $0 \leqslant t \leqslant 2$ $y \geqslant 0$; for $-2 \leqslant t \leqslant 0$ $y \leqslant 0$ (Fig. 62).

The desired area is found as the doubled area of the upper half of the figure bounded by the loop:

$$S = 2\int_{-a}^{3a} y\, dx = 2\int_0^2 y(t)\, x'(t)\, dt = 2\int_0^2 b(4t - t^3)\, a \cdot 2t\, dt$$

$$= 4ab\int_0^2 (4t^2 - t^4)\, dt = 4ab\left(\frac{4}{3}t^3 - \frac{t^5}{5}\right)\Big|_0^2 = \frac{256}{15}ab.$$

The area of a figure bounded by the graph of a continuous function $r = r(\varphi)$ and two rays $\varphi \quad \alpha$, $\varphi \quad \beta$, where φ and r are polar coordinates, or the area of a *curvilinear sector* bounded by an arc of the

graph of the curve $r = r(\varphi)$, $\alpha \leqslant \varphi \leqslant \beta$, is computed by the formula

$$S = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\varphi. \tag{5}$$

Example 5. Find the area of a circular lune bounded by arcs of the circles $r = 2a \cos \varphi$, $r = 2a \sin \varphi$, $0 \leqslant \varphi \leqslant \pi/2$ (Fig. 63).

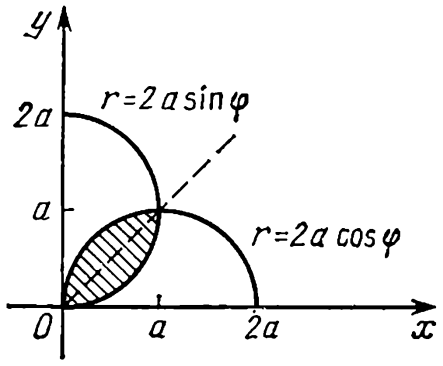

Fig. 63.

The circles intersect for $\varphi = \pi/4$; the figure under consideration is symmetric about the ray $\varphi = \pi/4$. Consequently, its area can be computed in the following way:

$$S = 2 \cdot \frac{1}{2} \int_{0}^{\pi/4} 4a^2 \sin^2 \varphi \, d\varphi = 2a^2 \int_{0}^{\pi/4} (1 - \cos 2\varphi) \, d\varphi$$

$$= 2a^2 \left(\varphi - \frac{1}{2} \sin 2\varphi \right) \Big|_{0}^{\pi/4} = \left(\frac{\pi}{2} - 1 \right) a^2.$$

6.1. Find the area of the figure bounded by the curve $y = \ln x$ and the straight lines $x = e$, $x = e^2$, $y = 0$.

6.2. Find the area of the figure bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

6.3. Find the area of the figure bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

6.4. Find the area of the figure bounded by the parabola $y = x^2 + 2x$ and the straight line $y = x + 2$.

6.5. Find the area of the figure bounded by the curves $y = \frac{27}{x^2 + 9}$ and $y = \frac{x^2}{6}$.

6.6. Find the area of the figure bounded by the curves $y^2 = 2px$ and $y^2 = \frac{4}{p}(x - p)^2$ $(p > 0)$.

6.7. Find the area of the figure bounded by the circles $x^2 + y^2 = a^2$, $x^2 + y^2 - 2ay = a^2$ and the straight line $y = a$.

6.8. Find the area of the figure bounded by the curves $y = \frac{a^3}{a^2+x^2}$, $y = \frac{a^2x}{a^2+x^2}$ and the y-axis.

6.9. Find the area of the figure bounded by the y-axis, the parabola $(x - a)^2 = 2p(y - b)$, and the tangent to this parabola at the point with abscissa $x = c$ $(c > a > 0$, $p > 0)$.

6.10. Find the area enclosed by the curves $y = e^x - 1$, $y = e^{2x} - 3$, $x = 0$.

6.11. Find the area of the region between the parabola $y = 3 + 2x - x^2$ and the x-axis.

6.12. Find the area of the figure bounded by the curve $y = \arcsin x$ and the straight lines $x = 0$, $y = \pi/2$.

6.13. Find the area of the upper lune bounded by the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 + 2ay = a^2$ $(a > 0)$.

6.14. Find the area of the region enclosed by the lines $(x - 1)(y + 2) = 2$ and $x + y = 2$.

6.15. Find the area enclosed by the curve $y = \ln x$, the tangent to this curve at the point $x = e$ and the x-axis.

6.16. Find the area of the figure bounded by the curves $y = \ln(x + 2)$, $y = 2 \ln x$, $y = 0$.

6.17. Find the area of each of the two parts into which the circle $x^2 + y^2 \leqslant 2ax$ is divided by the parabola $y^2 = 2ax - a^2$.

6.18. Find the area of the lune bounded by the hyperbola $x^2 - y^2 = a^2$ and the parabola $y^2 = \frac{3}{2}ax$.

6.19. Find the area of the hyperbolic segment with altitude h and base $2r$ (the semitransverse axis of the hyperbola being equal to a).

6.20. Find the area of the region bounded by the curve $a^2y^2 = \frac{x^5}{2a-x}$ and its asymptote $(a > 0)$.

6.21. Find the area of the region enclosed by the lines $x^2 - y^2 = a^2$, $(x^2 - a^2)^3 y^2 = a^8$ and the x-axis $(x > 0)$.

6.22. Find the area of each of the two parts into which the circle $x^2 + y^2 \leqslant 2ax$ is separated by the hyperbola $4x^2 - 3y^2 = a^2$.

6.23. Find the area of the elliptic segment with altitude h and base $2r$ (the semimajor axis of the ellipse is equal to a, the base of the segment being parallel to the minor axis).

6.24. Find the area enclosed by the curves $y = \frac{a^3}{a^2+x^2}$, $y = \frac{a^2x}{a^2+x^2}$ and the x-axis ($a > 0$).

6.25. Find the area of the figure bounded by the curve $y^2 = \frac{x^4}{a^2-x^2}$ and its asymptotes.

6.26. Find the area enclosed by the astroid $x = a\cos^3 t$, $y = a\sin^3 t$.

6.27. Find the area enclosed by the loop of the curve $x = \frac{1}{3}t(3 - t^2)$, $y = t^2$.

6.28. Find the area of the figure bounded by one arch of the cycloid $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$ and the x-axis.

6.29. Find the area enclosed by the loop of the curve $x = a(t^2 + 1)$, $y = b(t^3 - 3t)$.

6.30. Find the area enclosed by the loop of the curve $x = 2t - t^2$, $y = 2t^2 - t^3$.

6.31. Find the area enclosed by the cardioid $r = a(1 + \sin\varphi)$.

6.32. Find the area enclosed by one leaf of the curve $r = a\sin 2\varphi$.

6.33. Find the area enclosed by the curve $r = a\sin 5\varphi$.

6.34. Find the area of the figure bounded by the curves $r = a\tan\varphi\sec\varphi$, $r = 2a\cos\varphi$ and the polar axis.

6.35. Find the area of the figure lying in the first quadrant and bounded by the curves $r = a\tan\varphi$, $r = \frac{a}{\cos\varphi}$ and the polar axis.

6.36. Find the area enclosed by two consecutive turns of the logarithmic spiral $r = e^{\varphi}$, beginning with $\varphi = 0$.

6.37. Find the area enclosed by the curves $r^2 = 2\cos 2\varphi$, $r = 1$ ($r \geqslant 1$).

6.38. Find the area enclosed by the curve $r = a\cos 3\varphi$.

6.39. Find the area enclosed by the lemniscate of Bernoulli $r^2 = a^2 \sin 2\varphi$.

6.40. Find the area enclosed by the circle $r = \sqrt{3} \sin \varphi$ and the cardioid $r = 1 - \cos \varphi$ (outside of the cardioid).

2. Arc Length of a Curve. If a smooth curve is given by the equation $y = f(x)$, then the length l of its arc is

$$l = \int_a^b \sqrt{1 + (y')^2}\, dx,$$

where a and b are the abscissas of the end points of the arc.

If a curve is represented by parametric equations $x = x(t)$, $y = y(t)$, $(t_1 \leqslant t \leqslant t_2)$, then

$$l = \int_{t_1}^{t_2} \sqrt{(x_t')^2 + (y_t')^2}\, dt.$$

The arc length of a space curve defined by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, $t_1 \leqslant t \leqslant t_2$, is expressed in a similar way:

$$l = \int_{t_1}^{t_2} \sqrt{(x_t')^2 + (y_t')^2 + (z_t')^2}\, dt.$$

If we are given a polar equation of a smooth curve $r = r(\varphi)$, $\alpha \leqslant \varphi \leqslant \beta$, then

$$l = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\varphi.$$

Example 6. Find the arc length of the semicubical parabola $y^2 = x^3$ from the origin to the point (4, 8).

We have

$$y = x^{3/2}, \quad y' = \frac{3}{2} x^{1/2}$$

$$l = \int_0^4 \sqrt{1 + \frac{9}{4} x}\, dx = \frac{4}{9} \cdot \frac{2}{3} \left(1 + \frac{9}{4} x\right)^{3/2} \Big|_0^4 = \frac{8}{27} (10\sqrt{10} - 1).$$

Example 7. Find the arc length of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$.

We have:

$$x'_t = -3a\cos^2 t \sin t,\ y'_t = 3a \sin^2 t \cos t,$$

$$\frac{1}{4} l = \int_0^{\pi/2} \sqrt{9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}\, dt$$

$$= 3a \int_0^{\pi/2} \sin t \cos t\, dt = 3a \cdot \frac{\sin^2 t}{2} \Big|_0^{\pi/2} = \frac{3a}{2},$$

whence $l = 6a$.

Example 8. Find the arc length of the cardioid $r = a(1 - \cos \varphi)$

We have:

$$r' = a \sin \varphi,$$

$$\frac{1}{2} l = \int_0^{\pi} \sqrt{a^2 (1 - \cos \varphi)^2 + a^2 \sin^2 \varphi}\, d\varphi$$

$$= a \int_0^{\pi} \sqrt{2(1 - \cos \varphi)}\, d\varphi = 2a \int_0^{\pi} \sin \frac{\varphi}{2}\, d\varphi = 4a,$$

whence $l = 8a$.

6.41. Find the arc length of the parabola $y = x^2$ from $x = 0$ to $x = 1$.

6.42. Find the arc length of the curve $y = \frac{1}{3}(3 - x)\sqrt{x}$ between the points of its intersection with the x-axis.

6.43. Find the length of the arc of the semicubical parabola $y^2 = \frac{8}{27p}(x - p)^3$ lying inside the parabola $y^2 = 2px$.

6.44. Find the length of the arc of the curve $y = a \ln (a^2 - x^2)$ $(a > 1)$ lying above the x-axis.

6.45. Find the arc length of the closed curve $8a^2y^2 = x^2(a^2 - x^2)$.

6.46*. Find the perimeter of the lune formed by two circles: $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ $(a > b > 0)$.

6.47. Find the arc length of the catenary $y = \frac{1}{2} \cosh 2x$ from $x = 0$ to $x = 3$.

6.48. Find the arc length of the curve $y = \frac{2}{\pi} \ln \sin \frac{\pi x}{2}$ from $x = \frac{1}{2}$ to $x = \frac{3}{2}$.

6.49. Find the length of the arc of the semicubical parabola $y^2 = \frac{5}{p}(x-p)^3$, cut off by the straight line $x = 2p$.

6.50. Find the arc length of the curve $x = a(3\cos t - \cos 3t)$, $y = a(3\sin t - \sin 3t)$ from $t = 0$ to $t = \frac{\pi}{2}$.

6.51. Find the arc length of the curve $x = e^t \cos t$, $y = e^t \sin t$ from $t = 0$ to $t = 1$.

6.52. Find the arc length of the loop of the curve $x = t^2$, $y = t\left(\frac{1}{3} - t^2\right)$.

6.53. Find the arc length of the curve $x = \frac{t^6}{6}$, $y = 2 - \frac{t^4}{4}$ between the points of its intersection with the coordinate axes.

6.54. Find the arc length of the loop of the curve $x = a(t^2+1)$, $y = \frac{a}{3}(t^3 - 3t)$.

6.55. On the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ find the point which divides the arc length of the first arch of the cycloid in a ratio 1 : 3 (as measured from the origin).

6.56. Find the length of the arc of the logarithmical spiral $r = e^{a\varphi}$ found inside the circle $r = 1$.

6.57. Find the length of the arc of the cardioid $r = 2(1 - \cos\varphi)$ found inside the circle $r = 1$.

6.58*. Find the arc length of the entire curve $r = a\cos^3\frac{\varphi}{3}$.

6.59. Find the length of the arc of the spiral of Archimedes $r = 5\varphi$ found inside the circle $r = 10\pi$.

6.60. Find the arc length of the entire curve $r = a\sin^4\frac{\varphi}{4}$.

In Problems 6.61 to 6.65 find the arc length of the given space curve between the given points or planes.

6.61. $x = at^2$, $y = a\left(t + \frac{1}{3}t^3\right)$, $z = a\left(t - \frac{1}{3}t^3\right)$ from $t = 0$ to $t = \sqrt{3}$.

6.62. $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$ between the planes $z = 0$ and $z = a$ $(a > 0)$.

6.63. $x^2 = 4y$, $9z^2 = 16xy$ between the planes $x = 0$ and $x = 4$.

6.64. $x = a\sqrt{t}\cos t$, $y = a\sqrt{t}\sin t$, $z = at$ from $t = 0$ to an arbitrary $t > 0$.

6.65. $x = t - \sin t$, $y = 1 - \cos t$, $z = 4\cos\frac{t}{2}$ between the two points of intersection of the curve with the xz-plane.

3. The Area of a Surface of Revolution. The area of a surface generated by revolving an arc of a curve defined by the function $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis is computed by the formula

$$Q_x = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

If an arc is represented by parametric equations $x = x(t)$, $y = y(t)$, $t_1 \leqslant t \leqslant t_2$, then

$$Q_x = 2\pi \int_{t_1}^{t_2} y(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt.$$

If an arc is given in polar coordinates $r = r(\varphi)$, $\alpha \leqslant \varphi \leqslant \beta$, then

$$Q_x = 2\pi \int_\alpha^\beta r\sin\varphi\sqrt{r^2+(r')^2}\,d\varphi.$$

If an arc of a curve revolves about an arbitrary axis, then the area of a surface of revolution is expressed by the integral

$$Q = 2\pi \int_A^B R\,dl,$$

where R is the distance from a point on the curve to the axis of revolution, dl is the differential of the arc, A and B are the limits of integration corresponding to the end points of the arc. Here R and dl must be expressed in terms of the variable of integration.

Example 9. Find the area of the surface formed by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.

We have:

$$y = (a^{2/3} - x^{2/3})^{3/2},$$

$$y' = \frac{3}{2}(a^{2/3}-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{(a^{2/3}-x^{2/3})^{1/2}}{x^{1/3}},$$

$$\sqrt{1+\frac{a^{2/3}-x^{2/3}}{x^{2/3}}} = \frac{a^{1/3}}{|x|^{1/3}}.$$

Consequently,

$$Q_x = 2\cdot 2\pi \int_0^a (a^{2/3} - x^{2/3})^{3/2} \cdot \frac{a^{1/3}}{x^{1/3}}\, dx = 4\pi a^{1/3} \int_0^a (a^{2/3} - x^{2/3})^{3/2}\, x^{-1/3}\, dx$$

$$= -4\pi a^{1/3}\, \frac{3}{2}\, \frac{(a^{2/3} - x^{2/3})^{5/2}}{\frac{5}{2}} \Bigg|_0^a = \frac{12}{5}\, \pi a^2.$$

Example 10. Find the area of the surface generated by revolving one arch of the cycloid $x = a\,(t - \sin t)$, $y = a\,(1 - \cos t)$ about the x-axis.

We have:

$$x_t' = a\,(1 - \cos t), \qquad y_t' = a \sin t,$$

$$\sqrt{(x_t')^2 + (y_t')^2} = \sqrt{a^2\,(1 - \cos t)^2 + a^2 \sin^2 t}$$

$$= a\,\sqrt{2\,(1 - \cos t)} = 2a \sin \frac{t}{2}.$$

Hence,

$$Q_x = 2\pi \int_0^{2\pi} a\,(1 - \cos t)\cdot 2a \sin \frac{t}{2}\, dt = 8\pi a^2 \int_0^{2\pi} \sin^3 \frac{t}{2}\, dt$$

$$= 8\pi a^2 \int_0^{2\pi} \left(1 - \cos^2 \frac{t}{2}\right) \sin \frac{t}{2}\, dt$$

$$= -16\pi a^2 \left(\cos \frac{t}{2} - \frac{\cos^3 \frac{t}{2}}{3}\right) \Bigg|_0^{2\pi} = \frac{64}{3}\, \pi a^2.$$

Example 11. Find the area of the surface generated by revolving the cardioid $r = 2a\,(1 + \cos \varphi)$ about the polar axis.

We have:

$$r' = -2a \sin \varphi,$$

$$\sqrt{r^2 + (r')^2} = \sqrt{4a^2\,(1 + \cos \varphi)^2 + 4a^2 \sin^2 \varphi} = 4a \cos \frac{\varphi}{2},$$

and, further,

$$Q_x = 2\pi \int_0^{\pi} 2a\,(1 + \cos \varphi) \sin \varphi \cdot 4a \cos \frac{\varphi}{2}\, d\varphi$$

$$= 64\pi a^2 \int_0^{\pi} \cos^4 \frac{\varphi}{2} \sin \frac{\varphi}{2}\, d\varphi = \frac{128}{5}\, \pi a^2.$$

6.66. Find the area of the surface (called the *catenoid*) generated by rotating an arc of the catenary $y = \frac{1}{2} \cosh 2x$, $0 \leqslant x \leqslant 3$, about the x-axis.

6.67. Find the surface area of the ellipsoid obtained by rotating the ellipse $4x^2 + y^2 = 4$ about: (a) the x-axis; (b) the y-axis.

6.68. Find the area of the surface obtained by rotating about the x-axis the arc of the curve $y = \frac{1}{3} x^3$ from $x = -1$ to $x = 1$.

6.69. Find the area of the surface generated by revolving (about the x-axis) the arc of the curve $y = \frac{1}{6} \sqrt{x} (x - 12)$ between the points of its intersection with the x-axis.

6.70. Find the area of the surface generated by rotating (about the y-axis) the arc of the semicubical parabola $9ay^2 = 4x^3$ cut off by the straight line $x = a$.

6.71. Find the area of the surface obtained by revolving the loop of the curve $9ay^2 = x(3a - x)^2$ about: (a) the x-axis; (b) the y-axis.

6.72. Find the area of the surface formed by rotating the arc of the curve $y = e^{-x/2}$, $0 \leqslant x < +\infty$, about the x-axis.

6.73. Find the area of the surface generated by revolving the arc of the curve $x = a(3 \cos t - \cos 3t)$, $y = a(3 \sin t - \sin 3t)$, $0 \leqslant t \leqslant \pi/2$, about: (a) the x-axis; (b) the y-axis.

6.74. Find the area of the surface obtained by rotating the loop of the curve $x = a(t^2 + 1)$, $y = \frac{at}{3}(3 - t^2)$ about the x-axis.

6.75. Find the area of the surface generated by revolving one arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about its axis of symmetry.

6.76. Find the area of the surface formed by rotating the arc of the involute of a circle $x = a(t \sin t + \cos t)$, $y = a(\sin t - t \cos t)$, $0 \leqslant t \leqslant \pi$, about the x-axis.

6.77. Find the area of the surface obtained by rotating the circle $r = 2a \sin \varphi$ about the polar axis.

6.78. Find the area of the surface generated by revolving the cardioid $r = a(1 + \cos \varphi)$ about the tangent at its vertex $(2a, 0)$.

6.79. Prove that the area of the surface formed by rotating the lemniscate $r^2 = a^2 \sin 2\varphi$ about the polar axis is equal to the surface area of the sphere of radius a.

6.80. Find the area of the surface generated by revolving the arc of the curve $r = a \sec^2 \frac{\varphi}{2}$, $0 \leqslant \varphi \leqslant \frac{\pi}{2}$, about the polar axis.

4. Volume of Body. If the area $S(x)$ of a section of a body by a plane perpendicular to the x-axis is a continuous function on the interval $[a, b]$, then the volume of the body is computed by the formula

$$V = \int_a^b S(x)\,dx. \tag{6}$$

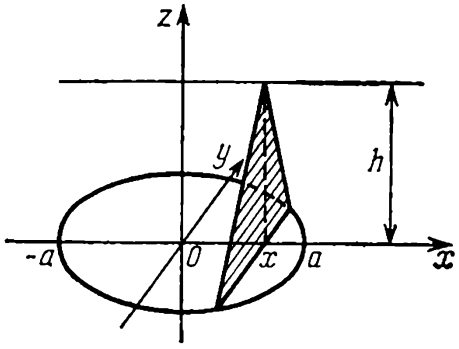

Fig. 64.

Example 12. The plane of an isosceles triangle moves in a perpendicular manner towards a fixed diameter of a circle of radius a. The base of the triangle is a chord of the circle and its vertex lies on a straight line, parallel to the fixed diameter at a distance h from the plane containing the circle. Find the volume of the body formed by the plane of the triangle during its motion from one end of the diameter to the other.

Choosing a coordinate system so that the centre of the circle turns out to lie at the origin and the fixed diameter on the x-axis we obtain the equation of the circle in the form $x^2 + y^2 = a^2$ (see Fig. 64).

The section of the body by a plane perpendicular to the x-axis is an isosceles triangle with base $2y = 2\sqrt{a^2 - x^2}$ and altitude h. We have:

$$S(x) = \frac{1}{2} \cdot \sqrt{a^2 - x^2}\; h - h\sqrt{a^2 - x^2} \quad (-a \leqslant x \leqslant a),$$

$$V = h \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = 2h \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= 2h \left(\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} \right) \Bigg|_0^a = 2h \cdot \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{1}{2} \pi a^2 h.$$

In case of solids of revolution we obtain a rather simple expression for the function $S(x)$. For instance, if a curvilinear trapezoid, bounded by a curve $y = f(x)$, $a \leqslant x \leqslant b$, rotates about the x- or y-axis, then the volumes of solids of revolution are computed by the respective

formulas:

$$V_x = \pi \int_a^b f^2(x)\, dx, \tag{7}$$

$$V_y = 2\pi \int_a^b x|f(x)|\, dx, \quad a \geqslant 0. \tag{8}$$

If a curvilinear sector, bounded by the curve $r = r(\varphi)$ and the rays $\varphi = \alpha$, $\varphi = \beta$, rotates about the polar axis, then the volume of the solid of revolution in question is equal to

$$V = \frac{2}{3}\pi \int_\alpha^\beta r^3 \sin\varphi\, d\varphi,.$$

Volumes of bodies are computed considerably simpler with the aid of multiple integrals. Therefore, presently we shall confine ourselves only to simplest problems.

Example 13. The figure bounded by the curves $y = \sqrt{2px}$ and $y = \frac{2}{\sqrt{p}}(x-p)^{3/2}$ revolves about the x-axis. Find the volume of the solid of revolution thus obtained (Fig. 65).

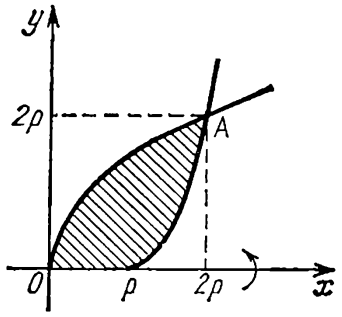

Fig. 65.

Let us first find the points of intersection of the curves:

$$\sqrt{2px} = \frac{2}{\sqrt{p}}(x-p)^{3/2}, \text{ or } 2p^2x = 4(x-p)^3;$$

obviously, the equation is satisfied by the value $x = 2p$, and then $y = 2p$, i.e. we have the point of intersection $(2p, 2p)$. The desired volume is determined as the difference between two volumes: the volume V_1 obtained by rotating a curvilinear trapezoid bounded by the parabola $y = \sqrt{2px}$ $(0 \leqslant x \leqslant 2p)$ and the volume V_2 generated by revolving a curvilinear trapezoid bounded by the semicubical parabola $y = \frac{2}{\sqrt{p}}(x-p)^{3/2}$ $(p \leqslant x \leqslant 2p)$.

Making use of formula (7), we obtain

$$V_x = V_1 - V_2 = \pi \int_0^{2p} y_1^2\, dx - \pi \int_p^{2p} y_2^2\, dx$$

$$= \pi \cdot 2p \int_0^{2p} x\, dx - \pi \cdot \frac{4}{p} \int_p^{2p} (x-p)^3\, dx$$

$$= 2\pi p \cdot \frac{x^2}{2}\Big|_0^{2p} - \frac{4\pi}{p} \cdot \frac{(x-p)^4}{4}\Big|_p^{2p} = 4\pi p^3 - \pi p^3 = 3\pi p^3.$$

Example 14. The figure enclosed between the curve $x = a \cos t$, $y = a \sin 2t$ $(0 \leqslant t \leqslant \pi/2)$ and the x-axis rotates about the y-axis. Find the volume of the revolution obtained.

It is obvious that $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant a$, and also that $y = 0$ for $t = 0$ and for $t = \pi/2$, that is, the figure under consideration is a curvilinear trapezoid. Further, for $t = 0$, $x = a$, and for $t = \pi/2$ $x = 0$. Consequently, the sought-for volume is expressed by formula (8). We have:

$$V_y = 2\pi \int_0^a x(t)\, y(t)\, dt = 2\pi \int_{\pi/2}^0 a \cos t \cdot a \sin 2t\, (-a \sin t)\, dt$$

$$= \pi a^3 \int_0^{\pi/2} \sin^2 2t\, dt = \frac{\pi a^3}{2} \int_0^{\pi/2} (1 - \cos 4t)\, dt$$

$$= \frac{\pi a^3}{2} \left(t - \frac{1}{4} \sin 4t \right) \Big|_0^{\pi/2} = \frac{\pi^2 a^3}{4}.$$

Example 15. The cardioid $r = a(1 - \cos \varphi)$ rotates about the polar axis. Find the volume of the solid of revolution thus generated.

$$V = \frac{2}{3} \pi \int_0^{\pi} a^3 (1 - \cos \varphi)^3 \sin \varphi\, d\varphi = \frac{2}{3} \pi a^3 \frac{(1 - \cos \varphi)^4}{4} \Big|_0^{\pi} = \frac{8}{3} \pi a^3.$$

6.81. Constructed on a chord of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$, parallel to the x-axis, is a square whose side is equal to the length of the chord and whose plane is perpendicular to the xy-plane. Find the volume of the body formed by the moving plane of the square if the chord displaces along the astroid.

6.82. Find the volume of the wedge cut off from a right circular cylinder of radius a by a plane passing through the diameter of the base at an angle α to the plane of the base.

6.83. Find the volume of the solid formed by revolving the figure bounded by the lines $2y = x^2$ and $2x + 2y - 3 = 0$ about the x-axis.

6.84. Find the volume of the solid obtained by rotating about the x-axis the figure enclosed by the lines $y = e^{-2x} - 1$, $y = e^{-x} + 1$, $x = 0$.

6.85. Find the volume of the solid generated by revolving the figure enclosed by the lines $y = x$, $y = x + \sin^2 x$ $(0 \leqslant x \leqslant \pi)$ about the y-axis.

6.86. Find the volume of the solid generated by revolving about the y-axis the figure bounded by the lines $y = \frac{x^2}{2} + 2x + 2$ and $y = 2$.

6.87. Find the volume of the solid obtained by rotating a parabolic segment with base $2a$ and altitude h about the altitude.

6.88. Find the volumes of the solids obtained by revolving the figure bounded by the curve $x = at^2$, $y = a \ln t$ $(a > 0)$ and the coordinate axes about: (a) the x-axis; (b) the y-axis.

6.89. Find the volume of the solid generated by revolving the figure enclosed between the curve $x = a \cos t$, $y = a \sin 2t$, and the x-axis $(0 \leqslant x \leqslant a)$ about the x-axis.

6.90. Find the volume of the solid obtained by revolving the astroid $x = a \cos^3 t$, $y = a \sin^3 t$ about the straight line $x = a$.

6.91. Find the volume of the solid obtained by revolving the curve $r = a \sin^2 \varphi$ about the polar axis.

6.92. Find the volume of the solid generated by rotating the lemniscate $r^2 = a^2 \cos 2\varphi$ about the polar axis.

SEC. 6.7.

APPLICATIONS OF THE DEFINITE INTEGRAL TO SOLUTION OF SOME PROBLEMS OF MECHANICS AND PHYSICS

1. Moments and Centres of Masses of Plane Figures. If an arc of a curve is given by an equation $y = f(x)$, $a \leqslant x \leqslant b$ and has density* $\rho = \rho(x)$, then the static moments of this arc M_x and M_y about the x- and y-axes are equal to (respectively)

$$M_x = \int_a^b \rho(x) f(x) \sqrt{1 + (f'(x))^2}\, dx,$$

$$M_y = \int_a^b \rho(x) x \sqrt{1 + (f'(x))^2}\, dx,$$

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* Everywhere in the problems where density is not indicated it is assumed that the curve is uniform and $\rho = 1$.

and the moments of inertia I_x and I_y with respect to the same coordinate axes are computed by the formulas

$$I_x = \int_a^b \rho(x) f^2(x) \sqrt{1+(f'(x))^2}\, dx,$$

$$I_y = \int_a^b \rho(x) x^2 \sqrt{1+(f'(x))^2}\, dx,$$

and the coordinates of the centre of masses $\bar{x}$ and $\bar{y}$ by the formulas

$$\bar{x} = \frac{M_y}{l} = \frac{1}{l} \int_a^b \rho(x) x \sqrt{1+(f'(x))^2}\, dx,$$

$$\bar{y} = \frac{M_x}{l} = \frac{1}{l} \int_a^b \rho(x) f(x) \sqrt{1+(f'(x))^2}\, dx,$$

where l is the mass of the arc, i.e.

$$l = \int_a^b \rho(x) \sqrt{1+(f'(x))^2}\, dx.$$

Example 1. Find the static moments and the moments of inertia about the x- and y-axes of the arc of the catenary $y = \cosh x$ for $0 \leqslant x \leqslant 1$.

We have: $y' = \sinh x$, $\sqrt{1+(y')^2} = \sqrt{1+\sinh^2 x} = \cosh x$. Consequently,

$$M_x = \int_0^1 \cosh^2 x\, dx = \frac{1}{2} \int_0^1 (1+\cosh 2x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sinh 2x\right)\Big|_0^1 = \frac{1}{4}(2+\sinh 2),$$

$$M_y = \int_0^1 x \cosh x\, dx = \int_0^1 x\, d(\sinh x) = x \sinh x \Big|_0^1 - \int_0^1 \sinh x\, dx$$

$$= \sinh 1 - \cosh x \Big|_0^1 = \sinh 1 - \cosh 1 + 1,$$

$$I_x = \int_0^1 \cosh^3 x\, dx = \int_0^1 (1+\sinh^2 x)\cosh x\, dx$$

$$= \left(\sinh x + \frac{\sinh^3 x}{3}\right)\Big|_0^1 = \sinh 1 + \frac{1}{3}\sinh^3 1,$$

$$I_y = \int_0^1 x^2 \cosh x\, dx = \int_0^1 x^2\, d(\sinh x) = x^2 \sinh x \Big|_0^1 - 2\int_0^1 x \sinh x\, dx$$

$$= \sinh 1 - 2\int_0^1 x\, d(\cosh x) = \sinh 1 - 2\left(x \cosh x \Big|_0^1 - \int_0^1 \cosh x\, dx\right)$$

$$= \sinh 1 - 2\cosh 1 + 2\sinh 1 = 3\sinh 1 - 2\cosh 1.$$

Example 2. Find the coordinates of the centre of masses of the arc of the circle $x = a\cos t$, $y = a\sin t$ situated in the first quadrant.

We have: $l = \frac{\pi a}{2}$, $0 \leqslant t \leqslant \frac{\pi}{2}$, $x_t' = -a\sin t$, $y_t' = a\cos t$,

$$\sqrt{(x_t')^2 + (y_t')^2} + \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a.$$

Hence we obtain:

$$M_x = a^2\int_0^{\pi/2} \cos t\, dt = a^2 \sin t \Big|_0^{\pi/2} = a^2,$$

$$M_y = a^2\int_0^{\pi/2} \sin t\, dt = -a^2\cos t \Big|_0^{\pi/2} = a^2,$$

$$\bar{x} = \frac{M_y}{l} = \frac{a^2}{\pi a/2} = \frac{2a}{\pi}, \quad \bar{y} = \frac{a^2}{\pi a/2} = \frac{2a}{\pi}.$$

In applications the following theorem turns out to be useful:

Guldin's theorem. *The area of a surface of revolution, formed by revolving a curve about a line in its plane not cutting the curve, is equal to the product of the length of the generating curve and the circumference of the circle described by its centre of masses (or centroid).*

Example 3. Find the coordinates of the centre of masses of the semi-circle $y = \sqrt{a^2 - x^2}$.

As a consequence of symmetry, $\bar{x} = 0$. Revolving the semi-circle about the x-axis, we obtain a sphere whose surface area is equal to $4\pi a^2$, and the length of the semi-circle is equal to πa. By Guldin's theorem, we have

$$4\pi a^2 = \pi a \cdot 2\pi \bar{y}.$$

Hence $\bar{y} = \frac{2a}{\pi}$, i.e. the centre of masses has the coordinates $C\left(0, \frac{2a}{\pi}\right)$.

7.1. Find the static moment of the sinusoid $y = \sin x$ $(0 \leqslant x \leqslant \pi)$ about the x-axis.

7.2. Find the static moment and the moment of inertia of the arc of the curve $y = e^x$ $(0 \leqslant x \leqslant 1)$ about the x-axis.

7.3. Find the static moment and the moment of inertia of one arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ with respect to the x-axis.

7.4. Find the static moment and the moment of inertia of the semi-circle of radius a about its diameter.

7.5. Find the static moments about the x- and y-axes of the arc of the circle $r = 2a \cos \varphi$ lying above the polar axis.

7.6. Find the centre of masses of the arc of the catenary $= a \cosh \frac{x}{a}$ $(0 \leqslant x \leqslant a)$.

7.7. Find the centre of masses of the arc of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$ situated above the x-axis.

7.8. Find the Cartesian coordinates of the centre of masses of the arc of the cardioid $r = a(1 + \cos \varphi)$ $(0 \leqslant \varphi \leqslant \pi)$.

7.9. Making use of Guldin's theorem, find the centre of masses of the arc of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$ lying in the first quadrant.

2. Physical Problems. Some applications of the definite integral to solution of physical problems are illustrated below in Examples 4-7.

Example 4. The velocity of rectilinear motion performed by a body is expressed by the formula $v = 2t + 3t^2$ (m/s). Find the path covered by the body during five seconds from the start.

Since the path covered by the body moving with velocity $v(t)$ during the time interval $[t_1, t_2]$ is expressed by the integral

$$S = \int_{t_1}^{t_2} v(t)\, dt,$$

we have:

$$S \int_0^5 (2t + 3t^2)\, dt = (t^2 + t^3) \Big|_0^5 = 150 \text{ (m)}.$$

Example 5. Find the work which must be done to lift a body of mass m from Earth's surface whose radius is R to an altitude h. Find the work done by a body moving away to infinity.

The work done by a variable force $f(x)$ acting along the x-axis on the interval $[a, b]$ is expressed by the integral

$$A = \int_a^b f(x)\, dx.$$

According to Newton's law of universal gravitation, the force F acting on a body of mass m is calculated by the formula

$$F = k\,\frac{mM}{r^2},$$

where M is the mass of the Earth, r is the distance of the mass m from the centre of the Earth, and k is the gravitational constant. Since on Earth's surface, that is, for $r = R$, we have $F = mg$, we may write $mg = k\,\frac{mM}{R^2}$. Hence we find $kM = gR^2$, and therefore

$$F = mg\,\frac{R^2}{r^2}.$$

Consequently, the required work is

$$A = \int_R^{R+h} F\,dr = \int_R^{R+h} mg\,R^2\,\frac{dr}{r^2} = mg\,R^2\left(-\frac{1}{r}\right)\Bigg|_h^{R+h} = mg\,R\,\frac{h}{R+h}.$$

Hence for $h \to \infty$ we have

$$\lim_{h\to\infty} A = mgR.$$

Example 6. Compute the kinetic energy of a homogeneous circular cone rotating with an angular velocity ω about its axis, given the radius R of the base of the cone, the altitude of the cone H and density γ.

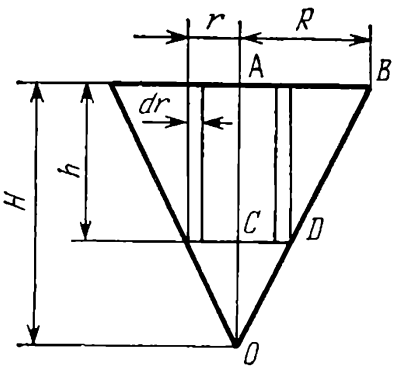

Fig. 66.

The kinetic energy of a body rotating about some axis with an angular velocity ω is equal to $\frac{1}{2}I\omega^2$, where I is the moment of inertia of the body about the axis of rotation. We take the mass of the hollow cylinder of altitude h, with inside radius r and thickness of the walls dr for an elementary mass dm (see Fig. 66). Then $dm = 2\pi rh\gamma\,dr$ $(0 \leqslant r \leqslant R)$. From the similarity of the triangles OCD and OAB we have

$$\frac{r}{R} = \frac{H-h}{H}, \text{ i. e. } h = H\left(1-\frac{r}{R}\right).$$

Consequently,

$$dm = 2\pi\gamma H\left(1-\frac{r}{R}\right)r\,dr,$$

and the elementary moment of inertia

$$dI = dm\cdot r^2 = 2\pi\gamma H\left(1-\frac{r}{R}\right)r^3\,dr.$$

Thus, the moment of inertia of the entire cone is

$$I=\int_0^R dI=\int_0^R 2\pi\gamma H\left(1-\frac{r}{R}\right)r^3\,dr=2\pi\gamma H\left(\frac{R^4}{4}-\frac{R^4}{5}\right)=\frac{1}{10}\pi\gamma HR^4,$$

and the kinetic energy of the cone is equal to

$$K=\frac{1}{20}\pi\gamma HR^4\omega^2.$$

Example 7. Find the pressure exerted by water on a vertical triangular plate with base a and altitude h, submerged into the water its vertex downward so that its base is found on the water surface.

Let us take advantage of Pascal's law, according to which the pressure P exerted by a liquid with specific gravity γ on a small area S at a submergence depth H is expressed by the formula

$$P=\gamma HS.$$

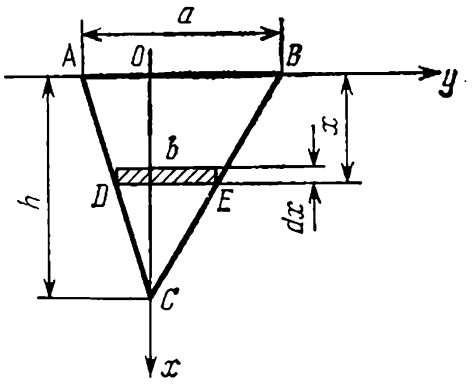

Fig. 67.

Introducing a coordinate system shown in Fig. 67, let us consider an elementary rectangular area found at a depth x having base b and altitude dx. From the similarity of the triangles CAB and CDE we have

$$\frac{b}{a}=\frac{h-x}{h},\quad \text{i.e. } b=\frac{a}{h}(h-x),$$

consequently,

$$dS=b\,dx=\frac{a}{h}(h-x)\,dx \text{ and } dP=x\,dS=\frac{ax}{h}(h-x)\,dx$$

(for water $\gamma=1$).

Hence, the pressure exerted by the water on the entire plate is equal to

$$P=\int_0^h x\,dS=\frac{a}{h}\int_0^h x(h-x)\,dx=\frac{a}{h}\left(\frac{h^3}{2}-\frac{h^3}{3}\right)=\frac{ah^2}{6}.$$

7.10. The velocity of a body launched with an initial velocity v_0 is equal to $v=v_0-gt$, where t is the time of climb, g is free fall acceleration (air resistance is neglected). What is the maximal height reached by the launched body?

7.11. A point on the x-axis performs harmonic oscillation about the origin with velocity $v=v_0\cos(\omega t+\varphi)$, where t is time, and v_0, ω, φ are constants. Find the law

of oscillation of the point and the mean value of the absolute magnitude of the velocity during the period of oscillation.

7.12. Two bodies move in one and the same straight line: one with velocity $v_1 = 3t^2 - 4t$ (m/s), the other with velocity $v_2 = 4(t + 3)$ (m/s). If at the initial instant they started from one and the same point, then at what instant and at what distance from the starting point will they meet once again?

7.13. The velocity of motion performed by a point is $v = 0.1\, te^{-0.02t}$(m/s). Find the path traversed by the point from the start to a full stop ($v(t_2) = 0$).

7.14*. What work must be done to stretch a spring by 5 cm if a force of 1 N stretches it by 1 cm?

7.15. Compute the work which must be done to bulk a heap of sand having a conical shape with the radius of base R and altitude H. The specific gravity of sand is γ.

7.16. Compute the work which must be done to pump the water out of the boiler having the shape of a paraboloid of revolution with the vertex directed upward. The radius of the base is R and the altitude is H.

7.17. Compute the work which must be done to construct a pyramid with a square base if the height of the pyramid is H, the side of the base is a, and the specific gravity of the material is γ.

7.18. Calculate the work which must be done to pump the water out of a container having the shape of a cone with its vertex directed upward. The radius of the base is R, the height is H.

7.19. Compute the work which must be done to pump the water out of a cistern bounded by the surfaces: $y^2 = 2pz$, $x = \pm a$, $z = p$ $(p > 0)$.

7.20*. An electric charge e_0 concentrated at the origin repels the charge e from the point $(a, 0)$ to the point $(b, 0)$. Determine the work A done by the repulsive force F. Find the work done as the charge e moves away to infinity.

7.21*. A cylinder with a moving piston is filled with vapour of volume $V_0 = 0.2$ m^3 and pressure $p_0 = 10{,}330$ N/m^2. What work must be done to halve the volume of the vapour at a constant temperature (an isothermic process)?

7.22*. Determine the work done during adiabatic compression of the air having the initial volume $V_0 = 8$ m^3

and pressure $p_0 = 10{,}000$ N/m^2 to the volume $V_1 = 2$ m^3.

7.23. Find the kinetic energy of a homogeneous ball of radius R and density γ rotating about its diameter with an angular velocity ω.

7.24. Find the kinetic energy of a plate having the shape of a parabolic segment and revolving about the axis of the parabola with a constant angular velocity ω. The base of the segment is a, its altitude h, the thickness of the plate d, and the density of the material is γ.

7.25. Find the kinetic energy of a triangular plate rotating about its base with an angular velocity ω. The base of the plate is a, altitude h, thickness l, density γ.

7.26. Find the kinetic energy of a homogeneous circular cylinder of density γ having the radius of the base R and altitude H rotating about its axis with an angular velocity ω.

7.27. Find the pressure of water on a vertical triangular plate with base a and altitude h submerged into the water so that its vertex is found on the water surface, its base being parallel to the water surface.

7.28. The end of a tube submerged into a liquid with specific gravity γ is shut with a round flap. Determine the pressure of the liquid on the flap if its radius is R, and its centre is found at a depth H.

7.29. Find the pressure exerted by a liquid of specific gravity γ on a vertical wall having the shape of a semiellipse whose major axis is found on the surface of the liquid. The semimajor axis of the ellipse is a, the semiminor axis is b.

7.30. Find the pressure exerted by a liquid with specific gravity γ contained in a circular cylinder on the lateral walls of the cylinder if the radius of its base is R and the altitude is H.

7.31. Find the mass of a rod of length $l = 5$ m whose linear density varies according to the law $\gamma = 1 + 0.1\ x^3$ (kg/m), where x is the distance from one of the ends of the rod.

7.32*. Find the amount of heat liberated by the alternating current $I = I_0 \cos \omega t$ during the period $2\pi/\omega$ in a conductor whose resistance is R.

7.33*. Within what time will the water contained in a cylindrical vessel with the area of the base $S = 100$ cm^2 and height $H = 20$ cm flow out through a hole of area $S_0 = 1$ cm^2 made in the bottom?

7.34.** For a steady-state laminar flow of a liquid through a pipe of a round section of radius a the flow velocity v at a point found at a distance r from the axis of the pipe is given by the formula $v = \frac{p}{4\mu l}(a^2 - r^2)$, where p is the difference between the liquid pressures at the ends of the pipe, μ is the viscosity coefficient, and l is the length of the pipe. Determine the rate of flow Q of the liquid, i.e. the amount of liquid flowing through the cross section of the pipe per unit time.

7.35*. With what force does a semi-ring of radius R and mass M attract a material point m situated at its centre?

7.36. During what time will the water flow out of a funnel whose height is $H = 50$ cm, the radius of the upper base $R = 5$ cm, and the radius of the lower base is $r = 0.2$ cm?

7.37. Determine the rate of flow of a liquid through a spillway of a rectangular cross section. The height of the spillway is h, the width a, and the coefficient μ.

SEC. 6.8.
NUMERICAL INTEGRATION OF FUNCTIONS OF ONE VARIABLE

Numerical integration consists in finding the integral $\int_a^b f(x)\,dx$ of a continuous function $f(x)$ *by the quadrature formula*

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} a_{nk} f(x_k),$$

where the coefficients a_{nk} are real numbers and the points x_k belong to $[a, b]$, $k = 1, 2, \ldots, n$. The form of the sum

$$S_n(f) = \sum_{k=1}^{n} a_{nk} f(x_k)$$

determines the method of numerical integration, and the difference

$$R_n(f) = \int_a^b f(x)\,dx - S_n(f)$$

—the error of the method.

For the *method of rectangles*

$$\int_a^b f(x)\,dx \approx h \sum_{k=1}^{n} f(x_k), \tag{1}$$

$h = \frac{b-a}{n}$ (separation step), $x_0 = a - \frac{h}{2}$, $x_k = x_{k-1} + h$ $(k = 1, 2, \ldots, n)$.

For the *method of trapezoids*

$$\int_a^b f(x)\,dx \approx h\left(\frac{f(a)+f(b)}{2} + \sum_{k=1}^{n-1} f(x_k)\right), \tag{2}$$

$$h = \frac{b-a}{n},\quad x_0 = a,\quad x_k = x_{k-1} + h \qquad (k = 1, \ldots, n).$$

For *Simpson's method*

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(a) + f(b) + 4\sum_{k=1}^{n} f(x_{2k-1}) + 2\sum_{k=1}^{n-1} f(x_{2k})\right), \tag{3}$$

$$h = \frac{b-a}{2n},\quad x_0 = a,\quad x_k = x_{k-1} + h \qquad (k = 1, 2, \ldots, 2n).$$

The right-hand parts of the formulas of rectangles (1), trapezoids (2), and Simpson's (3) are integral sums; they tend to the given integral as $h \to 0$. But for a fixed h each of them differs from the corresponding integral by the quantity $R_n(f)$. Given the limiting absolute error $\varepsilon > 0$, we choose the parameter n, or, which is the same, the subinterval h for which the inequality

$$|R_n(f)| < \varepsilon$$

is fulfilled.

The quantities $R_n(f)$ (in the assumption that the derivatives entering into these formulas exist) are characterized by the following equalities:

$R_n(f) = -\frac{b-a}{24} f''(\xi)\,h^2$, $\xi \in [a, b]$ for the rectangle method,

$R_n(f)=\frac{b-a}{12}f''(\xi)h^2$, $\xi\in[a, b]$ for the trapezoid method,

$R_n(f)=-\frac{b-a}{180}f^{(IV)}(\xi)h^4$, $\xi\in[a, b]$ for Simpson's method.

Example 1. Find $\ln 2$ within 10^{-4} from the relation $\ln 2=\int\limits_{0.5}^{1}\frac{dx}{x}$, using Simpson's method for evaluating the integral.

For the integrand $f(x)=\frac{1}{x}$ on the interval $\left[\frac{1}{2}, 1\right]$ we have $f^{(IV)}(x)=\frac{24}{x^5}$, whence $|f^{(IV)}(x)|<24\cdot 2^5$. Taking into account that $a=\frac{1}{2}$, $b=1$, $h=\frac{1}{4n}$, we obtain

$$|R_n(f)|<\frac{1}{2\cdot 180}24\cdot 2^5\left(\frac{1}{4n}\right)^4,$$

or

$$|R_n(f)|<\frac{1}{120n^4}.$$

To achieve the preassigned accuracy, it is necessary that the inequality

$$\frac{1}{120n^4}<10^{-4}, \text{ or } n^4>\frac{10^3}{12},$$

be fulfilled which will take place for $n^4>100$. Therefore, we should choose $n=4$. Finding $h=\frac{1}{16}=0.0625$, we shall compute the values of the given function with an accuracy automatically exceeding 10^{-4}. We shall obtain the following table:

$x_0=0.5$	$f(x_0)=2$		
$x_1=0.5625$		$f(x_1)=1.7777777$	
$x_2=0.6250$			$f(x_2)=1.6$
$x_3=0.6875$		$f(x_3)=1.4545454$	
$x_4=0.7500$			$f(x_4)=1.3333333$
$x_5=0.8125$		$f(x_5)=1.2307692$	
$x_6=0.8750$			$f(x_6)=1.1428571$
$x_7=0.9375$		$f(x_7)=1.0666666$	
$x_8=1$	$f(x_8)=1$		
	$\sigma_1=3$	$\sigma_2=5.5297589$	$\sigma_3=4.0761904$

Computing the sum

$$\sigma = \sigma_1 + 4\sigma_2 + 2\sigma_3 = 33.271415$$

and $\frac{h}{3} = 0.0208333$, and applying Simpson's formula (3), we obtain the following result:

$$\ln 2 = 0.6931.$$

Another technique of estimation of the error of the method of numerical integration consists in that we use the asymptotic equality

$$\int_a^b f(x)\,dx - S_{n_{\nu+1}}(f) = \frac{S_{n_{\nu+1}}(f) - S_{n_\nu}(f)}{\lambda^m - 1} + o\left(n_{\nu+1}^{-m}\right),$$

where

$$n_{\nu+1} = \lambda n_\nu \ (\lambda > 1), \ \nu = 1, 2, 3, \ldots,$$

and $m = 2$ for the methods of rectangles and trapezoids, $m = 4$ for Simpson's method. The computations by the formulas for finding the sum $S_n(f)$ are carried out for $n = n_1, n_2, n_3, \ldots$ until the relation

$$\frac{|S_{n_{\nu+1}}(f) - S_{n_\nu}(f)|}{\lambda^m - 1} < \varepsilon \tag{4}$$

is fulfilled.

This method is called *Runge's rule*. The criterion of its applicability is the relation

$$\frac{|S_{n_{\nu+1}}(f) - S_{n_\nu}(f)|}{|S_{n_\nu}(f) - S_{n_{\nu-1}}(f)|} \approx \lambda^{-m}.$$

The number $\lambda > 1$ is chosen arbitrarily, but preferably to be equal to 2 or 3.

Example 2. Applying the trapezoid method, evaluate the integral

$$\int_0^1 \frac{x\,d}{\sqrt{1+x^3}}$$

to within 10^{-4}.

Let us set $n_1 = 10$, $n_2 = 20$ and evaluate the integrand function $f(x) = \frac{1}{\sqrt{1+x^3}}$ at the respective nodes $x_k^{(1)} = x_0 + kh_1 = \frac{k}{10}$ $(k = 1, 2, \ldots, 10)$ and $x_k^{(2)} = x_0 + kh_2 = \frac{k}{20}$ $(k = 1, 2, \ldots, 20)$ (see Table 6.1).

We first find the sum $S_{n_1} = \sigma_1 \cdot h_1 = 0.9091616$, where $\sigma_1 = \frac{f(x_0^{(1)}) + f(x_{10}^{(1)})}{2} + \sum_{k=1}^{9} f(x_k^{(1)})$ and $h_1 = \frac{1}{10}$. Applying the tra-

Table 6.1

$x_0^{(1)}=0$	$f(x_0^{(1)})=1$	$x_0^{(2)}=0$	
		$x_1^{(2)}=0.05$	$f(x_1^{(2)})=0.9999376$
$x_1^{(1)}=0.1$	$f(x_1^{(1)})=0.9995004$	$x_2^{(2)}=0.1$	
		$x_3^{(2)}=0.15$	$f(x_3^{(2)})=0.9983168$
$x_2^{(1)}=0.2$	$f(x_2^{(1)})=0.9960238$	$x_4^{(2)}=0.2$	
		$x_5^{(2)}=0.25$	$f(x_5^{(2)})=0.9922778$
$x_3^{(1)}=0.3$	$f(x_3^{(1)})=0.9867674$	$x_6^{(2)}=0.3$	
		$x_7^{(2)}=0.35$	$f(x_7^{(2)})=0.9792281$
$x_4^{(1)}=0.4$	$f(x_4^{(1)})=0.9694584$	$x_8^{(2)}=0.4$	
		$x_9^{(2)}=0.45$	$f(x_9^{(2)})=0.9573324$
$x_5^{(1)}=0.5$	$f(x_5^{(1)})=0.9428091$	$x_{10}^{(2)}=0.5$	
		$x_{11}^{(2)}=0.55$	$f(x_{11}^{(2)})=0.9259358$
$x_6^{(1)}=0.6$	$f(x_6^{(1)})=0.9068453$	$x_{12}^{(2)}=0.6$	
		$x_{13}^{(2)}=0.65$	$f(x_{13}^{(2)})=0.8857451$
$x_7^{(1)}=0.7$	$f(x_7^{(1)})=0.8629030$	$x_{14}^{(2)}=0.7$	
		$x_{15}^{(2)}=0.75$	$f(x_{15}^{(2)})=0.8386278$
$x_8^{(1)}=0.8$	$f(x_8^{(1)})=0.8132501$	$x_{16}^{(2)}=0.8$	
		$x_{17}^{(2)}=0.85$	$f(x_{17}^{(2)})=0.7871027$
$x_9^{(1)}=0.9$	$f(x_9^{(1)})=0.7605057$	$x_{18}^{(2)}=0.9$	
		$x_{19}^{(2)}=0.95$	$f(x_{19}^{(2)})=0.7337535$
$x_{10}^{(1)}=1$	$f(x_{10}^{(1)})=0.7071068$	$x_{20}^{(2)}=1$	
	$\sigma_1=9.0916166$		$\sigma_2=9.0982576$

pezoid formula once again, we find

$$S_{n_2}=(\sigma_1+\sigma_2)\,h_2=0.9094937,$$

where $h_2=\frac{1}{20}$ and $\sigma_2=\sum_{k=1}^{10} f(x_{2k-1}^{(2)})$. From the relation $x_k^{(1)}=x_{2k}^{(2)}$, $k=0, 1, \ldots, 10$, it is seen that in order to find S_{n_2}, there is no need to calculate each of the twenty-one values of the function anew, it remains only to add ten new values forming the sum σ_2 to the earlier found values entering into the sum σ_1. Setting in the left-hand member of inequality (4) $\lambda=2$, $m=2$ and taking into account the values S_{n_1} and S_{n_2}, we get

$$\frac{S_{n_2}-S_{n_1}}{3}=0.0001106.$$

Therefore we have (to within 10^{-4}):

$$\int_0^1 \frac{dx}{\sqrt{1+x^3}} = 0.9094.$$

In Problems 8.1 to 8.24 evaluate the indicated definite integrals within 10^{-4} utilizing one of the following techniques: (a) the method of rectangles, (b) the method of trapezoids, (c) Simpson's method.

8.1. $\int_2^{3.5} \frac{dx}{\sqrt{5+4x-x^2}}$. 8.2. $\int_0^1 \frac{x^3\,dx}{x^8+1}$. 8.3. $\int_0^1 \frac{x\,dx}{x^2+3x+2}$.

8.4. $\int_0^2 \frac{dx}{4+x^2}$. 8.5. $\int_1^4 \frac{1+\sqrt{x}}{x^2}\,dx$. 8.6. $\int_0^2 \sqrt{1+x^3}\,dx$.

8.7. $\int_0^2 \sqrt{1+x^5}\,dx$. 8.8. $\int_0^{0.5} \frac{dx}{\sqrt{1+x^4}}$. 8.9. $\int_0^{0.6} \frac{dx}{\sqrt{1-x^4}}$.

8.10. $\int_2^{10} \frac{dx}{\sqrt[3]{1+x^2}}$. 8.11. $\int_0^1 \sqrt{x(1-x)}\,dx$.

8.12. $\int_0^1 x \ln(1+x)\,dx$. 8.13. $\int_0^1 e^{x^2}\,dx$. 8.14. $\int_0^1 e^{x^3}\,dx$.

8.15. $\int_0^{0.5} e^{\sqrt{x}}\,dx$. 8.16. $\int_2^3 \frac{dx}{\ln x}$. 8.17. $\int_0^1 \frac{\ln(1+x^2)}{1+x^2}\,dx$.

8.18. $\int_0^{3.1416} \ln(5+4\cos x)\,dx$. 8.19. $\int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$.

8.20. $\int_0^1 \sqrt[3]{x}\cos x\,dx$. 8.21. $\int_0^{\pi} \sqrt{\sin x}\sin\frac{x}{2}\,dx$.

8.22. $\int_0^1 \frac{\arctan x}{x}\,dx$. 8.23. $\int_{0.4}^{0.6} \frac{e^x}{x}\,dx$. 8.24. $\int_0^{0.8} \frac{\sin x}{x}\,dx$.

Answers

1.1. $\frac{x^8}{4}+C$. **1.2.** $3x\sqrt[3]{x}+C$. **1.3.** $3\ln|x|-\frac{5}{x}+C$. **1.4.** $\frac{x^3}{3}+$ $+\frac{5}{2}x^2-\ln|x|+C$. **1.5.** $x+6\sqrt{x}+3\ln x-\frac{2}{\sqrt{x}}+C$. **1.6.** $\sin x+C$. **1.7.** $\frac{2}{b}\sqrt{a+bx}+C$. **1.8.** $-\frac{1}{3}e^{2-3x}+C$. **1.9.** $-\frac{3}{\ln 5}\cdot 5^{-x/3}+C$. **1.10.** $\frac{1}{4}\tan 4x+C$. **1.11.** $\frac{x^3}{3}+\frac{x^2}{2}+x+$ $+2\ln|x-1|+C$. **1.12.** $\frac{1}{8}\sin 8x+C$. **1.13.** $x-\frac{1}{2}\cos 2x+$ $+C$. **1.14.** $\frac{1}{2}\sin 2x+C$. **1.15.** $x^3+x^2+\ln|x|+C$. **1.16.** $-\frac{1}{x^2}-\frac{1}{x^3}+C$. **1.17.** $\frac{2}{3}x\sqrt{mx}+C$. **1.18.** $\frac{nx^{\frac{n-1}{n}}}{n-1}+C$. **1.19.** $3\sqrt[3]{x}-\frac{4}{5}x\sqrt[4]{x}-4\sqrt[4]{x}+C$. **1.20.** $2\sqrt{ax}+2x+\frac{2}{3}\frac{x\sqrt{x}}{\sqrt{a}}+C$. **1.21.** $\frac{x^3}{3}+2\ln|x|+C$. **1.22.** $\frac{2^x e^x}{\ln 2+1}+C$. **1.23.** $\frac{1}{\ln 2}\cdot 2^x+$ $+x^3+C$. **1.24.** $x^2+3\sin x+C$. **1.25.** $2\cot x-\ln\left|\tan\frac{x}{2}\right|+$ $+C$. **1.26.** $3\tan x+2\cot x+C$. **1.27.** $-\cot x-\tan x+C=$ $=-\frac{2}{\sin 2x}+C$. **1.28.** $\frac{1}{2}(x-\sin x)+C$. **1.29.** (a) $-x+\tan x+C$. (b) $x-\tanh x+C$. *Hint.* Use the identities: (a) $\tan^2 x=\sec^2 x-$ -1; (b) $1-\tanh^2 x=\operatorname{sech}^2 x$. **1.30.** $\tan x+C$. **1.31.** $\frac{\pi}{2}x+C$. **1.32.** $x+\cos x+C$. **1.33.** $\frac{1}{2}\arctan\frac{x}{2}+C$. **1.34.** $\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5}+x}{\sqrt{5}-x}\right|+$ $+C$. **1.35.** $\arcsin\frac{x}{\sqrt{3}}+C$. **1.36.** $\ln|x+\sqrt{x^2+3}|-\ln|x+$ $+\sqrt{x^2-3}|+C$. **1.37.** $\ln|x|+2\arctan x+C$. **1.38.** $\frac{x^3}{3}+(a+b)\times$ $\times\frac{x^2}{2}+abx+C$. **1.39.** $ax+\frac{9}{4}a^{2/3}x^{4/3}+\frac{9}{5}a^{1/3}x^{5/3}+\frac{x^2}{2}+C$. **1.40.** $x+3\ln|\tan x+\sec x|-2\tan x+C$. **1.41.** (a) $-\cot x-x+C$; (b) $x-\coth x+C$. **1.42.** $\ln|x+\sqrt{x^2-7}|+C$. **1.43.** $x-$ $-\frac{1}{4\sqrt{2}}\ln\left|\frac{x-2\sqrt{2}}{x+2\sqrt{2}}\right|+C$. **1.44.** $\frac{2}{3}\sqrt{(3+x)^3}+C$. **1.45.** $-\frac{3}{16}\times$ $\times(3-4\sin x)^{4/3}+C$. **1.46.** $\frac{\cosh^2 x}{2}+C$. **1.47.** $-\frac{1}{3\tan^3 x}+C$.

1.48. $-\frac{1}{\ln x}+C$. 1.49. $\frac{1}{b}\ln|a+bx|+C$. 1.50. $-\frac{1}{b}\ln|a-b\tan x|+$ $+C$. 1.51. $-\frac{\sqrt{2}}{3}\ln\left|2-3\sin\frac{x}{\sqrt{2}}\right|+C$. 1.52. $\ln|\sin x|+C$. 1.53. $\frac{3^{4x}}{4\ln 3}+C$. 1.54. $\frac{1}{a}\sin(ax+b)+C$. 1.55. $-\cos(\ln x)+$ $+C$. 1.56. $-2\cos\sqrt{x}+C$. 1.57. $\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|+C$. 1.58. $-\frac{1}{3}\coth 3x+C$. 1.59. $\frac{3}{4}\sqrt[3]{(x^2-1)^2}+C$. 1.60. $-\frac{1}{2\ln 5}\times$ $\times 5^{-x^2}+C$. 1.61. $\frac{1}{4}\ln\left|\frac{1+2x}{1-2x}\right|+C$. 1.62. $-\frac{1}{a}\operatorname{arc\,tan}(e^{-ax})+C$. 1.63. $\frac{1}{\sqrt{3}}\arcsin\frac{x\sqrt{3}}{\sqrt{5}}+C$. 1.64. $\frac{1}{3}\ln|3x+\sqrt{9x^2-1}|+C$. 1.65. $-\ln(\cos x+\sqrt{\cos^2 x+4})+C$. 1.66. $\frac{1}{4}\operatorname{arc\,tan}(x^4)+C$. 1.67. $\frac{1}{2}\times$ $\times\ln(x^2+\sqrt{x^4+1})+C$. 1.68. $\frac{1}{b^2}\ln|a^2+b^2x|+C$. 1.69. $\frac{1}{2a\cos^2 ax}+$ $+C$. 1.70. $\frac{\cosh^3 x}{3}+C$. 1.71. $\frac{1}{7-e^x}+C$. 1.72. $\ln|\cos x|+C$. 1.73. $\frac{1}{4}\ln|\sinh 4x|+C$. 1.74. $-\frac{a^{1/x}}{\ln a}+C$. 1.75. $\frac{1}{2}\tanh(x^2+$ $+1)+C$. 1.76. $\frac{1}{2\sqrt{a^2-b^2}}\ln\left|\frac{\sqrt{a-b}\,x-\sqrt{a+b}\,x}{\sqrt{a-b}\,x+\sqrt{a+b}\,x}\right|+C$. 1.77. $\frac{1}{2\sqrt{7}}\operatorname{arc\,tan}\frac{2x}{\sqrt{7}}+C$. 1.78. $\frac{1}{8}\ln(4x^2+7)+C$. 1.79. $\frac{1}{3}\ln|x^3+\sqrt{x^6+1}|+C$. 1.80. $\frac{1}{\ln a}\ln(a^x+\sqrt{a^{2x}-1})+C$. 1.81. $\ln|x+2|+\frac{3}{x+2}+C$. *Hint.* $\frac{x-1}{(x+2)^2}=\frac{(x+2)-3}{(x+2)^2}=\frac{1}{x+2}-$ $-\frac{3}{(x+2)^2}$. 1.82. $x-\sqrt{3}\operatorname{arc\,tan}\frac{x}{\sqrt{3}}+C$. 1.83. $x-\ln|x^2-4|+$ $+\frac{7}{4}\ln\left|\frac{x-2}{x+2}\right|+C$. 1.84. $\frac{1}{4ab}\ln\left|\frac{ax^2-b}{ax^2+b}\right|+C$. 1.85. $\frac{1}{48}\times$ $\times\ln\left|\frac{3+2x^4}{3-2x^4}\right|+C$. 1.86. $\frac{1}{5}\ln|x^5+5x-8|+C$. 1.87. $\frac{1}{25}\times$ $\times\sqrt[4]{(5x^4-3)^5}+C$. 1.88. $3\ln(x+\sqrt{x^2+1})-\frac{1}{2}\ln(x^2+1)+C$. 1.89. $-\frac{1}{4}\sqrt{1-4x^2}+\frac{1}{2}\arcsin(2x)+C$. 1.90. $\frac{a}{b}\operatorname{arc\,tan}\frac{bx}{a}+$

$+\frac{1}{b}\ln(bx+\sqrt{a^2+b^2x^2})+C$. **1.91.** $-\frac{5}{\sqrt[5]{a^x}\ln a}+C$. **1.92.** $\frac{3}{4}\times$ $\times\sqrt[3]{(4+e^x)^4}+C$. **1.93.** $\ln(e^x+\sqrt{e^{2x}+4})+C$. **1.94.** $x-$ $-\frac{1}{\ln 2}\ln(2^x+1)+C$. *Hint.* $\frac{1}{2^x+1}=\frac{(1+2^x)-2^x}{2^x+1}=1-\frac{2^x}{2^x+1}$. **1.95.** $e^{\arcsin x}-\sqrt{1-x^2}+\arcsin x+C$. **1.96.** $e^{\sqrt{x^2-1}}+C$. **1.97.** $-\frac{2}{3}\sqrt{(3-\cosh x)^3}+C$. **1.98.** $-\frac{1}{2}\sqrt{1-4\ln x}+C$. **1.99.** $\frac{1}{2}\arcsin(2\ln x)+C$. **1.100.** $\frac{x}{2}-\frac{\sin 2x}{4}+C$. *Hint.* $\sin^2 x=$ $=\frac{1-\cos 2x}{2}$. **1.101.** $\frac{x}{2}+\frac{\sin 2x}{4}+C$. $\cos^2 x=\frac{1+\cos 2x}{2}$. **1.102.** $\sqrt{2}\ln\left|\tan\frac{x}{2\sqrt{2}}\right|+C$. **1.103.** $x+\frac{1}{a}\sin^2 ax+C$. **1.104.** $\frac{1}{3}\ln\left|\tan\left(\frac{x^3}{6}+\frac{\pi}{4}\right)\right|+C$. **1.105.** $\frac{7}{2}x+\frac{1}{2}\ln\left|\tan\left(x+\right.\right.$ $\left.\left.+\frac{\pi}{4}\right)\right|+\frac{3}{2}\sin 2x+\frac{\sin 4x}{8}+C$. **1.106.** $-2\sqrt{3-\cos^2 x}+C$. **1.107.** $-\ln(\cos^2 x+\sqrt{\cos^4 x+3})+C$. **1.108.** $\ln|\tan x|+C$. *Hint.* $\frac{1}{\sin x\cos x}=\frac{2}{\sin 2x}$. **1.109.** $-\frac{1}{\sqrt{3}}\ln|\cos\sqrt{3}x|+C$. **1.110.** $\frac{1}{a}\times$ $\times\ln\cosh ax+C$. **1.111.** $\frac{1}{a}\tan(ax+b)-x+C$. **1.112.** $-\frac{1}{3}\times$ $\times\cot(x^3-3)-\frac{x^3}{3}+C$. **1.113.** $e^{\sec x}+C$. **1.114.** $\frac{1}{2}\arctan\cos x+C$. **1.115.** $\frac{1}{2}\ln\left|\frac{1+\ln x}{1-\ln x}\right|+C$. **1.116.** $\frac{1}{6}\ln(1+3e^{2x})+C$. **1.117.** $\frac{1}{12}e^{4x^3}+C$. **1.118.** $2e^{\sqrt{x}}+C$. **1.119.** $\frac{1}{6}x^3+\frac{1}{4}x^2+\frac{1}{2}x+$ $+\ln(x-1)+C$. **1.120.** $\frac{3}{2}\sqrt[3]{(x+1)^2}-3\sqrt[3]{x+1}+C$. **1.121.** $\frac{1}{6}\times$ $\times\sqrt{(4x-1)^3}+C$. **1.122.** $\frac{1}{8}\sqrt[3]{(6x+5)^4}$. **1.123.** $-\frac{1}{b}\cos(a+bx)+C$. **1.124.** $\frac{1}{3}\ln\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+C$. **1.125.** $\frac{1}{2}\ln\left|\frac{x}{2+\sqrt{4-x^2}}\right|+C$. **1.126.** $2\ln(\sqrt{x}+1)+C$. **1.127.** $e^x-\ln(e^x+1)+C$. **1.128.** $\frac{1}{25}\times$ $\times\left(\frac{(5x-1)^{21}}{21}+\frac{(5x-1)^{20}}{20}\right)+C$. **1.129.** $-2\sqrt{1-e^x}+\frac{2}{3}\times$

$\times \sqrt{(1-e^x)^3} - \frac{2}{5}\sqrt{(1-e^x)^5} + C$. **1.130.** $2\left(\frac{\sqrt{(x+1)^3}}{3} - \frac{x+1}{2} + \right.$
$\left. + 2\sqrt{x+1} - 2\ln\left(\sqrt{x+1}+1\right)\right) + C$. **1.131.** $\frac{1}{2(3-x)^6} - \frac{1}{5(3-x)^5} +$
$+ C$. **1.132.** $\frac{\sqrt{3}}{3}\ln\frac{\sqrt{3+e^x}-\sqrt{3}}{\sqrt{3+e^x}+\sqrt{3}} + C$. **1.133.** $\ln\left|\frac{x}{1+\sqrt{x^2-1}}\right| +$
$+ C$. **1.134.** $x\arccos x - \sqrt{1-x^2} + C$. **1.135.** $x\sin x + \cos x + C$.
1.136. $\frac{x^2}{2}\ln x - \frac{x^2}{4} + C$. **1.137.** $\frac{3}{2}\sqrt[3]{x^2}\ln x - \frac{9}{4}\sqrt[3]{x^2} + C$.
1.138. $\left(\frac{x^3}{3} - \frac{x^2}{2} + x\right)\ln x - \frac{x^3}{9} + \frac{x^2}{4} - x + C$. **1.139.** $(2-x^2)\times$
$\times\cos x + 2x\sin x + C$. **1.140.** $-(x^2+2x+2)e^{-x} + C$. **1.141.** $(x^3 -$
$- 3x^2 + 6x - 6)e^x + C$. **1.142.** $-\frac{e^{-x^2}}{2}(x^2+1) + C$. *Hint.* Set $u = x^2$,
$dv = xe^{-x^2}dx$. **1.143.** $-\frac{1}{x}(\ln^2 x + 2\ln x + 2) + C$. **1.144.** $\frac{1}{2}\times$
$\times (x^2+1)\arctan x - \frac{x}{2} + C$. **1.145.** $\frac{x}{2\cos^2 x} - \frac{1}{2}\tan x + C$.
1.146. $\frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + C$. **1.147.** $\frac{(x-\sqrt{1-x^2})e^{\arccos x}}{2} +$
$+ C$. **1.148.** $x\ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + C$. **1.149.** $\frac{x^4}{4}\ln x - \frac{x^4}{16} + C$.
1.150. $\frac{3^x}{\ln^2 3}(x\ln 3 - 1) + C$. **1.151.** $(x^2-2x+1)\sin x + 2(x-1)\times$
$\times\cos x + C$. **1.152.** $x\tan x + \ln|\cos x| + C$. **1.153.** $\frac{x}{2}(\sin(\ln x) + \cos\times$
$\times(\ln x)) + C$. **1.154.** $2e^{\sqrt{x}}(\sqrt{x}-1) + C$. *Hint.* Make the substitution
$x = t^2$ and integrate by parts. **1.155.** $\frac{1+x^2}{2}(\arctan x)^2 - x\arctan x +$
$+ \frac{1}{2}\ln(1+x^2) + C$. **1.156.** $-\frac{\arcsin x}{x}\ln\left|\frac{x}{1+\sqrt{1-x^2}}\right| + C$.
1.157. $-x\cot x + \ln|\cos x| - \frac{x^2}{2} + C$. **1.158.** $e^{-x}\frac{2\sin x - \cos 2x - 5}{10} +$
$+ C$. **1.159.** $-\frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x + C$. *Hint.* Set $u = x$, $dv =$
$= \frac{x\,dx}{(x^2+1)^2}$. **1.160.** $I_n = \int\frac{dx}{(x^2+a^2)^n} = \frac{1}{a^2}\int\frac{(a^2+x^2)-x^2}{(x^2+a^2)^n}dx =$
$= \frac{1}{a^2}\int\frac{dx}{(x^2+a^2)^{n-1}} - \frac{1}{a^2}\int x\frac{x\,dx}{(x^2+a^2)^n} = \frac{1}{a^2}I_{n-1} - \frac{1}{a^2}\times$

$\times\left(-\frac{x}{2(n-1)(x^2+a^2)^{n-1}}+\frac{1}{2(n-1)}I_{n-1}\right)=\frac{1}{2(n-1)a^2}\times$ $\times\left(\frac{x}{(x^2+a^2)^{n-1}}+(2n-3)I_{n-1}\right)$. Hence, $I_2=\frac{1}{2a^2}\left(\frac{1}{x^2+a^2}+\right.$ $\left.+\frac{1}{a}\arctan\frac{x}{a}\right)+C$, $I_3=\frac{1}{4a^2}\left(\frac{x}{(x^2+a^2)^2}+\frac{3x}{2a^2(x^2+a^2)}+\frac{3}{2a^3}\times\right.$ $\left.\times\arctan\frac{x}{a}\right)+C$. **1.161.** We set $u=\sqrt{x^2+a}$, $dv=dx$. Then $du=$ $=\frac{x\,dx}{\sqrt{x^2+a}}$, $v=x$. We have $\int\sqrt{x^2+a}\,dx=x\sqrt{x^2+a}-$ $-\int\frac{x^2\,dx}{\sqrt{x^2+a}}=x\sqrt{x^2+a}-\int\frac{(x^2+a)-a}{\sqrt{x^2+a}}\,dx=x\sqrt{x^2+a}-$ $-\int\sqrt{x^2+a}\,dx+a\int\frac{dx}{\sqrt{x^2+a}}$. Hence, $\int\sqrt{x^2+a}\,dx=\frac{x}{2}\sqrt{x^2+a}+$ $+\frac{a}{2}\ln|x+\sqrt{x^2+a}|+C$. **1.162.** We put $u=x$, $dv=\frac{x\,dx}{\sqrt{a^2-x^2}}$. Then $du=dx$, $v=-\sqrt{a^2-x^2}$. We have $\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx=-x\times$ $\times\sqrt{a^2-x^2}+\int\sqrt{a^2-x^2}\,dx=-x\sqrt{a^2-x^2}+\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx=$ $=-x\sqrt{a^2-x^2}+a^2\int\frac{dx}{\sqrt{a^2-x^2}}-\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx$. Hence, $\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$. **1.163.** $\left(\frac{x^2}{2}-\right.$ $\left.-\frac{1}{4}\right)\arcsin x+\frac{x}{4}\sqrt{1-x^2}+C$. **1.164.** $(\ln(\ln x)-1)\ln x+C$. **1.165.** $\frac{x^3}{3}\arctan x-\frac{x^2}{6}-\frac{1}{6}\ln(x^2+1)+C$. **1.166.** $-2\sqrt{1-x}\times$ $\times\arcsin\sqrt{x}+2\sqrt{x}+C$. **1.167.** $\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$. *Hint.* See the solution of Problem 1.161.

2.1. $\frac{1}{6}\ln\left|\frac{x-1}{x+5}\right|+C$. **2.2.** $\frac{1}{\sqrt{6}}\arctan\frac{2(x-1)}{\sqrt{6}}+C$.

2.3. $\arcsin\frac{x-4}{4}+C$. **2.4.** $\ln\left|x+\frac{p}{2}+\sqrt{x^2+px+q}\right|+C$.

2.5. $\frac{1}{6}\ln\left|\frac{x-6}{x}\right|+C$. **2.6.** $\frac{\sqrt{3}}{3}\arcsin\frac{(x+1)\sqrt{3}}{\sqrt{7}}+C$.

2.7. $\frac{1}{2}\ln|x^2-5x+4|+\frac{5}{6}\ln\left|\frac{x-4}{x-1}\right|+C$. **2.8.** $\frac{1}{2}\ln(x^2-3x+$

$+3)+\sqrt{3}\arctan\frac{2x-3}{\sqrt{3}}+C$. **2.9.** $-\sqrt{2-x-x^2}+\frac{7}{2}\times$ $\times\arcsin\frac{2x+1}{3}+C$. **2.10.** $\sqrt{x^2-6x+1}+C$. **2.11.** $\frac{1}{4}\times$ $\times\arctan\frac{x^2+3}{2}+C$. **2.12.** $\frac{1}{2\ln 3}\ln\left|\frac{3^x-3}{3^x-1}\right|+C$. **2.13.** $2\ln(x^2-$ $-2x+6)+\frac{1}{\sqrt{5}}\arctan\frac{x-1}{\sqrt{5}}+C$. **2.14.** $\sqrt{x^2+x+1}-$ $-\frac{1}{2}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+C$. **2.15.** $\ln\left|\frac{x}{1+4x+\sqrt{x^2+8x+1}}\right|+$ $+C$. **2.16.** $-\frac{1}{2}\sqrt{\frac{5-x}{x-1}}+C$. **2.17.** $-\frac{\sqrt{2x^2-x+1}}{x}-$ $-\frac{1}{2}\ln\frac{2-x+\sqrt{2x^2-x+1}}{|x|}+C$. **2.18.** $-\frac{\sqrt{x^2+5}}{9(x+2)}-\frac{2}{27}\times$ $\times\ln\frac{5-2x+3\sqrt{x^2+5}}{|x+2|}+C$. **2.19.** $\frac{1}{7}\ln\left|\frac{x-3}{x+4}\right|+C$. **2.20.** $-\frac{1}{6}\ln|x|-\frac{7}{2}\ln|x-2|+\frac{17}{3}\ln|x-3|+C$. **2.21.** $x-$ $-\frac{1}{2}\ln|x|-\frac{3}{4}\ln|x+2|+\frac{5}{4}\ln|x-2|+C$. **2.22.** $\frac{x^2}{2}+\frac{1}{x+1}+$ $+\frac{2}{(x+1)^2}+C$. **2.23.** $-\frac{4}{3(x-1)}+\frac{20}{9}\ln|x-1|+\frac{7}{9}\ln|x+2|+C$. **2.24.** $-\frac{1}{2(x^2-5x+4)^2}+C$. **2.25.** $\frac{1}{4}\ln\frac{x^2}{x^2+2}+C$. **2.26.** $\frac{1}{4\sqrt{2}}\times$ $\times\ln\left|\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right|+\frac{\sqrt{2}}{4}\arctan\frac{x\sqrt{2}}{1-x^2}+C$. **2.27.** $-\frac{1}{4}\times$ $\times\left(\frac{x+1}{(x^2+1)^2}+\frac{3x}{2(x^2+1)}+\frac{3}{2}\arctan x\right)+C$. *Hint.* $\int\frac{x-1}{(x^2+1)^3}dx=$ $=\int\frac{x}{(x^2+1)^3}dx-\int\frac{dx}{(x^2+1)^3}=-\frac{1}{4(x^2+1)^2}-\int\frac{dx}{(x^2+1)^3}$. Further, evaluate $\int\frac{dx}{(x^2+1)^3}$ by the recurrence formula derived in Problem 1.160. **2.28.** $\frac{1}{9}\ln|x-1|-\frac{1}{18}\ln(x^2+x+1)+$ $+\frac{x+1}{3(x^2+x+1)}+\frac{\sqrt{3}}{9}\arctan\frac{2x+1}{\sqrt{3}}+C$. *Hint.* $\frac{x}{(x+1)(x^2+x+1)^2}=$ $=\frac{A}{x-1}+\frac{Bx+C}{(x^2+x+1)^2}+\frac{Dx+E}{x^2+x+1}$. Using the method of unde-

termined coefficients, we get $\frac{x}{(x-1)(x^2+x+1)^2} = \frac{1}{9(x-1)} - \frac{1}{3}\frac{x-1}{(x^2+x+1)^2} - \frac{1}{9}\frac{x+2}{x^2+x+1}$. To find $\int \frac{dx}{(x^2+x+1)^2} = \int \frac{dx}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}$ make the substitution $x+\frac{1}{2}=t$ and then apply the recurrence formula derived in Problem 1.160. **2.29.** $\frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right|+C$. **2.30.** $\frac{1}{2}\ln|x+1|-2\ln|x-2|+\frac{5}{2}\ln|x-3|+C$. **2.31.** $\frac{1}{24}\ln\frac{(x+2)^2}{x^2-2x+4}+\frac{\sqrt{3}}{12}\arctan\frac{x-1}{\sqrt{3}}+C$. **2.32.** $\frac{3}{x-2}-\frac{2}{x-3}+\ln\left|\frac{x-3}{x-2}\right|+C$. **2.33.** $x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|-\arctan x+C$. **2.34.** $\frac{x}{2(1+x^2)}+\frac{\arctan x}{2}+C$. **2.35.** $-\frac{1}{a^2x}-\frac{1}{a^3}\arctan\frac{x}{a}+C$. *Hint.* $\frac{1}{x^2(x^2+a^2)}=\frac{1}{a^2}\frac{(a^2+x^2)-x^2}{x^2(x^2+a^2)}$. **2.36.** $\frac{1}{4a^3}\ln\left|\frac{x-a}{x+a}\right|-\frac{1}{2a^3}\arctan\frac{x}{a}+C$. *Hint.* $\frac{1}{(x^2+a^2)(x^2-a^2)}=\frac{1}{2a^2}\frac{(x^2+a^2)-(x^2-a^2)}{(x^2+a^2)(x^2-a^2)}$. **2.37.** $\frac{1}{4\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|-\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+C$. **2.38.** $\ln|x|-\frac{1}{6}\ln(x^6+1)+\frac{1}{6(x^6+1)}+C$. *Hint.* $\frac{1}{x(x^6+1)^2}=\frac{(1+x^6)-x^6}{x(x^6+1)^2}$. **2.39.** $\frac{1}{4x^4}+\frac{1}{2x^2}+\ln|x|-\frac{1}{2}\ln(x^2+1)+C$. **2.40.** $\frac{1}{12}\ln(x^4+1)(x^4-2)^2+C$. *Hint.* Set $x^4=t$. **2.41.** $-\frac{1}{6(x+1)^6}+\frac{3}{7(x+1)^7}-\frac{1}{4(x+1)^8}+C$. **2.42.** $\frac{1}{6}\times\ln|x^3-1|-\frac{1}{12}\ln(x^6+x^3+2)+\frac{1}{6\sqrt{7}}\arctan\frac{2x^3+1}{\sqrt{7}}+C$. **2.43.** $-\cos x+\frac{\cos^3 x}{3}+C$. **2.44.** $\frac{1}{7\cos^7 x}-\frac{1}{5\cos^5 x}+C$. **2.45.** $\sin x-\sin^3 x+\frac{3}{5}\sin^5 x-\frac{1}{7}\sin^7 x+C$. **2.46.** $\frac{3x}{8}+\frac{\sin x}{2}+\frac{\sin 2x}{16}+C$. **2.47.** $\frac{x}{8}-\frac{\sin 4x}{32}+C$. **2.48.** $\frac{x}{16}-\frac{\sin 4x}{64}-\frac{\sin^3 2x}{48}+C$. **2.49.** $-\cot x-\frac{2}{3}\cot^3 x-\frac{1}{5}\cot^5 x+C$. **2.50.** $\frac{1}{5}\times$

$\times \tan^5 x+\frac{1}{3}\tan^3 x+C$. **2.51.** $-\frac{1}{2\tan^2 x}+3\ln|\tan x|+$ $+\frac{3}{2}\tan^2 x+\frac{1}{4}\tan^4 x+C$. **2.52.** $-\frac{1}{3\tan^3 x}-\frac{2}{\tan x}+\tan x+C$. **2.53.** $\frac{\sqrt{2}}{2}\left(\ln\left|\tan\frac{x}{2}\right|-\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right)+C$. **2.54.** $\frac{\sin x}{4\cos^4 x}+$ $+\frac{3}{8}\frac{\sin x}{\cos^2 x}+\frac{3}{8}\ln|\tan x+\sec x|+C$. **2.55.** $\frac{1}{2}\tan^2 x+\ln|\cos x|+$ $+C$. **2.56.** $x+2\cot\frac{x}{2}-\cot^2\frac{x}{2}-\frac{2}{3}\cot^3\frac{x}{2}-2\ln\left|\sin\frac{x}{2}\right|+C$. **2.57.** $-2\sqrt{\cot x}+C$. **2.58.** $\sin x-\frac{2}{3}\sin^3 x+\frac{\sin^5 x}{5}+C$. **2.59.** $-2\sqrt{\cos x}+\frac{2}{5}\sqrt{\cos^5 x}+C$. **2.60.** $\frac{5}{16}x-\frac{1}{8}\sin 4x+$ $+\frac{1}{96}\sin^3 4x+\frac{3}{128}\sin 8x+C$. **2.61.** $\tan x+\frac{1}{3}\tan^3 x+C$. **2.62.** $3\ln\left|\tan\frac{x}{3}\right|-\frac{3}{2\tan^2\frac{x}{3}}+C$. **2.63.** $\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|-$ $-\frac{\cos x}{2\sin^2 x}+C$. **2.64.** $\frac{1}{2}\sin x+\frac{1}{12}\sin 3x+\frac{1}{20}\sin 5x+C$. **2.65.** $-\frac{\cos 8x}{16}+\frac{\cos 2x}{4}+C$. **2.66.** $-\frac{\sin 25x}{50}+\frac{\sin 5x}{10}+C$. **2.67.** $\frac{3}{5}\sin\frac{5x}{6}+3\sin\frac{x}{6}+C$. **2.68.** $\frac{3}{2}\cos\frac{x}{3}-\frac{1}{2}\cos x+C$. **2.69.** $\frac{\sin\frac{x}{2}}{2}+\frac{\sin 5x}{20}+\frac{\sin 7x}{28}+C$. **2.70.** $\frac{1}{24}\cos 6x-\frac{1}{16}\cos 4x-$ $-\frac{1}{8}\cos 2x+C$. **2.71.** $\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+\tan\frac{x}{2}}{\sqrt{5}-\tan\frac{x}{2}}\right|+$ **2.72.** $\arctan\left(\tan\frac{x}{2}-1\right)+C$. **2.73.** $\frac{1}{\cos x}-\tan x+x+C$. *Hint.* Multiply the numerator and denominator of the integrand by $(1-$ $-\sin x)$. **2.74.** $\frac{1}{4\sqrt{7}}\ln\left|\frac{2\tan x-\sqrt{7}}{2\tan x+\sqrt{7}}\right|+C$. **2.75.** $-\frac{1}{2}\times$ $\times\arctan\left(\frac{\cos x-1}{2}\right)+C$. **2.76.** $-\frac{1}{4}\ln(1+4\cos^2 x)+C$.

2.77. $\frac{2\sqrt{3}}{3}\arctan\left(\frac{2\tan\frac{x}{2}-1}{\sqrt{3}}\right)+C$. 2.78. $\frac{2}{5\left(\tan\frac{x}{2}-1\right)}-$

$-\frac{2}{5\sqrt{15}}\arctan\frac{4\tan\frac{x}{2}+1}{\sqrt{15}}+C$. *Hint.* $\frac{1}{(\sin x+4)(\sin x-1)}-$

$=\frac{(\sin x+4)-(\sin x-1)}{5(\sin x+4)(\sin x-1)}$. 2.79. $\ln|\sin x-\cos x|+C$. 2.80. $\frac{1}{4}\times$

$\times\ln\left|\frac{\tan x+2}{\tan x+6}\right|+C$. 2.81. $\frac{\sinh 6x}{12}+\frac{x}{2}+C$. 2.82. $\frac{\cosh^3 2x}{6}-$

$-\frac{\cosh 2x}{2}+C$. 2.83. $\frac{\sinh 4x}{32}-\frac{x}{8}+C$. 2.84. $\frac{3x}{8}+\frac{\sinh 2x}{4}+$

$+\frac{\sinh 4x}{32}+C$. 2.85. $-2\coth 2x+C$. 2.86. $\frac{1}{4}\ln\left|\frac{\tanh x-2}{\tanh x+2}\right|+$

$+C$. *Hint.* Divide the numerator and denominator of the integrand by $\cosh^2 x$. 2.87. $-\frac{1}{\sinh x}-\coth x+C$. *Hint.* Multiply the numerator and denominator of the integrand fraction by $\cosh x+1$. 2.88. $2\sqrt{2}\sinh\frac{x}{2}+C$. 2.89. $\ln|\sinh x|-$

$-\frac{\coth^2 x}{2}+C$. 2.90. $x-\tanh x-\frac{\tanh^3 x}{3}+C$.

2.91. $\arctan\frac{\sqrt{1+x}}{2}+C$. 2.92. $\frac{3}{20}\sqrt[3]{(2x-3)^5}+\frac{9}{8}\sqrt[3]{(2x-3)^2}+C$.

2.93. $2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln|\sqrt[6]{x}-1|+C$. 2.94. $6\arctan\sqrt[6]{x+a}-$

$-6\ln\sqrt[6]{x+a}+3\ln|1+\sqrt[3]{x+a}|+C$. 2.95. $\frac{3}{16}\sqrt{\left(\frac{x+1}{x-1}\right)^4}-$

$-\frac{3}{28}\sqrt[3]{\left(\frac{x+1}{x-1}\right)^7}+C$. 2.96. $\ln\left|\frac{\sqrt[4]{x}-1}{\sqrt[4]{x}+1}\right|-2\arctan\sqrt[4]{x}+C$.

2.97. $6\sqrt[6]{x}-12\arctan\frac{\sqrt[6]{x}}{2}+C$. 2.98. $\ln\left|\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x+1}-\sqrt{x-1}}\right|-$

$-2\arctan\sqrt{\frac{x-1}{x+1}}+C$. 2.99. $\frac{1}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}\sqrt{4-x^2}}{x+\sqrt{3}\sqrt{4-x^2}}\right|+C$.

2.100. $\ln\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}+C$. 2.101. $\sqrt{x^2-1}-\arccos\frac{1}{x}+C$.

2.102. $\frac{\sqrt{(a^2-x^2)^3}}{3}-a^2\sqrt{a^2-x^2}+C$. 2.103. $\frac{x+1}{2}\sqrt{1-2x-x^2}+$

$+\arcsin\frac{x+1}{\sqrt{2}}+C$. **2.104.** $6\arcsin\frac{x+1}{2}-\frac{\sqrt{3-2x-x^2}}{4}(x^3+$

$+3x^2-7x-9)+C$. **2.105.** $\frac{x^3}{3\sqrt{(1+x^2)^3}}+C$. **2.106.** $\ln|x+$

$+\sqrt{x^2-1}|-\frac{\sqrt{x^2-1}}{x}+C$. **2.107.** $\frac{x-1}{2}\sqrt{x^2-2x+10}+$

$+\frac{9}{2}\ln(x-1+\sqrt{x^2-2x+10})+C$. **2.108.** $\frac{x-2}{2}\sqrt{4x-x^2}+$

$+2\arcsin\frac{x-2}{2}+C$. **2.109.** $-\frac{\sqrt{x^2+5}}{x}+\ln(x+\sqrt{x^2+5})+C$.

2.110. $\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+C$. **2.111.** $\frac{x}{9\sqrt{x^2+9}}+C$.

2.112. $\frac{1}{8}(2x^3-5x)\sqrt{x^2-1}+\frac{3}{8}\ln(x^2+\sqrt{x^2-1})+C$.

3.1. $\frac{1}{2}\ln(x^2+2x+4)+\frac{2}{\sqrt{3}}\arctan\frac{x+1}{\sqrt{3}}+C$. **3.2.** $\frac{x^2}{2}+x+$

$+\ln|x^2-x-1|+\frac{2}{\sqrt{5}}\ln\left|\frac{2x-1-\sqrt{5}}{2x-1+\sqrt{5}}\right|+C$. **3.3.** $-\frac{1}{5(x-2)}+$

$+\frac{1}{25}\ln\left|\frac{x+3}{x-2}\right|+C$. **3.4.** $-\frac{2}{9}\ln\frac{|x-1|}{\sqrt{x^2+x+1}}+\frac{2\sqrt{3}}{9}\times$

$\times\arctan\frac{2x+1}{\sqrt{3}}+\frac{1}{3}\frac{x}{1-x^3}+C$. **3.5.** $\frac{1}{4}\left(2\ln\frac{x^4+1}{x^4}-\frac{1}{x^4}-\right.$

$\left.-\frac{1}{x^4+1}\right)+C$. **3.6.** $\sqrt{x^2+x+2}-\frac{1}{2}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+2}\right)+$

$+C$. **3.7.** $-\sqrt{6+4\ln x-\ln^2 x}+2\arcsin\frac{\ln x-2}{\sqrt{10}}+C$. **3.8.** $-\frac{1}{2}\times$

$\times\ln\left|\frac{2x+2+\sqrt{x^2+8x+4}}{x}\right|+C$. **3.9.** $\frac{\sqrt{(x^2-4)^3}}{3}+C$.

3.10. $\frac{\sqrt{(x^2+4x-5)^3}}{3}-(x+2)\sqrt{x^2+4x-5}+9\ln(x+2+$

$+\sqrt{x^2+4x-5})+C$. **3.11.** $\frac{x+2}{2}\sqrt{x^2+4x+5}+\frac{1}{2}\ln(x+2+$

$+\sqrt{x^2+4x+5})+C$. **3.12.** $\frac{1}{15}\arctan\frac{5x}{3\sqrt{16-x^2}}+C$. **3.13.** $\frac{1}{2}\times$

$\times\ln(x^2+\sqrt{x^4+16})+C$. **3.14.** $\frac{x}{4\sqrt{x^2+4}}+C$. **3.15.** $2\sqrt{x}+$

$+4\sqrt[4]{x}-\frac{8}{\sqrt{3}}\arctan\frac{2\sqrt[4]{x}-1}{\sqrt{3}}+C$. **3.16.** $-\sqrt{\frac{1-x}{1+x}}+C$.

3.17. $-x+\tan x+\sec x+C$. **3.18.** $x\tan\frac{x}{2}+2\ln\left|\cos\frac{x}{2}\right|+C$.

3.19. $\frac{1}{3(1-\sin x)^3}+C$. **3.20.** $\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)+C$.

3.21. $\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|+C$. **3.22.** $2\tan x-\frac{3}{4}\sqrt[3]{\tan^4 x}+C$.

3.23. $\arcsin\left(\frac{\sec x}{\sqrt{5}}\right)+C$. **3.24.** $-\frac{\sin^2 x}{2}+5\sin x-24\ln(\sin x+$

$+5)+C$. **3.25.** $\ln|\tan x|+\tan^2 x+\frac{\tan^4 x}{4}+C$. **3.26.** $\tan x+$

$+\frac{2}{3}\tan^3 x+\frac{\tan^5 x}{5}+C$. **3.27.** $-\frac{\cos x}{4\sin^4 x}-\frac{3\cos x}{8\sin^2 x}+\frac{3}{8}\times$

$\times\ln\left|\tan\frac{x}{2}\right|+C$. **3.28.** $-\frac{x\cos 3x}{6}+\frac{\sin 3x}{18}+\frac{x\cos x}{2}-\frac{\sin x}{2}+$

$+C$. **3.29.** $\ln|\tanh x|+C$. **3.30.** $\arctan(\tanh x)+C$.

3.31. $\ln(\cosh x)-\frac{\tanh^2 x}{2}-\frac{\tanh^4 x}{4}+C$. **3.32.** $2\sinh\sqrt{1+x}+$

$+C$. **3.33.** $x\tanh x-\ln(\cosh x)+C$. **3.34.** $\frac{x}{2}-$

$-\frac{x\cos(2\ln x)+2x\sin 2(\ln x)}{10}+C$. **3.35.** $\frac{e^{2x}}{4}(2x-1)+C$.

3.36. $-\frac{1}{2}e^{-x^2}+C$. **3.37.** $\frac{1}{6}\ln\left|\frac{e^x-1}{e^x+1}\right|+C$. **3.38.** $\frac{1}{\ln a-\ln b}\times$

$\times\left(\frac{a^x}{b^x}-\frac{b^x}{a^x}\right)-2x+C$. **3.39.** $\frac{1}{2}e^{\arcsin x}(x+\sqrt{1-x^2})+C$.

3.40. $2\sqrt{e^x-1}-2\arctan\sqrt{e^x-1}+C$. **3.41.** $-\frac{\sqrt{1-x^2}}{x}\arcsin x+$

$+\frac{1}{2}(\arcsin x)^2+\ln|x|+C$. **3.42.** $x-e^{-x}\arcsin(e^x)-\ln(1+$

$+\sqrt{1-e^{2x}})+C$, $x\leqslant 0$. **3.43.** $-\frac{x^2}{6}-\left(x-\frac{x^3}{3}\right)\arctan x+$

$+\frac{1}{2}(\arctan x)^2+\frac{2}{3}\ln(1+x^2)+C$. **3.44.** $\frac{x}{4}+\frac{x^3}{12}+\frac{1}{4}(1+x^2)\times$

$\times\arctan x+C$. **3.45.** $-\frac{\ln(1+x+x^2)}{1+x}-\frac{1}{2}\frac{\ln(1+x)^2}{1+x+x^2}+$

$+\sqrt{3}\arctan\frac{1+2x}{\sqrt{3}}+C$. **3.46.** $-x^2+\frac{x^2}{2}\ln(4+x^4)+2\arctan\times$ $\times\frac{x^2}{2}+C$. **3.47.** $-\frac{x^2+7}{9}\sqrt{x^2+1}+\frac{(x^2+1)^{3/2}}{3}\ln\sqrt{x^2-1}-$ $-\frac{1}{3}\ln\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}+C$, $|x|>1$. **3.48.** $\left(\frac{3-x}{1-x}-\ln\frac{x}{\sqrt{1-x}}\right)\times$ $\times\sqrt{1-x^2}-\frac{1}{2}\arcsin x-\ln\frac{1+\sqrt{1-x^2}}{x}+C$, $0<x<1$. **3.49.** x^x+C, $x>0$. **3.50.** $x-\ln(1+e^x)-2e^{-x/2}\arctan e^{x/2}-$ $-(\arctan e^{x/2})^2+C$.

4.1. $\frac{35}{2}$. *Hint.* Divide the interval $[0, 5]$ into n equal parts.

4.2. 1. *Hint.* Divide the interval $\left[0, \frac{\pi}{2}\right]$ into n equal parts. Apply the formula: $\cos\alpha+\cos 2\alpha+\ldots+\cos n\alpha=\frac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}$.

4.3. $e^{10}-1$. *Hint.* Divide the interval $[0, 10]$ into n equal parts. **4.4.** 2/3. *Hint.* Divide the interval $[1, 3]$ into n parts so that the abscissas of the division points form a geometric progression. **4.5.** 15/4. **4.6.** 9/2. **4.7.** 5. **4.8.** 19/15. **4.9.** $3\frac{57}{64}$. **4.10.** 45/4. **4.11.** 1. **4.12.** 1. **4.13.** e^2-e. **4.14.** 7/ln 2. **4.15.** ln 2.5. **4.16.** (ln 3)/2. **4.17.** $\pi/12$. **4.18.** $\frac{\pi}{8}-\frac{1}{4}$. **4.19.** $\frac{\pi}{6}+\frac{8\sqrt{3}}{27}$. **4.20.** $\frac{1}{3}(2-3\cosh 2+\cosh^3 2)$. **4.21.** $\frac{1}{4}\arctan\frac{4}{7}$. **4.22.** $\ln\frac{2+\sqrt{5}}{1+\sqrt{2}}$. **4.23.** $\frac{11}{2}+7\ln 2$. **4.24.** $2\ln\frac{4}{3}-\frac{1}{2}$. **4.25.** $\frac{1}{2}(e-\sqrt[4]{e})$. **4.26.** sin 1. **4.27.** $\frac{\pi}{4}$. **4.28.** $\frac{2}{3}$. **4.29.** $\frac{1}{12}\sinh 2+\frac{1}{6}$. **4.30.** $\frac{1}{6}\ln\frac{2}{5}$. **4.31.** $\frac{\pi}{2\sqrt{2}}$. **4.32.** $2-\ln 5$. **4.33.** $\frac{\pi}{4}$. **4.34.** $\frac{\pi}{4}$. The sum $S_n=\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots$

$$\ldots+\frac{n}{n^2+n^2}=\frac{1}{n}\left(\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\ldots\right.$$

$$\left.\ldots+\frac{1}{1+\left(\frac{n}{n}\right)^2}\right)$$

may be regarded as an integral sum for the

function $f(x)=\frac{1}{1+x^2}$ on the interval $[0, 1]$. Therefore $\lim\limits_{n\to\infty} S_n = \int\limits_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \frac{\pi}{4}$. **4.35.** 1. **4.36.** $\frac{2}{3}(2^{3/2}-1)$. **4.37.** $\frac{19}{6}$. **4.38.** $\frac{45}{4}$. **4.39.** 7. **4.40.** $\frac{16}{3}$. **4.41.** $1-\frac{\sqrt{2}}{2}$. **4.42.** $\frac{1}{e}-\frac{1}{e^2}$. **4.43.** $2\ln\frac{3}{2}$. **4.44.** $4-3\ln 3$. **4.45.** (a) Minus. *Hint.* Separate the interval of integration into the subintervals $[-2, -1]$ and $[-1, 1]$ and take advantage of properties (1) and (9); (b) plus; (c) minus. **4.46.** (a) The second; (b) the first; (c) the second. **4.47.** (a) $\frac{1}{4}$; (b) $\frac{3}{4}$; (c) $\frac{2}{\pi}$; (d) $\frac{4}{3\pi}$. **4.48.** $\frac{2}{\pi} I_0 \cos\varphi$. **4.49.** $2\sqrt{7} < I < 6$. **4.50.** $\frac{2\pi}{\sqrt{7}} < I < \frac{2\pi}{\sqrt{3}}$. **4.51.** (a) $\frac{2}{3}(2\sqrt{2}-1) < I < \frac{2\sqrt{2}}{3}(2\sqrt{2}-1)$; (b) $|I| < \frac{\sqrt{30}}{4}$. **4.52.** (a) $\frac{4}{3} < I < \frac{2}{3}\sqrt{5}$; (b) $I < \sqrt{2.125}$. **4.53.** (a) $\frac{dI}{d\beta} = \frac{e^\beta}{\beta}$; (b) $\frac{dI}{d\alpha} = -\frac{e^\alpha}{\alpha}$. **4.54.** $x = \frac{\pi}{2}(2k+1)$, $k = 0, 1, 2, \ldots$. **4.55.** $\frac{\sin x}{x}$. **4.56.** $\frac{\sin x}{2\sqrt{x}} + \frac{1}{x^2}\sin\frac{1}{x^2}$. **4.57.** $-\frac{1}{\sqrt{1+x^3}}$. **4.58.** $\frac{x^2-x}{\ln x}$. **4.60.** No. **4.61.** $\frac{2}{3}\left(3+\ln\frac{2}{5}\right)$. **4.62.** $\ln\frac{3}{2}$. **4.63.** $\frac{1}{4}(2+\sinh 2)$. **4.64.** $\frac{2}{\sqrt{5}} \times \arctan\frac{1}{\sqrt{5}}$. **4.65.** $\frac{\pi}{3\sqrt{3}}$. **4.66.** π. **4.67.** $\frac{\pi}{6}$. **4.68.** $\frac{\pi}{6}$. **4.69.** $\frac{1}{3}(2\sqrt{3}-\pi)$. **4.70.** $\frac{\sqrt{3}-1}{2}$. **4.71.** $\frac{1}{32}(\pi+2)$. **4.72.** $2\left(\ln 2-\frac{1}{4}\right)$. **4.73.** $\ln\frac{\sqrt{5}+3}{2}$. **4.74.** $\frac{1}{6}$. **4.75.** $4-\pi$. **4.76.** $\frac{81}{16}\pi$. **4.80.** 1. **4.81.** $\pi\sqrt{2}-4$. **4.82.** $\frac{1}{18}(5\pi\sqrt{3}-9\ln 3)$. **4.83.** $e-2$. **4.84.** $\frac{4}{25}(e^{3\pi/4}+1)$. **4.85.** $\sqrt{2}-\frac{2}{\sqrt{3}}+\ln\frac{2+\sqrt{3}}{1+\sqrt{2}}$.

4.86. $\frac{e^2+1}{4}$. 4.87. $\frac{\pi}{4}-\frac{1}{2}$. 4.88. $\frac{\pi^2-8}{32}$. 4.89. $\frac{1}{2}(e^{\pi/2}-1)$.

4.90. $I_{2k}=\frac{1\cdot 3\cdot 5\ \ldots\ (2k-1)}{2\cdot 4\cdot 6\ \ldots\ 2k}\cdot\frac{\pi}{2}$ $(n=2k)$;

$$I_{2k+1}=\frac{2\cdot 4\cdot 6\ \ldots\ 2k}{3\cdot 5\cdot 7\ \ldots\ (2k+1)}\ (n=2k+1);\ I_7=\frac{16}{35};\ I_8=\frac{35}{256}\pi.$$

4.91. $I_4=24-\frac{65}{e}$.

5.1. $\frac{1}{2}$. 5.2. Diverges. 5.3. $\frac{\pi}{\sqrt{2}}$. 5.4. $\frac{2}{5}$. 5.5. Diverges. 5.6. $1+\ln 2$. 5.7. $\frac{1}{3}$. 5.8. $\frac{1}{2}$. 5.9. Diverges. 5.10. Converges. 5.11. Diverges. 5.12. Converges. 5.13. Diverges. 5.14. Converges. 5.15. Converges. 5.16. Converges. 5.17. Diverges. 5.18. Diverges. 5.19. Converges. 5.20. Converges. 5.21. Diverges. 5.22. Diverges. 5.23. $\frac{5}{2}(\sqrt[5]{3}+1)$. 5.24. Diverges. 5.25. π. 5.26. $\frac{\pi}{3}$. 5.27. $\frac{16}{3}$. 5.28. $2\sqrt{2}$. 5.29. Diverges. 5.30. π. 5.31. Converges. 5.32. Diverges. 5.33. Converges. 5.34. Diverges. 5.35. Diverges. 5.36. Converges. 5.37. Converges. 5.38. Diverges. 5.39. Converges. 5.40. Diverges.

6.1. e^2. 6.2. πab. 6.3. 16/3. 6.4. 9/2. 6.5. $\frac{3}{2}(3\pi-2)$. 6.6 $\frac{56}{15}p^2$. 6.7. a^2. 6.8. $\frac{a^2}{4}(\pi-2\ln 2)$. 6.9. $\frac{c^3}{6p}$. 6.10. $2\ln 2-\frac{1}{2}$. 6.11. 32/3. 6.12. 1. 6.13. a^2. 6.14. $1.5-2\ln 2$. 6.15. $\frac{e}{2}-1$. 6.16. $4\ln 2-1$. 6.17. $\frac{\pi a^2}{2}-\frac{2}{3}a^2$ and $\frac{\pi a^2}{2}+\frac{2}{3}a^2$. 6.18. $a^2\left(\frac{2}{\sqrt{3}}+\ln(2+\sqrt{3})\right)$. 6.19. $r(a+h)-\frac{a^2r}{\sqrt{2ah+h^2}}\ln\frac{a+h+\sqrt{2ah+h^2}}{a}$. 6.20. $5\pi a^2$. 6.21. $\frac{a^2}{2}(3\sqrt{2}-2-\ln(1+\sqrt{2}))$. 6.22. $\frac{\pi a^2}{2}+a^2-\frac{a^2}{2\sqrt{3}}\ln(2+\sqrt{3})$ and $\frac{\pi a^2}{2}-a^2+\frac{a^2}{2\sqrt{3}}\ln(2+\sqrt{3})$. 6.23. $\frac{a^2r}{\sqrt{2ah-h^2}}\arccos\left(1-\frac{h}{a}\right)-(a-h)r$. 6.24. $\frac{a^2}{4}(\pi+2\ln 2)$. 6.25. πa^2. 6.26. $\frac{3}{8}\pi a^2$. 6.27. $\frac{8\sqrt{3}}{5}$. 6.28. 12π. 6.29. $\frac{24}{5}ab\sqrt{3}$. 6.30. $\frac{8}{15}$. 6.31. $\frac{3}{2}\pi a^2$. 6.32. $\frac{\pi a^2}{8}$. 6.33. $\frac{\pi a^2}{4}$. 6.34. $\frac{a^2}{4}\left(\pi+\frac{4}{3}\right)$.

6.35. $\frac{\pi a^2}{4}$. 6.36. $\frac{1}{4}(e^{4\pi}-1)^2$. 6.37. $\sqrt{3}-\frac{\pi}{3}$. 6.38. $\frac{\pi a^2}{4}$. 6.39. a^2. 6.40. $\frac{3}{4}\sqrt{3}$. 6.41. $\frac{1}{2}\sqrt{5}+\frac{1}{4}\ln(2+\sqrt{5})$. 6.42. $2\sqrt{3}$. 6.43. $2p(3\sqrt{3}-1)$. 6.44. $4a\ln(a+\sqrt{a^2-1})-2\sqrt{a^2-1}$. 6.45. $\pi a\sqrt{2}$. 6.46. $\pi a-2(a-b)\arctan\frac{a}{b}$. *Hint.* Pass to polar coordinates. 6.47. $\frac{1}{2}\cdot\sinh 6$. 6.48. $\frac{4}{\pi}\ln\tan\frac{3\pi}{8}=\frac{4}{\pi}\ln(1+\sqrt{2})$. 6.49. $\frac{134}{27}p$. 6.50. $6a$. 6.51. $\sqrt{2}(e-1)$. 6.52. $\frac{4\sqrt{3}}{9}$. 6.53. $\frac{13}{3}$. 6.54. $4a\sqrt{3}$. 6.55. $x=a\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$, $y=\frac{3}{2}a$. 6.56. $\frac{\sqrt{1+a^2}}{a}$. 6.57. $8(2-\sqrt{3})$. 6.58. $\frac{3}{2}\pi a$. *Hint.* $0\leqslant\varphi\leqslant 3\pi$. 6.59. $5\pi\sqrt{1+4\pi^2}+\frac{5}{2}\ln(2\pi+\sqrt{1+4\pi^2})$. 6.60. $\frac{16}{3}a$. 6.61. $2a\sqrt{6}$. 6.62. $a\sqrt{3}$. 6.63. 8. 6.64. $\frac{1}{3}a\sqrt{t}(3+2t)$. 6.65. $8\sqrt{2}$. 6.66. $\frac{\pi}{8}(\sinh 12+12)$. 6.67. (a) $8\pi+\frac{4\pi}{\sqrt{3}}\ln(2+\sqrt{3})$; (b) $2\pi+\frac{8\pi^2}{3\sqrt{3}}$. 6.68. $\frac{2\pi}{9}\times(2\sqrt{2}-1)$. 6.69. 48π. 6.70. $\frac{16}{15}\pi a^2(\sqrt{2}+1)$. 6.71. (a) $3\pi a^2$; (b) $\frac{56}{5}\pi a^2\sqrt{3}$. 6.72. $\pi\left(\sqrt{5}+4\ln\frac{1+\sqrt{5}}{2}\right)$. 6.73. (a) $9\pi^2a^2$; (b) $24\pi a^2$. 6.74. $3\pi a^2$. 6.75. $\frac{8}{3}\pi a^2(3\pi-4)$. 6.76. $6\pi^2a^2$. 6.77. $4\pi^2a^2$. 6.78. $\frac{96}{5}\pi a^2$. 6.80. $\frac{8}{3}\pi a^2(2\sqrt{2}-1)$. 6.81. $\frac{128}{105}a^3$. 6.82. $\frac{2}{3}a^3\tan\alpha$. 6.83. $\frac{272}{15}\pi$. 6.84. $\frac{11}{4}\pi$. 6.85. $\frac{\pi^3}{2}$. 6.86. $\frac{64}{3}\pi$. 6.87. $\frac{\pi a^2h}{2}$. 6.88. (a) $\frac{\pi a^3}{2}$; (b) $\frac{\pi a^3}{4}$. 6.89. $\frac{8}{15}\pi a^3$. 6.90. $\frac{3}{4}\pi^2a^3$. 6.91. $\frac{64}{105}\pi a^3$. 6.92. $\frac{\pi a^3}{12}(3\sqrt{2}\ln(\sqrt{2}+1)-2)$.

7.1. $\sqrt{2}+\ln(1+\sqrt{2})$. 7.2. $M_x=\frac{1}{2}(e\sqrt{1+e^2}-\sqrt{2}+\ln(\sqrt{2}-1)\times$

$\times(e+\sqrt{1+e^2}))$, $I_x=\frac{1}{3}((1+e^2)^{3/2}-\sqrt{8})$. **7.3.** $M_x=\frac{32}{3}a^2$, $I_x=\frac{256}{15}a^3$. **7.4.** $M_x=2a^2$, $I_x=\frac{\pi a^3}{2}$. **7.5.** $M_x=2a^2$, $M_y=\pi a^2$. **7.6.** $\bar{x}=$ $=\frac{a(\sinh 1-\cosh 1+1)}{\sinh 1}=a\left(1-\tanh\frac{1}{2}\right)$, $\bar{y}=\frac{a(2+\sinh 2)}{4\sinh 1}=$ $=\frac{a}{2}(\operatorname{csch} 1+\cosh 1)$. **7.7.** $\bar{x}=0$, $\bar{y}=\frac{2}{5}a$. **7.8.** $\bar{x}=\bar{y}=\frac{4}{5}a$. **7.9.** $\bar{x}=y=\frac{2}{5}a$. **7.10.** $\frac{v_0^2}{2g}$. **7.11.** $x=\frac{v_0}{\omega}\sin(\omega t+\varphi)$; $v_{av}=\frac{2}{\pi}v_0$. **7.12.** $t=6$ s, $s=144$ m. **7.13.** 250 m. **7.14.** 0.125 J. *Hint.* According to Hooke's law, the force is proportional to elongation of a spring. **7.15.** $\frac{1}{12}\gamma\pi R^2H^2$. **7.16.** $^1\ \pi R^2H^2$. **7.17.** $\frac{1}{12}\gamma a^2H^2$. **7.18.** $\frac{1}{4}\pi H^2R^2$. **7.19.** $\frac{16}{15}\sqrt{2}ap^3$. **7.20.** $e_0e\left(\frac{1}{a}-\frac{1}{b}\right)$; $\frac{e_0e}{a}$. *Hint.* According to Coulomb's law, the force of interaction of charges in vacuum is $F=\frac{e_0e}{x^2}$, where x is the distance between the charges. **7.21.** 2066 ln 2. *Hint.* In an isothermic process $pv=p_0v_0$. The work $A=\int\limits_{v_1}^{v_2}p\,dv$, where v_1 and v_2 are the initial and final values of the volume, respectively. **7.22.** $\frac{p_0v_0}{k-1}\left(\left(\frac{v_0}{v_1}\right)^{k-1}-1\right)$. *Hint.* In an adiabatic process $pv^k=$ $=p_0v_0^k$, where $k\approx 1.4$ (Poisson's law). The work $A=\int\limits_{v_0}^{v_1}\frac{p_0v_0^k}{v^k}\,dv$. **7.23.** $\frac{4}{15}\pi\gamma\omega^2R^5$. **7.24.** $\frac{1}{60}\omega^2\gamma\,dha^3$. **7.25.** $\frac{1}{24}\gamma alh^3\omega^2$. **7.26.** $\frac{1}{4}\pi\omega^2\gamma R^4H$. **7.27.** $\frac{ah^2}{3}$. **7.28.** $\gamma\pi R^2H$. **7.29.** $\frac{2}{3}\gamma ab^2$. **7.30.** $\gamma\pi RH^2$. **7.31.** 20.625 kg. **7.32.** $\frac{0.24I_0^2R\pi}{\omega}$. *Hint.* According to the Joule-Lenz law, the quantity of heat liberated by direct current during time t is $Q=0.24I^2Rt$. **7.33.** $\frac{S}{\mu S_0}\sqrt{\frac{2H}{g}}\approx 5.6$ min. *Hint.* According to Torricelli's law, the velocity of efflux of water through an orifice made at a distance x from its free surface is equal to $v=\mu\sqrt{2gx}$, where $\mu\approx 0.6$, g is free-fall acceleration.

7.34. $\frac{\pi p a^4}{8\mu l}$. *Hint.* $Q = \int\limits_0^a v \cdot 2\pi r \, dr = \frac{2\pi p}{4\mu l} \int\limits_0^a (a^2 - r^2)\, r\, dr =$

$= \frac{\pi p}{2\mu l} \left(-\frac{(a^2 - r^2)^2}{4} \right) \Big|_0^a = \frac{\pi p a^4}{8\mu l}$. **7.35.** $\frac{2kmM}{\pi R^2}$. *Hint.* Use the law of universal gravitation. **7.36.** $\frac{R^2}{3r^2} \sqrt{\frac{2H}{g}} \approx 11$ min.

7.37. $\frac{2}{3} \mu a h \sqrt{2gh}$.

8.1. 0.5236. **8.2.** 0.1963. **8.3.** 0.1178. **8.4.** 0.3926. **8.5.** 1.7500. **8.6.** 3.2413. **8.7.** 4.2218. **8.8.** 0.4969. **8.9.** 0.6082. **8.10.** 2.6291. **8.11.** 0.3927. **8.12.** 0.2500. **8.13.** 1.4627. **8.14.** 1.3419. **8.15.** 0.8120. **8.16.** 1.1184. **8.17.** 0.1728. **8.18.** 4.3555. **8.19.** 0.6205. **8.20.** 0.6076. **8.21.** 1.5708. **8.22.** 0.9160. **8.23.** 0.6651. **8.24.** 0.7721.

Chapter 7

DIFFERENTIAL CALCULUS: FUNCTIONS OF SEVERAL VARIABLES

SEC. 7.1. BASIC CONCEPTS

1. The Concept of a Function of Several Variables. Let us recall that any ordered set of n real numbers $x_1, \ldots, x_n$ is denoted $(x_1, \ldots, x_n)$ or $P(x_1, \ldots, x_n)$ and is called a point of the n-dimensional arithmetic space $\mathbb{R}^n$, the numbers $x_1, \ldots, x_n$ being termed the coordinates of the point $P = P(x_1, \ldots, x_n)$. The distance between the points $P(x_1, \ldots, x_n)$ and $P'(x_1', \ldots, x_n')$ is determined by the formula

$$\rho(P, P') = \sqrt{(x_1 - x_1')^2 + \ldots + (x_n - x_n')^2}.$$

Let $D \subset \mathbb{R}^n$ be an arbitrary set of points of the n-dimensional arithmetic space. If each point $P(x_1, \ldots, x_n) \in D$ is associated with some quite definite real number $f(P) = f(x_1, \ldots, x_n)$, then we say that a *numerical function* f: $\mathbb{R}^n \to \mathbb{R}$ of n variables $x_1, \ldots, x_n$ is defined on the set D. The latter is called the domain of definition, and the set $E = \{u \in \mathbb{R} \mid u = f(P), P \in D\}$ is termed the range of values of the function $u = f(P)$.

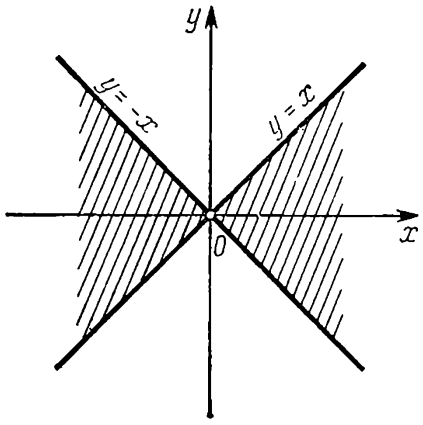

Fig. 68.

In a particular case, $n = 2$, the function of two variables $z = f(x, y)$ may be regarded as a function of points in the plane in a three-dimensional geometrical space with a fixed coordinate system $Oxyz$. The graph of this function is the set of points

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\},$$

representing, generally speaking, some surface in $\mathbb{R}^3$.

Example 1. Find the domain of definition of the function

$$z = \arcsin \frac{y}{x}.$$

The given function is defined for

$$-1 \leqslant \frac{y}{x} \leqslant 1, \quad x \neq 0.$$

Consequently, $-x \leqslant y \leqslant x$ for $x > 0$ and $x \leqslant y \leqslant -x$ for $x < 0$. The domain of definition of the function is shown in Fig. 68; it also contains the boundaries except for the origin.

Example 2. Let $f(x, y)=\dfrac{x^2-y^2}{xy}$. Find $f(3, -2)$, $f(y, x)$, $f\left(\dfrac{1}{x}, \dfrac{1}{y}\right)$.

We have:

$$f(3, -2)=\frac{3^2-(-2)^2}{3(-2)}=-\frac{5}{6}, \quad f(y, x)=\frac{y^2-x^2}{xy},$$

$$f\left(\frac{1}{x}, \frac{1}{y}\right)=\frac{\left(\frac{1}{x}\right)^2-\left(\frac{1}{y}\right)^2}{\frac{1}{x}\cdot\frac{1}{y}}=\frac{y^2-x^2}{xy}=-f(x, y).$$

1.1. Express the area S of a triangle as a function of the lengths of two of its sides x and y if its perimeter is equal to $2p$. Find the domain of definition of this function.

1.2. Express the volume V of a circular cone as a function of the area S of its lateral surface and the length l of its generatrix. Find the domain of definition of this function.

1.3. Express the area S of an equilateral trapezoid as a function of the lengths of its sides if x and y are the lengths of the bases, and z is the length of the lateral side. Find the domain of definition of this function.

In Problems 1.4 to 1.18 find the domain of definition of the given function of two variables (R is constant).

1.4. $z=\sqrt{R^2-x^2-y^2}$. **1.5.** $z=\sqrt{x^2+y^2-R^2}$.

1.6. $z=\dfrac{1}{\sqrt{R^2-x^2-y^2}}$. **1.7.** $z=\dfrac{1}{\sqrt{x^2+y^2-R^2}}$.

1.8. $z=(2x+3y-1)/(x-y)$.

1.9. $z=\sqrt{1-(x^2+y^2)}$. **1.10.** $z=\ln(-x-y)$.

1.11. $z=\sqrt{\dfrac{x^2+y^2-x}{2x-x^2-y^2}}$. **1.12.** $z=y\sqrt{\cos x}$.

1.13. $z=\sqrt{\log_a(x^2+y^2)}$. **1.14.** $z=\arccos\dfrac{x}{x+y}$.

1.15. $z=\sqrt{9-x^2-y^2}+\sqrt{x^2+y^2-4}$.

1.16. $z=\arcsin\dfrac{x}{y^2}+\arcsin(1-y)$.

1.17. $f(r, \varphi) = r\sqrt{\sin\varphi}$. **1.18.** $f(r, \varphi) = r\sqrt{\cos 2\varphi}$.

In Problems 1.19 to 1.21 find the domain of definition of the given function of three variables.

1.19. $u = \sqrt{x^2+y^2+z^2-R^2}$ (R is constant).

1.20. $u = \arcsin \dfrac{\sqrt{x^2+y^2}}{z}$.

1.21. $u = \ln(1 - x^2 - y^2 + z^2)$.

In Problems 1.22 and 1.23 find the domain of definition of the indicated function of n variables.

1.22. $u = \sqrt{1-x_1^2} + \sqrt{1-x_2^2} + \ldots + \sqrt{1-x_n^2}$.

1.23. $u = \sqrt{1 - \dfrac{x_1^2}{a_1^2} - \dfrac{x_2^2}{a_2^2} - \ldots - \dfrac{x_n^2}{a_n^2}}$.

1.24. Given the function $f(x, y) = \dfrac{2x-3y}{3x-2y}$. Find $f(2, 1)$, $f(1, 2)$, $f(3, 2)$, $f(2, 3)$, $f(a, a)$, $f(a, -a)$.

1.25. Given the function $f(x, y) = \dfrac{2xy}{x^2+y^2}$. Find $f(-3, 4)$ and $f\left(1, \dfrac{y}{x}\right)$.

1.26. Find $f(x)$, if $f\left(\dfrac{y}{x}\right) = \dfrac{\sqrt{x^2+y^2}}{x}$ $(x>0)$.

1.27. Let $z = x + y + f(x - y)$. Find the functions f and z, if $z = x^2$ for $y = 0$.

1.28.** Find $f(x, y)$, if $f\left(x+y, \dfrac{y}{x}\right) = x^2 - y^2$.

1.29. Given the functions: $f(x, y) = x^2 + y^2$, $\varphi(x, y) = x^2 - y^2$. Find (a) $f(\varphi(x, y), y^2)$; (b) $\varphi(f(x, y), \varphi(x, y))$.

1.30. Given the function $\varphi(x, y) = e^x \cos y$, $\psi(x, y) = e^x \sin y$. Prove that

(a) $\varphi^2(x, y) - \psi^2(x, y) = \varphi(2x, 2y)$;

(b) $2\varphi(x, y)\psi(x, y) = \psi(2x, 2y)$.

1.31. Given the following functions: $f(x, y) = x^2 - y^2$, $\varphi(x) = \cos x$, $\psi(x) = \sin x$. Find: (a) $f(\varphi(x), \psi(x))$; (b) $\varphi(f(x, y))$.

2. Limit and Continuity of a Function. The number A is said to be the *limit* of the function $u = f(P)$ as the point $P(x_1, x_2, \ldots, x_n)$ tends to the point $P_0(a_1, a_2, \ldots, a_n)$ if for any $\varepsilon > 0$ there exists

$\delta > 0$ such that the condition

$$0 < \rho(P, P_0) = \sqrt{(x_1 - a_1)^2 + \ldots + (x_n < a_n)^2}\ \delta$$

implies

$$| f(x_1, x_2, \ldots, x_n) - A | < \varepsilon.$$

And we write:

$$A = \lim_{P \to P_0} f(P) = \lim_{\substack{x_1 \to a_1 \\ x_2 \to a_2 \\ \cdots\cdots \\ x_n \to a_n}} f(x_1, x_2, \ldots, x_n).$$

Example 3. Find out whether the function $\dfrac{x^2 - y^2}{x^2 + y^2}$ has a limit as $x \to 0$, $y \to 0$.

Let the point $P(x, y)$ tend to the point $P_0(0, 0)$. Considering the change of x and y along the straight line $y = kx$, we obtain

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2 - k^2 x^2}{x^2 + k^2 x^2} = \lim_{x \to 0} \frac{1 - k^2}{1 + k^2} = \frac{1 - k^2}{1 + k^2}.$$

The result has different values depending on k and, therefore, the function has no limit.

A function $u = f(P)$ is said to be *continuous at a point* P_0 if the following three conditions are fulfilled:

(1) the function $f(P)$ is defined at the point P_0;

(2) there exists the limit $\lim\limits_{P \to P_0} f(P)$;

(3) $\lim\limits_{P \to P_0} f(P) = f(P_0)$.

A function is said to be *continuous in a domain* if it is continuous at each point of this domain. If at least one of the conditions (1)-(3) is violated at the point P_0, then P_0 is called a point of discontinuity of the function $f(P)$. Points of discontinuity can be isolated, form lines of discontinuity, surfaces of discontinuity, and so on.

Example 4. Find the points of discontinuity of the function

$$u = \frac{1 - xyz}{2x + 3y - z + 4}.$$

The given function is not defined at the points where the denominator vanishes. Therefore it has a surface of discontinuity—the plane $2x + 3y - z + 4 = 0$.

In Problems 1.32 to 1.36 find the indicated limits.

1.32. $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{xy}{3 - \sqrt{xy + 9}}$. **1.33.** $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{\sin xy}{xy}$.

1.34. $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{\sin xy}{y}$. **1.35.** $\lim\limits_{\substack{x \to 0 \\ y \to 0}} (1 + x^2 + y^2)^{\frac{1}{x^2 + y^2}}$.

1.36. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}$.

1.37. Show that the function $z=\dfrac{x}{y-x}$ can tend to any limit as $x\to 0$ and $y\to 0$. Give examples of such approach of the point (x, y) to the point $(0, 0)$ for which $\lim z=3$, $\lim z=2$, $\lim z=1$, $\lim z=-2$.

1.38. Show that there is no $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y)$ for the function $f(x, y)=\dfrac{x-y}{x+y}$ by computing the iterated limits

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x, y)\right),\quad \lim_{y\to 0}\left(\lim_{x\to 0} f(x, y)\right).$$

1.39. Show that for the function $f(x, y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2}$ the iterated limits

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x, y)\right)=\lim_{y\to 0}\left(\lim_{x\to 0} f(x, y)\right)=0$$

exist and are equal to each other, nevertheless $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y)$ is not existent.

1.40. Find out whether the function $\sin\ln(x^4+y^2)$ has a limit as $x\to 0$, $y\to 0$.

1.41. Find out whether the function $\dfrac{x^2+y^4}{x^4+y^2}$ has a limit as $x\to\infty$, $y\to\infty$.

1.42*. Show that at the point $(0, 0)$ the function

$$f(x, y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & \text{if } x^2+y^2\neq 0,\\ 0, & \text{if } x=y=0\end{cases}$$

is continuous along each ray $x=t\cos\alpha$, $y=t\sin\alpha$ $(0\leqslant t<+\infty)$, passing through this point, i.e. $\lim\limits_{t\to 0} f(t\cos\alpha, t\sin\alpha)=f(0, 0)$, but this function is not continuous at the point $(0, 0)$.

1.43. Show that at the point $(0, 0)$ the following functions are continuous with respect to either of the variables x and y,

but are discontinuous with respect to both variables simultaneously:

$$\text{(a) } f(x, y)=\begin{cases}\dfrac{xy}{(x^2+y^2)^2}, & \text{if } x^2+y^2\neq 0,\\ 0, & \text{if } x=y=0,\end{cases}$$

$$\text{(b) } f(x, y)=\begin{cases}\dfrac{x-y}{(x+y)^3}, & \text{if } x^2+y^2\neq 0,\\ 0, & \text{if } x=y=0.\end{cases}$$

In Prolems 1.44 to 1.49 find the points of discontinuity of the given functions of two variables.

1.44. $z=\dfrac{1}{(x-1)^2+(y+1)^2}$. **1.45.** $z=\dfrac{1}{\sin^2 \pi x+\sin^2 \pi y}$.

1.46. $z=\dfrac{1}{\sin x \sin y}$. **1.47.** $z=\ln (1-x^2-y^2)$.

1.48. $z=\dfrac{x^2+y^2}{(x+y)\,(y^2-x)}$.

1.49. $z=\dfrac{1}{(x^2+y^2-1)\,(x^2-y^2-1)}$.

In Problems 1.50 to 1.54 find the points of discontinuity of the indicated functions of three variables.

1.50. $u=\dfrac{1}{xyz}$. **1.51.** $u=\dfrac{1}{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1}$.

1.52. $u=\dfrac{1}{x^2+y^2-z^2}$. **1.53.** $u=\dfrac{1}{x^2+y^2-z^2-1}$.

1.54. $u=\dfrac{1}{x^2+y^2-z^2+1}$.

3. Partial Derivatives. Let $(x_1, \ldots, x_k, \ldots, x_n)$ be an arbitrary fixed point from the domain of definition of the function $u=f(x_1, \ldots, x_n)$. Assigning an increment Δx_k to the value of the variable x_k $(k=1, 2, \ldots, n)$, consider the limit

$$\lim_{\Delta x_k \to 0} \frac{f(x_1, \ldots, x_k+\Delta x_k, \ldots, x_n)-f(x_1, \ldots, x_k, \ldots, x_n)}{\Delta x_k}.$$

This limit is called the *partial derivative* (of the *first order*) of the given function with respect to the variable x_k at a point $(x_1, \ldots, x_n)$ and is denoted by the symbol $\dfrac{\partial u}{\partial x_k}$ or $f'_{x_k}(x_1, \ldots, x_n)$.

Partial derivatives are computed by usual differentiation rules and formulas (all the variables, except x_k, being regarded as constants).

Example 5. Find the partial derivatives of the function

$$z = \operatorname{arc\,tan} \frac{y}{x}.$$

Regarding y as a constant, we obtain

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}.$$

Regarding x as a constant, we get

$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}.$$

The function $u = f(x_1, x_2, \ldots, x_n)$ is said to be a *homogeneous* function of degree m if for any real number $t \neq 0$ the following equality holds true

$$f(tx_1, tx_2, \ldots, tx_n) = t^m \cdot f(x_1, x_2, \ldots, x_n).$$

If a homogeneous function $u = f(x_1, x_2, \ldots, x_n)$ of degree m has partial derivatives with respect to each of the variables, then the following relation (*Euler's theorem*) is fulfilled:

$$x_1 f'_{x_1}(x_1, x_2, \ldots, x_n) + x_2 f'_{x_2}(x_1, x_2, \ldots, x_n) + \ldots$$
$$\ldots + x_n f'_{x_n}(x_1, x_2, \ldots, x_n) = mf(x_1, x_2, \ldots, x_n).$$

Example 6. Verify Euler's theorem if

$$f(x, y) = Ax^2 + 2Bxy + Cy^2.$$

We have

$$f(tx, ty) = A(tx)^2 + 2B(tx)(ty) + C(ty)^2 = t^2 f(x, y).$$

Consequently, $m = 2$;

$$f'_x(x, y) = 2(Ax + By), \quad f'_y(x, y) = 2(Bx + Cy),$$

$$xf'_x(x, y) + yf'_y(x, y) \equiv 2x(Ax + By) + 2y(Bx + Cy) = 2f(x, y).$$

Partial derivatives of the second order of a function $u = f(x_1, x_2, \ldots, x_n)$ are defined as partial derivatives of its partial derivatives of the first order. Second-order derivatives are denoted as follows:

$$\frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_k}\right) = \frac{\partial^2 u}{\partial x_k^2} = f''_{x_k x_k}(x_1, x_2, \ldots, x_k, \ldots, x_n),$$

$$\frac{\partial}{\partial x_l}\left(\frac{\partial u}{\partial x_k}\right) = \frac{\partial^2 u}{\partial x_k \partial x_l} = f''_{x_k x_l}(x_1, x_2, \ldots, x_k, \ldots, x_l, \ldots, x_n)$$

and so on.

Partial derivatives of higher orders (the third, fourth, etc.) are defined and denoted in a similar way.

The result of repeated differentiation of a function with respect to different variables does not depend on the order of differentiation, provided that the "mixed" partial derivatives thus obtained are continuous.

Example 7. Find the partial derivatives of the second order of the function $z = \arctan \frac{y}{x}$.

We have (see Example 5)

$$\frac{\partial z}{\partial x} = -\frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}.$$

Differentiating for the second time, we obtain:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2},$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = -\frac{1\cdot(x^2+y^2)-2y\cdot y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2)-2x\cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\left(\text{here we have got convinced that } \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}\right),$$

$$\frac{\partial^2 x}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2}.$$

In Problems 1.55 to 1.65 find the partial derivatives of the first and second orders of the given functions.

1.55. $z = x^5 + y^5 - 5x^3y^3$. **1.56.** $z = xy + \frac{y}{x}$.

1.57. $z = \frac{xy}{\sqrt{x^2+y^2}}$. **1.58.** $z = xe^{-xy}$.

1.59. $z = \frac{\cos y^2}{x}$. **1.60.** $z = y^x$.

1.61. $z = \ln(x^2+y^2)$. **1.62.** $z = \arcsin \frac{y}{\sqrt{x^2+y^2}}$.

1.63. $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$. **1.64.** $u = \left(\frac{y}{x}\right)^z$.

1.65. $u = xy^2z^3t^4 + 3x - 4y + 2z - t + 1$.

1.66. Find $f'_x(3, 2)$, $f'_y(3, 2)$, $f''_{xx}(3, 2)$, $f''_{xy}(3, 2)$, $f''_{yy}(3, 2)$ if $f(x, y) = x^3y + xy^2 - 2x + 3y - 1$.

1.67. Find $f'_x(1, 2)$, $f'_y(1, 2)$, $f''_{xx}(1, 2,)$, $f''_{xy}(1, 2)$, $f''_{yy}(1, 2)$ if $f(x, y) = \int\limits_x^{x^2+y^2} e^t\, dt$.

1.68. Show that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$ if $z = x \sin(ax + by)$.

1.69. Show that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$ if $z = \cos \frac{y}{x} \arccos \frac{x}{y}$.

1.70. Find $f'''_{xxx}(0, 1)$, $f'''_{xxy}(0, 1)$, $f'''_{xyy}(0, 1)$, $f'''_{yyy}(0, 1)$ if $f(x, y) = e^{x^2 y}$.

1.71. Find $\frac{\partial^4 u}{\partial x\, \partial y\, \partial \xi\, \partial \eta}$ if $u = \ln \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}$.

1.72. Find $\frac{\partial^2 u}{\partial x^3\, \partial y^3}$ if $u = x^3 \sin y + y^3 \sin x$.

1.73. Find $\frac{\partial^{p+q} u}{\partial x^p\, \partial y^q}$ if $u = (x - x_0)^p (y - y_0)^q$.

In Problems 1.74 to 1.77 verify Euler's theorem on homogeneous functions.

1.74. $z = x^3 + x^2 y - y^3$. **1.75.** $z = \frac{y}{x^3 - y^3}$.

1.76. $z = \arctan \frac{y}{x}$. **1.77.** $u = \frac{x + y + z}{\sqrt[3]{x^2 + y^2 + z^2}}$.

1.78. Compute

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

if $x = r \cos\theta \cos\varphi$, $y = r \cos\theta \sin\varphi$, $z = r \sin\theta$.

1.79. Show that $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial y} + x + z = 0$ if $z = 4e^{-2y} + (2x + 4y - 3)\, e^{-y} - x - 1$.

1.80. Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$ if $u = \ln(x^3 + y^3 + z^3 - 3xyz)$.

1.81. Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = 0$ if $u = \frac{x - y}{z - t} + \frac{t - x}{y - z}$.

1.82. Show that the function $u = A \sin \lambda x \cos a\lambda t$ satisfies the *equation of string oscillation*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

1.83. Show that the function $u = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-x_0)^2}{4a^2 t}}$ satisfies the *heat equation*

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

1.84. Show that the function $u = \frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$ satisfies *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

1.85*. Show that the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if} \quad x^2+y^2 \neq 0, \\ 0 & \text{if} \quad x = y = 0 \end{cases}$$

has the partial derivatives $f'_x(x, y)$ and $f'_y(x, y)$ at the point (0, 0), although it is discontinuous at this point.

1.86*. Show that for the function

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if} \quad x^2+y^2 \neq 0, \\ 0 & \text{if} \quad x = y = 0 \end{cases}$$

the value of the second mixed derivative at the point (0, 0) depends on the order of differentiation, namely, $f''_{xy}(0, 0) = -1$, $f''_{yx}(0, 0) = 1$.

4. Differential of a Function and Its Application. The *total increment* of a function $u = f(x_1, x_2, \ldots, x_n)$ at a point $P(x_1, x_2, \ldots, x_n)$, corresponding to the increments $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, is defined as the difference

$$\Delta u = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) - f(x_1, x_2, \ldots, x_n).$$

The function $u = f(P)$ is said to be *differentiable* at the point $(x_1, x_2, \ldots, x_n)$ if everywhere in some neighbourhood of this point the total increment of the function can be represented in the form

$$\Delta u = A_1 \Delta x_1 + A_2 \Delta x_2 + \ldots + A_n \Delta x_n + o(\rho),$$

where $\rho = \sqrt{\Delta x_1^2 + \Delta x_2^2 + \ldots + \Delta x_n^2}$, A_1, A_2, ..., A_n are numbers independent of Δx_1, Δx_2, ..., Δx_n.

The *differential du of the first order* of a function $u = f(x_1, x_2, \ldots, x_n)$ at a point $(x_1, x_2, \ldots, x_n)$ is defined as the principal part of the total increment of this function at the point under consideration, linear with respect to Δx_1, Δx_2, ..., Δx_n, that is,

$$du = A_1 \Delta x_1 + A_2 \Delta x_2 + \ldots + A_n \Delta x_n.$$

By definition, the differentials of independent variables are regarded to be equal to their increments:

$$dx_1 = \Delta x_1, \quad dx_2 = \Delta x_2, \ldots, \quad dx_n = \Delta x_n.$$

For the differential of a function $u = f(x_1, x_2, \ldots, x_n)$ the following formula holds true:

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_n} dx_n. \tag{1}$$

Functions u, v of several variables obey the ordinary differentiation rules:

$$d(u + v) = du + dv,$$
$$d(uv) = v\,du + u\,dv,$$
$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

Example 8. Find the total increment and the differential of the function $f(x, y) = x^2 y$ at the point (x, y).

$$f(x + \Delta x, y + \Delta y) = (x + \Delta x)^2 (y + \Delta y),$$
$$\Delta f(x, y) = (x + \Delta x)^2 (y + \Delta y) - x^2 y$$
$$= 2xy \cdot \Delta x + x^2 \Delta y + 2x \Delta x \Delta y + y \Delta x^2 + \Delta x^2 \Delta y,$$
$$df(x, y) = 2xy \Delta x + x^2 \Delta y.$$

Example 9. Find the differential of the function

$$f(x, y, z) = \frac{z}{\sqrt{x^2 + y^2}}.$$

First method. We have

$$\frac{\partial f}{\partial x} = -\frac{xz}{(x^2 + y^2)^{3/2}}, \quad \frac{\partial f}{\partial y} = -\frac{yz}{(x^2 + y^2)^{3/2}},$$
$$\frac{\partial f}{\partial z} = \frac{1}{\sqrt{x^2 + y^2}}.$$

By formula (1), we obtain

$$df(x, y, z) = -\frac{xz}{(x^2 + y^2)^{3/2}} dx - \frac{yz}{(x^2 + y^2)^{3/2}} dy + \frac{1}{\sqrt{x^2 + y^2}} dz$$
$$= \frac{(x^2 + y^2)\,dz - z(x\,dx + y\,dy)}{(x^2 + y^2)^{3/2}}.$$

Second method. Applying the differentiation rules, we have:

$$df(x, y, z) = \frac{\sqrt{x^2+y^2}\,dz - z \cdot d\sqrt{x^2+y^2}}{x^2+y^2}$$

$$= \frac{\sqrt{x^2+y^2}\,dz - z \cdot \dfrac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{(x^2+y^2)\,dz - z\,(x\,dx+y\,dy)}{(x^2+y^2)^{3/2}}.$$

For a sufficiently small $\rho = \sqrt{\Delta x_1^2 + \Delta x_2^2 + \ldots + \Delta x_n^2}$ the following approximate equalities take place for a differentiable function $u = f(x_1, x_2, \ldots, x_n)$:

$$\Delta u \approx du,$$

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) \approx f(x_1, x_2, \ldots, x_n) + df(x_1, x_2, \ldots, x_n).$$

Example 10. Compute approximately

$$\sqrt{(4.05)^2+(3.07)^2}.$$

The desired number will be regarded as the value of the function $f(x, y) = \sqrt{x^2 + y^2}$ for $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, if $x_0 = 4$, $y_0 = 3$, $\Delta x = 0.05$, $\Delta y = 0.07$. We have:

$$f(4, 3) = \sqrt{4^2+3^2} = 5,$$

$$\Delta f(x, y) \approx df(x, y) = \frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}},$$

$$\Delta f(4, 3) \approx \frac{4 \cdot 0.05 + 3 \cdot 0.07}{5} \approx 0.08.$$

Consequently,

$$\sqrt{(4.05)^2 + (3.07)^2} \approx 5 + 0.08 = 5.08.$$

The differential of the second order d^2u of a function $u = f(x_1, x_2, \ldots, x_n)$ is defined as the differential of its differential of the first order regarded as a function of the variables $x_1, x_2, \ldots, x_n$ with the values of $dx_1, dx_2, \ldots, dx_n$ fixed:

$$d^2u = d(du).$$

The differential of the third order is defined in a similar way:

$$d^3u = d(d^2u).$$

In general,

$$d^mu = d(d^{m-1}u).$$

The differential of order m of a function $u = f(x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n$ are independent variables, is expressed by the

symbolic formula

$$d^m u = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \ldots + \frac{\partial}{\partial x_n} dx_n \right)^m u, \qquad (2)$$

which is formally expanded according to the binomial law.

For instance, in the case of a function $z = f(x, y)$ of two independent variables x and y the following formulas are valid for the differentials of the second and third orders, respectively:

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \, \partial y} dx \, dy + \frac{\partial^2 z}{\partial y^2} dy^2, \qquad (3)$$

$$d^3 z = \frac{\partial^3 z}{\partial x^3} dx^3 + 3 \frac{\partial^3 z}{\partial x^2 \, \partial y} dx^2 \, dy + 3 \frac{\partial^3 z}{\partial x \, \partial y^2} dx \, dy^2 + \frac{\partial^3 z}{\partial y^3} dy^3. \qquad (4)$$

Example 11. Find $d^2 z$ if $z = e^{xy}$.

By the differential rules, we have

$$dz = e^{xy} \cdot d(xy) = e^{xy} (y \, dx + x \, dy).$$

We differentiate for the second time, taking into consideration that dx and dy are independent of x and y (i.e. regarding dx and dy as constants):

$$d^2 z = e^{xy} \, d(xy) \cdot (y \, dx + x \, dy) + e^{xy} \cdot d(y \, dx + x \, dy)$$
$$= e^{xy} (y \, dx + x \, dy)^2 + e^{xy} 2 \, dx \, dy = e^{xy} ((y \, dx + x \, dy)^2 + 2 \, dx \, dy).$$

1.87. Find the total increment and the differential of the function $z = x^2 - xy + y^2$ if x varies from 2 to 2.1, and y from 1 to 1.2.

1.88. Find the total increment and the differential of the function $z = \log (x^2 + y^2)$ if x changes from 2 to 2.1, and y from 1 to 0.9.

In Problems 1.89 to 1.94 find the differential of the given function.

1.89. $z = \ln \left(y + \sqrt{x^2 + y^2} \right)$. **1.90.** $z = \tan \frac{y^2}{x}$.

1.91. $z = \ln \cos \frac{x}{y}$. **1.92.** $u = (xy)^z$.

1.93. $f(x_1, x_2, x_3, x_4) = x_1^{x_2 - x_3} \cdot \ln x_4$.

1.94. Find $df(1, 2, 1)$ if $f(x, y, z) = \frac{z}{x^2 + y^2}$.

In Problems 1.95 to 1.97 carry out approximate calculations.

1.95. $(2.01)^{3.03}$. **1.96.** $\sqrt{(1.02)^3 + (1.97)^3}$.

1.97. $\sin 28^\circ \cdot \cos 61^\circ$.

1.98. A cylindrical glass has the following inner dimensions: the radius of the base $R = 2.5$ m, height $H = 4$ m, and wall thickness $l = 1$ dm. Find approximately the volume of the material spent for the manufacture of the glass.

1.99. A rectangular parallelepiped has the following dimensions: $a = 2$ m, $b = 3$ m, $c = 6$ m. Find approximately the quantity by which the length of the diagonal of the parallelepiped will change if a is increased by 2 cm, b by 1 cm, and c is reduced by 3 cm.

1.100. In a frustum of a cone the radii of the bases are: $R = 20$ cm and $r = 10$ cm, the altitude $h = 30$ cm. What will the approximate change in the volume of the frustum be if R is increased by 2 mm, r by 3 mm, and h is decreased by 1 mm?

In Problems 1.101 to 1.109 find the differentials of the first and second orders of the given functions (x, y, z independent variables).

1.101. $z = x^3 + 3x^2y - y^3$. **1.102.** $z = \frac{y}{x} - \frac{x}{y}$.

1.103. $z = \sqrt{x^2 + 2xy}$. **1.104.** $z = \frac{y}{\sqrt{x^2 + y^2}}$.

1.105. $z = (x + y)\, e^{xy}$. **1.106.** $z = x \ln \frac{y}{x}$.

1.107. $z = \arctan \frac{x}{x + y}$.

1.108. $u = xy + yz + zx$. **1.109.** $u = e^{xyz}$.

1.110. Find d^3z if $z = e^y \sin x$.

1.111. Find d^3u if $u = x^3 + y^3 + z^3 - 3xyz$.

1.112. Find d^6u if $u = \ln (x + y + z)$.

1.113. Find $d^m u$ if $u = e^{ax + by + cz}$.

SEC. 7.2.

DIFFERENTIATION OF COMPOSITE AND IMPLICIT FUNCTIONS

1. Composite Functions of One and Several Independent Variables. If $z = f(x_1, x_2, \ldots, x_n)$ is a differentiable function of the variables $x_1, x_2, \ldots, x_n$ which are, in turn, differentiable functions of the independent variable t:

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \ldots, \quad x_n = \varphi_n(t),$$

then the derivative of the composite function $u = f(\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))$ is computed by the formula

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \cdot \frac{dx_2}{dt} + \ldots + \frac{\partial u}{\partial x_n} \cdot \frac{dx_n}{dt}. \tag{1}$$

In particular, if t coincides, say, with the variable x_1, then the "total" derivative of the function u with respect to x_1 is equal to

$$\frac{du}{dx_1} = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \cdot \frac{dx_2}{dx_1} + \ldots + \frac{\partial u}{\partial x_n} \cdot \frac{dx_n}{dx_1}. \tag{2}$$

Example 1. Find $\frac{du}{dt}$ if $u = xyz$, where

$$x = t^2 + 1, \quad y = \ln t, \quad z = \tan t.$$

By formula (1), we have

$$\frac{du}{dt} = yz \cdot 2t + xz \cdot \frac{1}{t} + xy \cdot \sec^2 t$$

$$= 2t \ln t \tan t + \frac{(t^2+1)\tan t}{t} + (t^2+1) \ln t \sec^2 t.$$

Example 2. Find $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$ if $z = y^x$, where $y = \varphi(x)$.

We have $\frac{\partial z}{\partial x} = y^x \ln y$. By formula (2), we obtain

$$\frac{dz}{dx} = y^x \ln y + xy^{x-1} \cdot \varphi'(x).$$

Let $u = f(x_1, x_2, \ldots, x_n)$, where $x_1 = \varphi_1(t_1, t_2, \ldots, t_m)$, $x_2 = \varphi_2(t_1, t_2, \ldots, t_m)$, $\ldots$, $x_n = \varphi_n(t_1, t_2, \ldots, t_m)$ ($t_1, t_2, \ldots, t_m$ independent variables). The partial derivatives of the function u with respect to $t_1, t_2, \ldots, t_m$ are expressed in the following way:

$$\begin{aligned}
\frac{\partial u}{\partial t_1} &= \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_1} + \ldots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_1}, \\
\frac{\partial u}{\partial t_2} &= \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_2} + \ldots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_2}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{\partial u}{\partial t_m} &= \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_m} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_m} + \ldots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_m}.
\end{aligned} \tag{3}$$

As is seen, expression (1) from Sec. 7.1 for the differential of the first order retains its form (*the invariant property of the form of the first differential*)

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_n} dx_n.$$

The expressions for the differentials of higher orders of a composite function, generally speaking, differ from the expressions of form (2) from the preceding section.

For instance, the differential of the second order is expressed by the formula

$$d^2u = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_n}dx_n\right)^2 u + \frac{\partial u}{\partial x_1}d^2x_1 + \frac{\partial u}{\partial x_2}d^2x_2 + \ldots + \frac{\partial u}{\partial x_n}d^2x_n. \quad (4)$$

Example 3. Find dz and d^2z if $z = f(u, v)$, where $u = \frac{1}{2}(x^2 - y^2)$, $v = xy$.

We have $dz = f'_u\,du + f'_v\,dv$, where

$$du = x\,dx - y\,dy,\quad dv = y\,dx + x\,dy.$$

Consequently,

$$dz = f'_u(x\,dx - y\,dy) + f'_v(y\,dx + x\,dy) = (xf'_u + yf'_v)\,dx + (xf'_v + yf'_u)\,dy.$$

Differentiating for the second time, we get

$$d^2z = d(f'_u)\cdot du + f'_u\cdot d(du) + d(f'_v)\,dv + f'_v\cdot d(dv)$$
$$= (f''_{uu}\,du + f''_{uv}\cdot dv)\,du + f'_u\cdot d^2u + (f''_{uv}\,du + f''_{vv}\,dv)\,dv + f'_v\cdot d^2v,$$

where

$$d^2u = dx^2 - dy^2,\quad d^2v = 2\,dx\,dy.$$

Consequently,

$$d^2z = (f''_{uu}(x\,dx - y\,dy) + f''_{uv}(y\,dx + x\,dy))(x\,dx - y\,dy) + f'_u(dx^2 - dy^2) + (f''_{uv}(x\,dx - y\,dy) + f''_{vv}(y\,dx + x\,dy))$$
$$\times(y\,dx + x\,dy) + f'_v\cdot 2\,dx\,dy = f''_{uu}(x\,dx - y\,dy)^2 + f''_{uv}(y\,dx + x\,dy)(x\,dx - y\,dy) + f'_u(dx^2 - dy^2)$$
$$+ f''_{uv}(x\,dx - y\,dy)(y\,dx + x\,dy) + f''_{vv}(y\,dx + x\,dy)^2 + 2f'_v\,dx\,dy = f''_{uu}(x^2\,dx^2 - 2xy\,dx\,dy + y^2\,dy^2)$$
$$+ 2f''_{uv}(xy(dx^2 - dy^2) + (x^2 - y^2)\,dx\,dy) + f''_{vv}(y^2\,dx^2 + 2xy\,dx\,dy + x^2\,dy^2) + f'_u(dx^2 - dy^2) + 2f'_v\,dx\,dy$$
$$= (x^2f''_{uu} + 2xyf''_{uv} + y^2f''_{vv} + f'_u)\,dx^2 + 2(xyf''_{vv} + (x^2 - y^2)f''_{uv} - xyf''_{uu} + f'_v)\,dx\,dy + (y^2f''_{uu} - 2xyf''_{uv} + x^2f''_{vv} - f'_u)\,dy^2.$$

2.1. Find $\frac{dz}{dt}$ if $z = e^{2x-3y}$, where $x = \tan t$, $y = t^2 - t$.

2.2. Find $\frac{dz}{dt}$ if $z = x^y$, where $x = \ln t$, $y = \sin t$.

2.3. Find $\frac{dz}{dt}$ if $z = \arctan \frac{y}{x}$, where $x = e^{2t} + 1$, $y = e^{2t} - 1$.

2.4. Find $\frac{du}{dt}$ if $u = \frac{yz}{x}$, where $x = e^t$, $y = \ln t$, $z = t^2 - 1$.

2.5. Find $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$ if $z = \ln(e^x + e^y)$, where $y = \frac{1}{3}x^3 + x$.

2.6. Find $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$ if $z = \arctan \frac{x+1}{y}$, where $y = e^{(x+1)^2}$.

2.7. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = u^2 \ln v$, where $u = \frac{y}{x}$, $v = x^2 + y^2$.

2.8. Find dz if $z = u^2 v - v^2 u$, where $u = x \sin y$, $v = y \cos x$.

2.9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = f(u, v)$, where $u = \frac{2y}{x+y}$, $v = x^2 - 3y$.

2.10. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = f(u, v)$, where $u = \ln(x^2 - y^2)$, $v = xy^2$.

2.11. Find dz if $z = f(u, v)$, where $u = \cos(xy)$, $v = x^5 - 7y$.

2.12. Find dz if $z = f(u, v)$ where $u = \sin \frac{x}{y}$, $v = \sqrt{x/y}$.

2.13. Find du if $u = f(x, y, z)$, where $x = s^2 + t^2$, $y = s^2 - t^2$, $z = 2st$.

2.14. Find $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ if $u = f(x_1, x_2, x_3, x_4)$, where $x_3 = g(x_1, x_2)$, $x_4 = h(x_1, x_2, x_3)$.

2.15. Show that the function $z = y \cdot \varphi(\cos(x - y))$ satisfies the equation $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{z}{y}$.

2.16. Show that the function $z = xf\left(\frac{y}{x}\right) - x^2 - y^2$ satisfies the equation $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - x^2 - y^2$.

2.17. Show that the function $z = \frac{y}{f(x^2 - y^2)}$ satisfies the equation $\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{z}{y^2}$.

2.18. Show that the function $u = \frac{1}{12}x^4 - \frac{1}{6}x^3(y+z) + \frac{1}{2}x^2yz + f(y-x, z-x)$ satisfies the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz.$$

2.19. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}$, $\frac{\partial^2 z}{\partial y^2}$ if $z = f(u, v)$, where $u = xy$, $v = x/y$.

2.20. Find $\frac{\partial^2 u}{\partial x \partial y}$ if $u = f(x, y, z)$, where $z = \varphi(x, y)$.

2.21. Find all the partial derivatives of the second order of the function $u = f(x, xy, xyz)$.

2.22. Show that the function $u = x\varphi(x+y) + y\psi(x+y)$ satisfies the equation $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$.

2.23. Show that the function $u = \varphi(xy) + \psi\left(\frac{x}{y}\right)$ satisfies the equation $x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$.

2.24. Find d^2u if $u = f(t)$, where $t = x^2 + y^2 + z^2$.

2.25. Find d^2u if $u = f(ax, by, cz)$.

2.26. Find d^2z if $z = f(u, v)$, where $u = x \sin y$, $v = y \cos x$.

2. Implicit Functions of One and Several Independent Variables. Let the equation $f(x, y) = 0$, where f is a differentiable function of the variables x and y, define y as a function of x. The first derivative of this implicit function $y = y(x)$ at a point x_0 is expressed by the formula

$$\left.\frac{dy}{dx}\right|_{x=x_0} = -\frac{f'_x(x_0, y_0)}{f'_y(x_0, y_0)} \tag{5}$$

on the condition that $f'_y(x_0, y_0) \neq 0$, where $y_0 = y(x_0)$, $f(x_0, y_0) = 0$.

The derivatives of higher orders are computed by consecutive differentiation of formula (5).

Example 4. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if

$$1 + xy - \ln(e^{xy} + e^{-xy}) = 0.$$

Let us denote the left-hand member of this equation by $f(x, y)$. Then

$$f'_x(x, y) = y - \frac{ye^{xy} - ye^{-xy}}{e^{xy} + e^{-xy}} = \frac{2ye^{-xy}}{e^{xy} + e^{-xy}}.$$

$$f'_y(x, y) = x - \frac{xe^{xy} - xe^{-xy}}{e^{xy} + e^{-xy}} = \frac{2xe^{-xy}}{e^{xy} + e^{-xy}}.$$

By formula (5), we obtain

$$\frac{dy}{dx} = -\frac{2ye^{-xy}}{2xe^{-xy}} = -\frac{y}{x}.$$

We differentiate for the second time, taking into account that y is a function of x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{y}{x}\right) = -\frac{x\dfrac{dy}{dx} - y}{x^2} = \frac{y - x\left(-\dfrac{y}{x}\right)}{x^2} = \frac{2y}{x^2}.$$

Let the equation $F(x_1, x_2, \ldots, x_n, u) = 0$, where F is a differentiable function of the variables $x_1, x_2, \ldots, x_n, u$ define u as a function of independent variables $x_1, x_2, \ldots, x_n$. The partial derivatives of this implicit function $u = u(x_1, x_2, \ldots, x_n)$ at a point $M^0(x_1^0, x_2^0, \ldots, x_n^0)$ are computed by the formulas

$$\left.\frac{du}{\partial x_k}\right|_{M=M^0} = -\frac{F'_{x_k}(x_1^0, x_2^0, \ldots, x_n^0, u^0)}{F'_u(x_1^0, x_2^0, \ldots, x_n^0, u^0)}$$

$$(k = 1, 2, \ldots, n) \qquad (6)$$

provided that $F'_u(x_1^0, x_2^0, \ldots, x_n^0, u^0) \neq 0$, where $u^0 = u(M^0)$ and $F(M^0, u^0) = 0$.

The partial derivatives of the function u can also be found in the following way: we compute the total differential of the function $F(x_1, x_2, \ldots, x_n, u)$, equate it to zero:

$$\frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \ldots + \frac{\partial F}{\partial x_n}dx_n + \frac{\partial F}{\partial u}du = 0$$

and hence express du.

Example 5. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if

$$x^3 + 2y^3 + z^3 - 3xyz - 2y + 3 = 0.$$

First method. Let us denote the left-hand side of the given equation by $F(x, y, z)$. Then

$$F'_x(x, y, z) = 3x^2 - 3yz,$$
$$F'_y(x, y, z) = 6y^2 - 3xz - 2,$$
$$F'_z(x, y, z) = 3z^2 - 3xy.$$

By formula (6), we obtain:

$$\frac{\partial z}{\partial x}=-\frac{F'_x(x, y, z)}{F'_z(x, y, z)}=-\frac{3x^2-3yz}{3z^2-3xy}=\frac{x^2-yz}{xy-z^2},$$

$$\frac{\partial z}{\partial y}=-\frac{F'_y(x, y, z)}{F'_z(x, y, z)}=-\frac{6y^2-3xz-2}{3z^2-3xy}=\frac{6y^2-3xz-2}{3(xy-z^2)}.$$

Second method. Differentiating the given equation:

$$3x^2\,dx+6y^2\,dy+3z^2\,dz-3yz\,dx-3xz\,dy-3xy\,dz-2dy=0,$$

we express dz:

$$dz=\frac{3(x^2-yz)\,dx+(6y^2-3xz-2)\,dy}{3(xy-z^2)}.$$

Comparing this expression with the formula $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$, we get

$$\frac{\partial z}{\partial x}=\frac{x^2-yz}{xy-z^2},\quad \frac{\partial z}{\partial y}=\frac{6y^2-3xz-2}{3(xy-z^2)}.$$

2.27. Find $\frac{dy}{dx}$ if $x^2e^{2y}-y^2e^{2x}=0$.

2.28. Find $\frac{dy}{dx}$ if $y\sin x-\cos(x-y)=0$.

2.29. Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ if $x+y=e^{x-y}$.

2.30. Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ if $x-y+\arctan y=0$.

2.31. Find $\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}}$, $\left.\frac{d^2y}{dx^2}\right|_{\substack{x=1\\y=1}}$, $\left.\frac{d^3y}{dx^3}\right|_{\substack{x=1\\y=1}}$ if $x^2+2xy+y^2-4x+2y-2=0$.

2.32. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(1, -2, 2)$ if $z^3-4xz+y^2-4=0$.

2.33. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z\ln(x+z)-\frac{xy}{z}=0$.

2.34. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x+y+z,\ x^2+y^2+z^2)=0$.

2.35. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $f(yz, e^{xz})=0$.

2.36. Find dz if $yz = \arctan(xz)$.

2.37. Find dz if $xz - e^{z/y} + x^3 + y^3 = 0$.

2.38. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x\,\partial y}$ if $x^2 - 2y^2 + z^2 - 4x + 2z - 5 = 0$.

2.39. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}$, $\frac{\partial^2 z}{\partial y^2}$ if $x + y + z = e^z$.

2.40. Find d^2z if $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

2.41. Show that the function z defined by the equation $\varphi(cx - az, cy - bz) = 0$, where φ is an arbitrary differentiable function of two variables, satisfies the equation

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c.$$

2.42. Show that the function z defined by the equation $(x - a\cos\alpha)^2 + (y - a\sin\alpha)^2 = \left(\frac{z-a}{m}\right)^2$, where a, α, m are constants, satisfies the equation

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = m^2.$$

2.43. Show that the function z defined by the equation $y = x\varphi(z) + \psi(z)$ satisfies the equation

$$\frac{\partial^2 z}{\partial x^2}\left(\frac{\partial z}{\partial y}\right)^2 - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial z}{\partial x}\right)^2 = 0.$$

3. Systems of Implicit and Parametrically Represented Functions. We shall confine ourselves here to consideration of functions of two independent variables. Let the system of two equations

$$\begin{aligned} F(x, y, u, v) &= 0, \\ G(x, y, u, v) &= 0 \end{aligned} \tag{7}$$

have the solution $x = x_0$, $y = y_0$, $u = u_0$, and $v = v_0$ and the functions F and G have continuous partial derivatives of the first order in a neighbourhood of the point $P_0(x_0, y_0, u_0, v_0)$, and the Jacobian

$$\frac{D(F, G)}{D(u, v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$$

is nonzero at the point P_0. Then in some neighbourhood of the point P_0 system (7) defines a unique system of continuous functions $u(x, y)$

and $v(x, y)$ having continuous partial derivatives and satisfying the conditions

$$u(x_0, y_0) = u_0, \quad v(x_0, y_0) = v_0.$$

The differentials of these functions du and dv (and, hence, their partial derivatives) can be found from the system of equations

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial v}dv = 0,$$

$$\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial u}du + \frac{\partial G}{\partial v}dv = 0.$$

Example 6. The functions u and v of independent variables x and y are represented implicitly by the system of equations

$$u + v = x, \quad u - yv = 0.$$

Find du, dv, d^2u, d^2v.

The Jacobian of the system $\dfrac{D(F, G)}{D(u, v)} = \begin{vmatrix} 1 & 1 \\ 1 & -y \end{vmatrix} = -y - 1$ is nonzero for $y \neq -1$. Differentiating, we find two equations relating the differentials of all the four variables:

$$du + dv = dx, \quad du - y\,dv - v\,dy = 0.$$

Solving this system with respect to du and dv for $y \neq -1$, we obtain

$$du = \frac{y\,dx + v\,dy}{1+y}, \quad dv = \frac{dx - v\,dy}{1+y}.$$

Repeated differentiation yields:

$$d^2u = \frac{(dx\,dy + dv\,dy)(1+y) - dy\,(y\,dx + v\,dy)}{(1+y)^2}$$

$$= \frac{\left(dx\,dy + \dfrac{dx - v\,dy}{1+y}\,dy\right)(1+y) - dy\,(y\,dx + v\,dy)}{(1+y)^2} =$$

$$= \frac{(1+y)\,dx\,dy + dx\,dy - v\,dy^2 - y\,dx\,dy - v\,dy^2}{(1+y)^2} = \frac{2\,(dx\,dy - v\,dy^2)}{(1+y)^2},$$

$$d^2v = \frac{-dv\,dy\,(1+y) - dy\,(dx - v\,dy)}{(1+y)^2}$$

$$= \frac{-\dfrac{dx - v\,dy}{1+y}\,dy\,(1+y) - dx\,dy + v\,dy^2}{(1+y)^2}$$

$$= \frac{-dx\,dy + v\,dy^2 - dx\,dy + v\,dy^2}{(1+y)^2} = \frac{2\,(v\,dy^2 - dx\,dy)}{(1+y)^2} = -d^2u.$$

Let a function z of independent variables x and y be represented parametrically by the equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

and

$$\frac{D(x, y)}{D(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

in the neighbourhood of a point $P(u_0, v_0)$. Then the differential dz of this function (and, hence, its partial derivatives) in the neighbourhood of the point P can be found from the system of equations

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv,$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv,$$

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv.$$

Example 7. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x = u \cos v, \quad y = u \sin v, \quad z = cv.$$

We have

$$\frac{D(x, y)}{D(u, v)} = \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} = u \neq 0 \text{ for } u \neq 0.$$

Differentiating, we find three equations relating the differentials of all the five variables

$$dx = \cos v\, du - u \sin v\, dv, \; dy = \sin v\, du + u \cos v\, dv, \; dz = c\, dv.$$

From the first two equations we find dv:

$$dv = \frac{\cos v\, dy - \sin v\, dx}{u}.$$

Substitute the found value of dv into the third equation, we obtain:

$$dz = \frac{c}{u}(\cos v\, dy - \sin v\, dx).$$

Hence

$$\frac{\partial z}{\partial x} = -\frac{c \sin v}{u}, \quad \frac{\partial z}{\partial y} = \frac{c \cos v}{u}.$$

2.44. The functions y and z of an independent variable x are given by the system of equations

$$7x^2 + y^2 - 3z^2 = -1, \; 4x^2 + 2y^2 - 3z^2 = 0.$$

Find $\frac{dy}{dx}$, $\frac{dz}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{\partial^2 z}{\partial x^2}$ for $x = 1$, $y = -2$, $z = 2$.

2.45. The functions y and z of an independent variable x are given by the system of equations

$$x^2 + y^2 - z^2 = 0, \quad x^2 + 2y^2 + 3z^2 = 1.$$

Find dy, dz, d^2y, d^2z.

2.46. The functions u and v of independent variables x and y are represented implicitly by the system of equations

$$xu + yv = 1, \quad x + y + u + v = 0.$$

Find du, dv, d^2u, d^2v.

2.47. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ if $uv = 3x - 2y + z$, $v^2 = x^2 + y^2 + z^2$.

2.48. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x = u + v$, $y = u - v$, $z = u^2v^2$.

2.49. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x = a\cos u \cosh v$, $y = b\sin u \cosh v$, $z = c\sinh v$.

2.50. Find dz if $x = e^u \cos v$, $y = e^u \sin v$, $z = uv$.

2.51. Find dz, if $x = u + v$, $y = u^2 + v^2$, $z = u^3 + v^3$ $(u \neq v)$.

4. Change of Variables in Differential Expressions. In practice we often have to express the derivatives entering into various differential expressions in terms of derivatives with respect to new variables.

Example 8. Transform the equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0,$$

setting $x = \cos t$.

Let us express the derivatives of y with respect to x in terms of derivatives of y with respect to t:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{dy}{dt}}{-\sin t},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\sin t \cdot \frac{d^2y}{dt^2} + \cos t \cdot \frac{dy}{dt}}{\sin^2 t \cdot (-\sin t)}$$

$$= \frac{1}{\sin^2 t} \cdot \frac{d^2y}{dt^2} - \frac{\cos t}{\sin^3 t} \cdot \frac{dy}{dt}.$$

Substituting the obtained expressions into the given equation and replacing x by $\cos t$, we get

$$(1-\cos^2 t)\left(\frac{1}{\sin^2 t}\cdot\frac{d^2y}{dt^2}-\frac{\cos t}{\sin^3 t}\cdot\frac{dy}{dt}\right)$$
$$-\cos t\left(-\frac{1}{\sin t}\cdot\frac{dy}{dt}\right)=0,$$

or $\dfrac{d^2y}{dt^2}=0.$

Example 9. Transform the equation

$$\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2=0,$$

by taking x for the function and y for the argument.

Let us express the derivatives of y with respect to x in terms of derivatives of x with respect to y:

$$\frac{dy}{dx}=\frac{1}{\dfrac{dx}{dy}},\quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{\dfrac{dx}{dy}}\right)=\frac{d}{dy}\left(\frac{1}{\dfrac{dx}{dy}}\right)\cdot\frac{dy}{dx}$$
$$=-\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^2}\cdot\frac{1}{\dfrac{dx}{dy}}=-\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}.$$

Substituting these expressions into the given equation we obtain:

$$-\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}+2y\cdot\frac{1}{\left(\dfrac{dx}{dy}\right)^2}=0,$$

or

$$\frac{d^2x}{dy^2}-2y\frac{dx}{dy}=0.$$

Example 10. Pass to polar coordinates in the expression

$$A=\frac{x+yy'}{xy'-y}.$$

Wa have

$$x=r\cos\varphi,\quad y=r\sin\varphi,$$
$$dx=\cos\varphi\,dr-r\sin\varphi\,d\varphi,\quad dy=\sin\varphi\,dr+r\cos\varphi\,d\varphi,$$

whence

$$y'=\frac{dy}{dx}=\frac{\sin\varphi\,dr+r\cos\varphi\,d\varphi}{\cos\varphi\,dr-r\sin\varphi\,d\varphi}.$$

Substituting the expressions for x, y, y' into A, we get

$$A=\frac{r\cos\varphi+r\sin\varphi\cdot\dfrac{\sin\varphi\,dr+r\cos\varphi\,d\varphi}{\cos\varphi\,dr-r\sin\varphi\,d\varphi}}{r\cos\varphi\cdot\dfrac{\sin\varphi\,dr+r\cos\varphi\,d\varphi}{\cos\varphi\,dr-r\sin\varphi\,d\varphi}-r\sin\varphi}=\frac{\dfrac{dr}{d\varphi}}{r}.$$

Example 11. Transform the equation

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=0,$$

by passing to new independent variables u and v if $u=xy$, $v=\dfrac{x}{y}$.

Expressing the partial derivatives of z with respect to x and y in terms of partial derivatives of z with respect to u and v, we have

$$\frac{\partial u}{\partial x}=y,\quad \frac{\partial v}{\partial x}=\frac{1}{y},\quad \frac{\partial u}{\partial y}=x,\quad \frac{\partial v}{\partial y}=-\frac{x}{y^2}.$$

By formula (3), we obtain

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}\cdot y+\frac{\partial z}{\partial v}\cdot\frac{1}{y},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial v}{\partial x}$$

$$=\left(\frac{\partial^2 z}{\partial u^2}y+\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{1}{y}\right)y+\left(\frac{\partial^2 z}{\partial u\,\partial v}y+\frac{\partial^2 z}{\partial v^2}\cdot\frac{1}{y}\right)\cdot\frac{1}{y}$$

$$=y^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{y^2}\frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}x-\frac{\partial z}{\partial v}\frac{x}{y^2},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=x\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{1}{y^2}+\frac{\partial z}{\partial v}\cdot\frac{2}{y^3}\right)$$

$$=x\left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{\partial v}{\partial y}\right.$$

$$\left.-\left(\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y}\right)\frac{1}{y^2}+\frac{\partial z}{\partial v}\cdot\frac{2}{y^3}\right)$$

$$=x\left(\frac{\partial^2 z}{\partial u^2}x-\frac{\partial^2 z}{\partial u\,\partial v}\frac{x}{y^2}-\left(\frac{\partial^2 z}{\partial u\,\partial v}x-\frac{\partial^2 z}{\partial v^2}\cdot\frac{x}{y^2}\right)\frac{1}{y^2}+\frac{\partial z}{\partial v}\cdot\frac{2}{y^3}\right)$$

$$=x^2\frac{\partial^2 z}{\partial u^2}-\frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}+\frac{2x}{y^3}\frac{\partial z}{\partial v}.$$

We then substitute the found expressions for the derivatives into the given equation:

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right)$$
$$-y^2\left(x^2\frac{\partial^2 z}{\partial u^2}-\frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}+\frac{2x}{y^3}\frac{\partial z}{\partial v}\right)=0.$$

After simplifications, we get for $x\neq 0$ and $y\neq 0$

$$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{2xy}\frac{\partial z}{\partial v},\quad \text{or}\quad \frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{2u}\frac{\partial z}{\partial v}.$$

Example 12. Transform the equation $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=(y-x)\,z$, by taking the quantities $u=x^2+y^2$, $v=\frac{1}{x}+\frac{1}{y}$ for the new independent variables and $w=\ln z-(x+y)$ for the new function.

Let us express the partial derivatives of z with respect to x and y in terms of partial derivatives of w with respect to u and v. To this end, we differentiate the given relations:

$$du=2\,(x\,dx+y\,dy),$$
$$dv=-\left(\frac{dx}{x^2}+\frac{dy}{y^2}\right),$$
$$dw=\frac{dz}{z}-(dx+dy).$$

Taking into consideration formula (1) from the preceding section, we have

$$\frac{\partial w}{\partial u}\,du+\frac{\partial w}{\partial v}\,dv=\frac{dz}{z}-(dx+dy),$$

or

$$2\frac{\partial w}{\partial u}\,(x\,dx+y\,dy)-\frac{\partial w}{\partial v}\left(\frac{dx}{x^2}+\frac{dy}{y^2}\right)=\frac{dz}{z}-(dx+dy),$$

whence

$$dz=z\left(\left(2x\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}+1\right)dx+\left(2y\frac{\partial w}{\partial u}-\frac{1}{y^2}\frac{\partial w}{\partial v}+1\right)dy\right).$$

Consequently,

$$\frac{\partial z}{\partial x}=z\left(2x\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}+1\right),$$
$$\frac{\partial z}{\partial y}=z\left(2y\frac{\partial w}{\partial u}-\frac{1}{y^2}\frac{\partial w}{\partial v}+1\right).$$

Substituting these expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ into the given equation, we obtain

$$yz\left(2x\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}+1\right)-xz\left(2y\frac{\partial w}{\partial u}-\frac{1}{y^2}\frac{\partial w}{\partial v}+1\right)=(y-x)z,$$

or $\frac{\partial w}{\partial v}=0$.

2.52. Transform the equation

$$x^4\frac{d^2y}{dx^2}+2x^3\frac{dy}{dx}-y=0$$

setting $x=1/t$.

2.53. Transform the equation

$$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2}=0$$

putting $x=\tan t$.

2.54. Transform the equation

$$3\left(\frac{d^2y}{dx^2}\right)^2-\frac{dy}{dx}\frac{d^3y}{dx^3}-\frac{\partial^2y}{\partial x^2}\left(\frac{dy}{dx}\right)^2=0$$

taking y for the argument.

2.55. Transform the equation

$$(xy'-y)^2=2xy\,(1+y'^2)$$

by passing to polar coordinates.

2.56. Transform the expression $w=x\frac{\partial y}{\partial x}+y\frac{\partial y}{\partial x}$ by passing to polar coordinates.

2.57. Transform the equation

$$(x+y)\frac{\partial z}{\partial x}-(x-y)\frac{\partial z}{\partial y}=0$$

by passing to new independent variables u and v if $u=\ln\sqrt{x^2+y^2}$, $v=\arctan\frac{y}{x}$.

2.58. Transform the equation $x\frac{\partial^2z}{\partial x^2}+y\frac{\partial^2z}{\partial x\,dy}=0$ by passing to new independent variables u and v if $u=y$, $v=y/x$.

2.59. Transform the expression $w = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ by passing to polar coordinates.

2.60. Transform the expression

$$w = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

by passing to spherical coordinates ($r = \rho \sin \theta$, $\varphi = \varphi$, $z = \rho \cos \theta$).

2.61. Transform the equation

$$(xy + z)\frac{\partial z}{\partial x} + (1 - y^2)\frac{\partial z}{\partial y} = x + yz$$

by taking $u = yz - x$, $v = xz - y$ for the new independent variables and $w = xy - z$ for the new function.

2.62. Transform the equation

$$\frac{\partial z}{\partial y} + \frac{1}{2}y\frac{\partial^2 z}{\partial y^2} = \frac{1}{x}$$

by taking $u = \frac{x}{y}$, $v = x$ for the new independent variables and $w = xz - y$ for the new function.

2.63. Transform the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\, \partial y} + \frac{\partial z}{\partial x} = z$$

by taking $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$ for the new independent variables and $w = ze^y$ for the new function.

SEC. 7.3.
APPLICATIONS OF PARTIAL DERIVATIVES

1. Taylor's Formula. If a function $f(P)$ is differentiable $m + 1$ times in some neighbourhood $U(P_0)$ of the point $P_0(x_1^0, \ldots, x_n^0)$, then for any point $P(x_1, \ldots, x_n) \in U(P_0)$ there holds *Taylor's formula*

$$f(P) = f(P_0) + \frac{df(P_0, \Delta x_1, \ldots, \Delta x_n)}{1!} + \frac{d^2 f(P_0, \Delta x_1, \ldots, \Delta x_n)}{2!} + \ldots$$

$$+ \frac{d^m f(P_0, \Delta x_1, \ldots, \Delta x_n)}{m!} + \frac{d^{m+1} f(\widetilde{P}, \Delta x_1, \ldots, \Delta x_n)}{(m+1)!}, \qquad (1)$$

where $\Delta x_1 = x_1 - x_1^0, \ldots, \Delta x_n = x_n - x_n^0$, and $\widetilde{P}$ is some point of the indicated surface.

For instance, in the case of a function $f(x, y)$ of two variables x and y Taylor's formula is written in the expanded form as follows:

$$f(x, y) = f(x_0, y_0) + \frac{1}{1!}\left(f'_x(x_0, y_0)(x-x_0) + f'_y(x_0, y_0)(y-y_0)\right)$$
$$+ \frac{1}{2!}\left(f''_{xx}(x_0, y_0)(x-x_0)^2 + 2f''_{xy}(x_0, y_0)(x-x_0)(y-y_0)\right.$$
$$\left. + f''_{yy}(x_0, y_0)(y-y_0)^2\right) + \ldots + \frac{1}{m!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^m$$
$$\times f(x_0, y_0) + \frac{1}{(m+1)!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^{m+1}$$
$$\times f(x_0 + \theta(x-x_0),\ y_0 + \theta(y-y_0)). \qquad (2)$$

The last term in formula (2) (the *remainder*) can be written in a shorter form

$$o(\rho^m), \quad \text{where} \quad \rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

(*Peano's form*).

In a particular case, for $x_0 = y_0 = 0$, formula (2) is called *Maclaurin's formula*.

Example 1. Applying Taylor's formula, expand the function $f(x, y) = x^3 - 5x^2 - xy + y^2 + 10x + 5y - 4$ in the neighbourhood of the point $(2, -1)$.

We have $f(2, -1) = 2$. Let us compute in succession the partial derivatives of the given function and their values at the point $(2, -1)$:

$$f'_x(x, y) = 3x^2 - 10x - y + 10, \qquad f'_x(2, -1) = 3;$$
$$f'_y(x, y) = -x + 2y + 5, \qquad f'_y(2, -1) = 1;$$
$$f''_{xx}(x, y) = 6x - 10, \qquad f''_{xx}(2, -1) = 2;$$
$$f''_{xy}(x, y) = -1, \qquad f''_{xy}(2, -1) = -1;$$
$$f''_{yy}(x, y) = 2, \qquad f''_{yy}(2, -1) = 2;$$
$$f'''_{xxx}(x, y) = 6, \qquad f'''_{xxx}(2, -1) = 6.$$

All consecutive derivatives are identically equal to zero. By formula (2), we obtain the desired expansion

$$f(x, y) = 2 + 3(x-2) + (y+1) + (x-2)^2 - (x-2)(y+1) + (y+1)^2 + (x-2)^3.$$

Example 2. Applying Taylor's formula, expand the function $f(x, y) = y^x$ in the neighbourhood of the point $(1, 1)$ up to second-order terms inclusive.

We have $f(1, 1) = 1$. Let us compute the partial derivatives of the first and second orders for the given function and their values at the point $(1, 1)$:

$$f'_x(x, y) = y^x \ln y, \qquad f'_x(1, 1) = 0;$$
$$f'_y(x, y) = xy^{x-1}, \qquad f'_y(1, 1) = 1;$$
$$f''_{xx}(x, y) = y^x \ln^2 y, \qquad f''_{xx}(1, 1) = 0;$$
$$f''_{xy}(x, y) = y^{x-1}(x \ln y + 1), \qquad f''_{xy}(1, 1) = 1;$$
$$f''_{yy}(x, y) = x(x-1)y^{x-2}, \qquad f''_{yy}(1, 1) = 0.$$

By formula (2), we get

$$f(x, y) = 1 + (y - 1) + (x - 1)(y - 1) + o(\rho^2),$$

where $\rho = \sqrt{(x-1)^2 + (y-1)^2}$.

3.1. Expand $f(x + h, y + k)$ in whole positive powers of h and k if $f(x, y) = xy^2$.

3.2. Find the increment obtained by the function $f(x, y) = -x^2 + 2xy + 3y^2 - 6x - 2y - 4$ when passing from the values $x = -2$, $y = 1$ to the values $x_1 = -2 + h$, $y_1 = 1 + k$.

3.3. Applying Taylor's formula, expand the function $f(x, y) = x^3 - 2y^3 + 3xy$ in the neighbourhood of the point (2, 1).

3.4. Expand $f(x + h, y + k, z + l)$ in whole positive powers of h, k, l if $f(x, y, z) = x^2 + 2y^2 + 3z^2 + xy - 2yz + 3x - y - 4z + 1$.

3.5. Applying Taylor's formula, expand the function $f(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$ in the neighbourhood of the point (1, —1, 2).

3.6. Applying Maclaurin's formula, expand the function $f(x, y) = e^y \cos x$ up to the terms of the third order inclusive.

3.7. Applying Maclaurin's formula, expand the function $f(x, y) = \sin x \sinh y$ up to the terms of the fourth order inclusive.

3.8. Applying Taylor's formula, expand the function $f(x, y) = y/x$ in the neighbourhood of the point (1, 1) up to the terms of the third order inclusive.

3.9. Applying Taylor's formula, expand the function $f(x, y, z) = \ln(xy + z^2)$ in the neighbourhood of the point (1, 1, 0) up to the terms of the second order inclusive.

3.10. Applying Taylor's formula, expand the implicit function $z(x, y)$ defined by the equation $z^3 + 3yz - 4x = 0$ in the neighbourhood of the point (1, 1) up to the terms of the second order inclusive if $z(1, 1) = 1$.

2. Extremum of a Function. A function $u = f(P)$ has a *maximum* (*minimum*) at a point $P_0(x_1^0, \ldots, x_n^0)$ if there exists a neighbourhood of the point P_0 such that for all of its points $P(x_1, \ldots, x_n)$ different from the point P_0 the inequality $f(P_0) > f(P)$ ($f(P_0) < f(P)$). The generic term for the maximum and minimum is the *extremum* (or *extreme value*).

Necessary condition for an extremum. If a *differentiable* function $f(P)$ reaches an extremum at a point P_0, then at this point

$$f'_{x_k}(P_0) = 0 \text{ for all } k = 1, 2, \ldots, n, \tag{3}$$

or $df(P_0, \Delta x_1, \ldots, \Delta x_n) = 0$ identically with respect to $\Delta x_1, \ldots, \Delta x_n$.

Points at which condition (3) is fulfilled are called the *stationary* points of the function $u = f(P)$. Hence, if P_0 is a point of extremum of the function $u = f(P)$, then either P_0 is a stationary point, or the function is not differentiable at this point.

Sufficient conditions for an extremum. Let $P_0(x_1^0, \ldots, x_n^0)$ be a stationary point of the function $u = f(P)$, and let this function be twice differentiable in some neighbourhood of the point P_0 and all of its second partial derivatives be continuous at the point P_0. Then:

(1) if the second differential $d^2u(P_0, \Delta x_1, \ldots, \Delta x_n)$ as a function of $\Delta x_1, \ldots, \Delta x_n$ does not change its sign for all possible sets of values of $\Delta x_1, \ldots, \Delta x_n$, which are not all zero, then the function $u = f(P)$ has an extremum at the point P_0, namely, a maximum for $d^2u(P_0, \Delta x_1, \ldots, \Delta x_n) < 0$ and a minimum for $d^2u(P_0, \Delta x_1, \ldots, \Delta x_n) > 0$;

(2) if $d^2u(P_0, \Delta x_1, \ldots, \Delta x_n)$ is a function of $\Delta x_1, \ldots, \Delta x_n$ with alternating signs, i.e. it attains both negative and positive values, then the point P_0 is not a point of extremum of the function $u = f(P)$;

(3) if $d^2u(P_0, \Delta x_1, \ldots, \Delta x_n) \geqslant 0$ or $d^2u(P_0, \Delta x_1, \ldots, \Delta x_n) \leqslant 0$, and there exist such sets of values of $\Delta x_1, \ldots, \Delta x_n$, not all zero, for which the value of the second differential vanishes, then the function $u = f(P)$ may have an extremum at the point P_0, but it may have none (in this case additional investigations are needed).

For a particular case of a function of two variables the sufficient conditions for an extremum may be formulated in the following way. Let $P_0(x_0, y_0)$ be a stationary point of a function $z = f(x, y)$, this function being twice differentiable in some neighbourhood of the point P_0, and all of its second partial derivatives being continuous at the point P_0. Let us introduce the following notation:

$A = f''_{xx}(x_0, y_0)$, $B = f''_{xy}(x_0, y_0)$, $C = f''_{yy}(x_0, y_0)$, $D = AC - B^2$.

Then:

(1) if $D > 0$, then the function $z = f(x, y)$ has an extremum at the point $P_0(x_0, y_0)$, namely: a maximum for $A < 0$ ($C < 0$) and a minimum for $A > 0$ ($C > 0$);

(2) if $D < 0$, then there is no extremum at the point $P_0(x_0, y_0)$;

(3) if $D = 0$, then an additional investigation is required.

Example 3. Investigate the function

$$z = x^3 + y^3 - 3xy$$

for an extremum.

First of all we find partial derivatives of the first order and form a system of equations of form (3):

$$\frac{\partial z}{\partial x} = 3(x^2 - y) = 0, \qquad \frac{\partial z}{\partial y} = 3(y^2 - x) = 0,$$

or

$$x^2 - y = 0,$$
$$y^2 - x = 0.$$

Solving this system, we find two stationary points:

$$P_1\,(0,\ 0) \text{ and } P_2\,(1,\ 1).$$

Further, we find the partial derivatives of the second order:

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x\,\partial y} = -3; \quad \frac{\partial^2 z}{\partial y^2} = 6y.$$

We then write the discriminant $D = AC - B^2$ for each stationa point.

For the point P_1

$$A = \frac{\partial^2 z}{\partial x^2}\Big|_{P_1} = 0, \quad B = \frac{\partial^2 z}{\partial x\,\partial y}\Big|_{P_1} = -3,$$
$$C = \frac{\partial^2 z}{\partial y^2}\Big|_{P_1} = 0, \quad D = -9 < 0.$$

Hence, there is no extemum at the point P_1.

For the point P_2

$$A = \frac{\partial^2 z}{\partial x^2}\Big|_{P_2} = 6, \quad B = \frac{\partial^2 z}{\partial x\,\partial y}\Big|_{P_2} = -3, \quad C = \frac{\partial^2 z}{\partial y^2}\Big|_{P_2} = 6,$$
$$D = 36 - 9 > 0, \quad A > 0.$$

Consequently, at the point P_2 the function has a minimum:

$$z_{\min} = z\Big|_{\substack{x=1\\y\ 1}} = 1 + 1 - 3 = -1.$$

In Problems 3.11 to 3.19 find the extrema of the giv functions of two variables.

3.11. $z = x^2 + xy + y^2 - 3x - 6y.$

3.12. $z = xy^2\,(1 - x - y)\ (x > 0,\ y > 0).$

3.13. $z = 3x^2 - x^3 + 3y^2 + 4y.$

3.14. $z = xy + \frac{50}{x} + \frac{20}{y}\ (x > 0,\ y > 0).$

3.15. $z = x^2 + y^2 - 2\ln x - 18\ln y\ (x > 0,\ y > 0).$

3.16. $z = x^3 + 3xy^2 - 15x - 12y.$

3.17. $z = 2x^3 - xy^2 + 5x^2 + y^2.$

3.18. $z = (2x^2 + y^2)\,e^{-(x^2+y^2)}.$

3.19. $z = 2 - \sqrt[3]{x^2 + y^2}.$

In Problems 3.20 to 3.22 find the extrema of the indicat functions of three variables.

3.20. $u = x^2 + y^2 + z^2 - 4x + 6y - 2z.$

3.21. $u = xy^2 z^3 (1 - x - 2y - 3z)$ $(x > 0,\ y > 0,\ z > 0)$.

3.22. $u = x + \frac{y}{x} + \frac{z}{y} + \frac{2}{z}$.

In Problems 3.23 and 3.24 find the extrema of the functions z represented implicitly.

3.23*. $x^2 + y^2 + z^2 + 4x - 2y - 4z - 7 = 0$.

3.24. $2x^2 + 2y^2 + z^2 + 8yz - z + 8 = 0$.

3. Conditional Extremum. A function $u = f(P) = f(x_1, \ldots, x_n)$ has a *conditional maximum* (*conditional minimum*) at a point P_0 $(x_1^0, \ldots, x_n^0)$ if there exists such a neighbourhood of the point P_0 for all points P of which $(P \neq P_0)$, satisfying the *constraint equations*

$$\varphi_k (P) = \varphi_k (x_1, \ldots, x_n) = 0 \quad (k = 1, 2, \ldots, m;\ m < n),$$

the inequality $f(P_0) > f(P)$ $(f(P_0) < f(P))$ is fulfilled.

The problem of finding a conditional extremum is reduced to investigating *Lagrange's function*

$$L (x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f (x_1, \ldots, x_n) + \sum_{k=1}^{m} \lambda_k \varphi_k (x_1, \ldots, x_n);$$

for an ordinary (unconditional) extremum; λ_k $(k = 1, 2, \ldots, m)$ are called *Lagrange's multipliers.*

The necessary conditions for a conditional extremum are expressed by a system of $n + m$ equations

$$\begin{aligned} &\frac{\partial L (P)}{\partial x_i} = 0 \quad (i = 1, 2, \ldots, n), \\ &\varphi_k (P) = 0 \quad (k = 1, 2, \ldots, m), \end{aligned} \tag{4}$$

from which we can find the unknowns

$$x_1, \ldots, x_n, \quad \lambda_1, \ldots, \lambda_m,$$

where $x_1, \ldots, x_n$ are the coordinates at which a conditional extremum is possible.

The sufficient conditions for a conditional extremum are connected with a study of the sign of the second differential of Lagrange's function $d^2 L (x_1^0, \ldots, x_n^0, \lambda_1^0, \ldots, \lambda_m^0, dx_1, \ldots, dx_n)$ for each system of values $x_1^0, \ldots, x_n^0, \lambda_1^0, \ldots, \lambda_m^0$, obtained from (4), provided that $dx_1, dx_2, \ldots, dx_n$ satisfy the equations

$$\sum_{j=1}^{n} \frac{\partial \varphi_k (x_1^0, \ldots, x_n^0)}{\partial x_j} dx_j = 0 \quad (k = 1, 2, \ldots, m) \tag{5}$$

for $dx_1^2 + dx_2^2 + \ldots + dx_n^2 \neq 0$. Namely the function $f(P)$ has a conditional maximum at the point $P_0(x_1^0, \ldots, x_n^0)$ if for all possible values of $dx_1, \ldots, dx_n$, satisfying conditions (5), which are not all zero, the inequality $d^2L(x_1^0, \ldots, x_n^0, \lambda_1^0, \ldots, \lambda_m^0, dx_1, \ldots, dx_n) < 0$, is fulfilled, and a conditional minimum if under these conditions $d^2L(x_1^0, \ldots, x_n^0, \lambda_1^0, \ldots, \lambda_m^0, dx_1, \ldots, dx_n) > 0$.

In case of a function $z = f(x, y)$ with constraint $\varphi(x, y) = 0$ Lagrange's function has the form

$$L(x, y, \lambda) = f(x, y) + \lambda \cdot \varphi(x, y).$$

System (4) consists of three equations:

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \varphi(x, y) = 0.$$

Let $P_0(x_0, y_0)$, λ_0 be an arbitrary solution of this system and

$$\Delta = -\begin{vmatrix} 0 & \varphi'_x(P_0) & \varphi'_y(P_0) \\ \varphi'_x(P_0) & L''_{xx}(P_0, \lambda_0) & L''_{xy}(P_0, \lambda_0) \\ \varphi'_y(P_0) & L''_{xy}(P_0, \lambda_0) & L''_{yy}(P_0, \lambda_0) \end{vmatrix}.$$

If $\Delta < 0$, then the function $z = f(x, y)$ has a conditional maximum at the point $P_0(x_0, y_0)$; if $\Delta > 0$, then a conditional minimum.

Example 4. Find the conditional extremum of the function $z = x + 2y$ for $x^2 + y^2 = 5$.

Let us form Lagrange's function:

$$L(x, y, \lambda) = x + 2y + \lambda(x^2 + y^2 - 5).$$

We have

$$\frac{\partial L}{\partial x} = 1 + 2\lambda x, \quad \frac{\partial L}{\partial y} = 2 + 2\lambda y.$$

System of equations (4) takes the form

$$\begin{aligned} 1 + 2\lambda x &= 0, \\ 2 + 2\lambda y &= 0, \\ x^2 + y^2 &= 5. \end{aligned}$$

The system has two solutions: $x_1 = -1$, $y_1 = -2$, $\lambda_1 = \frac{1}{2}$; $x_2 = 1$, $y_2 = 2$, $\lambda_2 = -\frac{1}{2}$. Since $\frac{\partial^2 L}{\partial x^2} = 2\lambda$, $\frac{\partial^2 L}{\partial x\, \partial y} = 0$, $\frac{\partial^2 L}{\partial y^2} = 2\lambda$, we have

$$d^2L = 2\lambda(dx^2 + dy^2).$$

For $\lambda = \frac{1}{2}$ $d^2L > 0$. Therefore the function has a conditional minimum at the point $P_1(-1, -2)$ and $z_{\min} = -5$. For $\lambda = -\frac{1}{2}$

$d^2L < 0$. Therefore the function has a conditional maximum at the point P_2 (1, 2) and $z_{max} = 5$.

Or in a different way:

$$\varphi(x, y) = x^2 + y^2 - 5,$$

$$\varphi'_x = 2x, \quad \varphi'_y = 2y, \quad \varphi'_x(-1, -2) = -2, \quad \varphi'_y(-1, -2) = -4,$$

$$L''_{xx} = 1, \quad L''_{xy} = 0, \quad L''_{yy} = 1 \text{ for } \lambda = \frac{1}{2};$$

consequently,

$$\Delta = -\begin{vmatrix} 0 & -2 & -4 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 20 > 0,$$

i.e. the function has a conditional minimum at the point P_1 (−1, −2). Analogously, for the point P_2 (1, 2)

$$\Delta = -\begin{vmatrix} 0 & 2 & 4 \\ 2 & -1 & 0 \\ 4 & 0 & -1 \end{vmatrix} = -20 < 0,$$

i.e. P_2 (1, 2) is a point of conditional maximum.

In Problems 3.25 to 3.33 find the conditional extrema of the given functions.

3.25. $z = x^2 + y^2 - xy + x + y - 4$ for $x + y + 3 = 0$.

3.26. $z = \frac{1}{x} + \frac{1}{y}$ for $x + y = 2$.

3.27. $z = \frac{x - y - 4}{\sqrt{2}}$ for $x^2 + y^2 = 1$.

3.28. $z = xy^2$ for $x + 2y = 1$.

3.29. $z = 2x + y$ for $x^2 + y^2 = 1$.

3.30. $u = 2x + y - 2z$ for $x^2 + y^2 + z^2 = 36$.

3.31. $u = x^2 + y^2 + z^2$ for $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$.

3.32. $u = xy^2z^3$ for $x + 2y + 3z = 12$ $(x > 0, y > 0, z > 0)$.

3.33. $u = xyz$ for $x + y + z = 4$, $xy + yz + zx = 5$.

3.34*. Prove the inequality

$$\frac{x^3 + y^3 + z^3}{3} \geqslant \left(\frac{x + y + z}{3}\right)^3$$

if $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

4. The Greatest and the Least Values of a Function. If a function $f(P)$ is differentiable in a bounded closed domain, then it attains its greatest (least) value either at a stationary point, or at a boundary point of the domain.

Example 5. Find the greatest and the least values of the function $z = x^3 + y^3 - 3xy$ in the domain

$$0 \leqslant x \leqslant 2, \quad -1 \leqslant y \leqslant 2.$$

The given domain is a rectangle.

(1) Find the stationary points (see Example 3): $P_1(0, 0)$ and $P_2(1, 1)$. The respective values of the function at these points are $z_1 = 0$, and $z_2 = -1$.

(2) Investigate the function on the boundaries of the given domain.

(a) For $x = 0$ we have $z = y^3$. This function is increasing monotonically and at the end points of the interval $[-1, 2]$ takes on the following values: $z|_{y=-1} = -1$, $z|_{y=2} = 8$.

(b) For $x = 2$ we have $z = 8 + y^3 - 6y$. Find the values of this function at the stationary point and at the end points of the interval $[-1, 2]$. We have $z' = 3y^2 - 6$; $z' = 0$ for $y^2 = 2$, or, in the given domain, for $y = \sqrt{2}$; $z|_{y=\sqrt{2}} = 8 + 2\sqrt{2} - 6\sqrt{2} = 8 - 4\sqrt{2}$; $z|_{y=-1} = 13$; $z|_{y=2} = 4$.

(c) For $y = -1$ we have $z = x^3 - 1 + 3x$ and $z' = 3x^2 + 3 > 0$. The function increases monotonically from $z|_{x=0} = -1$ to $z_{x=2} = 13$.

(d) For $y = 2$ we have $z = x^3 + 8 - 6x$; $z' = 3x^2 - 6$; $z' = 0$ for $x = \sqrt{2}$; $z|_{x=\sqrt{2}} = 8 - 4\sqrt{2}$; $z|_{x=0} = 8$, $z|_{x=2} = 6$.

(3) Comparing all found values of the given function, we conclude that $z_{\text{greatest}} = 13$ at the point $(2, -1)$; $z_{\text{least}} = -1$ at the points $(1, 1)$ and $(0, -1)$.

Example 6. For what dimensions does an open rectangular bath of a given capacity V have the least surface area? Find this area.

The bath has the shape of a rectangular parallelepiped. Let its dimensions be equal to x, y, z. Since the volume $V = xyz$ is given, $z = \dfrac{V}{xy}$. The surface area of the bath is

$$S = S(x, y) = 2(xz + yz) + xy = 2(x + y)\frac{V}{xy} + xy = 2V\left(\frac{1}{y} + \frac{1}{x}\right) + xy.$$

The problem is reduced to finding the minimum of the function $S(x, y)$ with $x > 0$, $y > 0$ (according to the sense of the problem).

Solving the system of equations

$$S'_x(x, y) = -\frac{2V}{x^2} + y = 0,$$

$$S'_y(x, y) = -\frac{2V}{y^2} + x = 0,$$

we find the stationary point $x_0 = y_0 = \sqrt[3]{2V}$. Let us now check whether the sufficient conditions of a minimum are fulfilled:

$$S''_{xx}(x, y) = \frac{4V}{x^3}, \quad S''_{xy}(x, y) = 1, \qquad S''_{yy}(x, y) = \frac{4V}{y^3}.$$

Consequently,

$$A = S''_{xx}(\sqrt[3]{2V}, \sqrt[3]{2V}) = 2,\ B = S''_{xy}(\sqrt[3]{2V}, \sqrt[3]{2V}) = 1,$$
$$C = S''_{yy}(\sqrt[3]{2V}, \sqrt[3]{2V}) = 2,\quad D = AC - B^2 = 4 - 1 > 0,\ A > 0.$$

Hence, the function $S(x, y)$ has a minimum for $x = y = \sqrt[3]{2V}$; then

$$z = \frac{V}{\sqrt[3]{4V^2}} = \frac{\sqrt[3]{2V}}{2};$$

$$S_{\min} = 2V\left(\frac{1}{\sqrt[3]{2V}} + \frac{1}{\sqrt[3]{2V}}\right) + \sqrt[3]{4V^2} = 3\sqrt[3]{4V^2}.$$

3.35. Find the greatest value of the function $z = x - 2y + 5$ in the indicated domains:

(a) $x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant 1$;

(b) $x \leqslant 0,\ y \geqslant 0,\ y - x \leqslant 1$.

3.36. Find the greatest and the least values of the function $z = x^2 + y^2 - xy - x - y$ in the domain $x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant 3$.

3.37. Find the greatest and the least values of the function $z = xy$ in the domain $x^2 + y^2 \leqslant 1$.

3.38. Find the greatest and the least values of the function $z = xy^2$ in the domain $x^2 + y^2 \leqslant 1$.

3.39. Represent the positive number a in the form of the product of four positive factors so that the sum of their inverses is the least.

3.40. Among all rectangular parallelepipeds having the given sum of the lengths of its edges $12a$, find the parallelepiped with the greatest volume.

3.41. Find the rectangular parallelepiped with the length of its diagonal d having the greatest volume.

3.42. Inside a quadrangle find the point the sum of the squares of whose distances from the vertices of the quadrangle would be the least.

3.43. In a hemisphere of radius R inscribe a rectangular parallelepiped of the greatest possible volume.

3.44. In a right circular cone with radius of the base R and altitude H inscribe a rectangular parallelepiped of the greatest volume.

3.45. Among all triangles with base a and angle α at the vertex find the triangle with the greatest area.

3.46*. On the ellipse $x^2 + 9y^2 = 9$ find the points greatest and least distant from the straight line $4x + 9y = 16$.

3.47*. Given on the ellipse $x^2 + 4y^2 = 4$ are two points: $A\ (-\sqrt{3},\ 0.5)$ and $B\ (1,\ \sqrt{3}/2)$. On the same ellipse find a third point C such that the area of the triangle ABC is the greatest.

3.48. Determine the external dimensions of a closed box with a given wall thickness δ and capacity V so that the least amount of material would be spent for its manufacture.

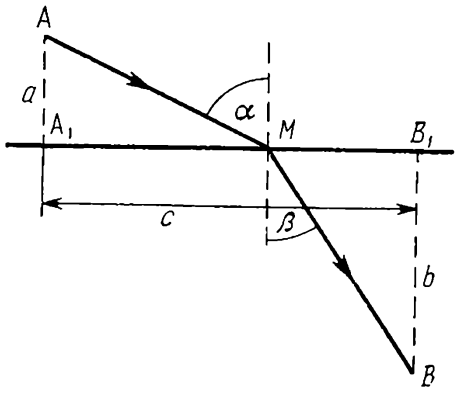

Fig. 69.

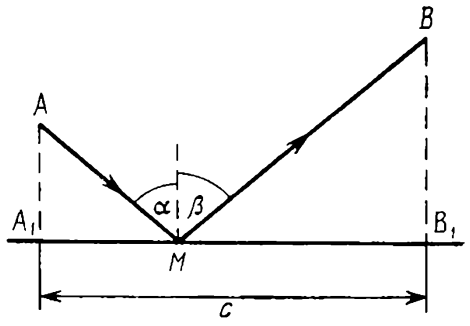

Fig. 70.

3.49. Given in the plane are n material points $P_1\ (x_1,\ y_1)$, $P_2\ (x_2,\ y_2)$, . . ., $P_n\ (x_n,\ y_n)$ with masses equal to m_1, m_2, ..., m_n, respectively. At what position of point $P\ (x,\ y)$ will the moment of inertia of the system about the point P be the least?

3.50*. Points A and B are situated in different optical media separated by the plane A_1B_1 (see Fig. 69). The velocity of light in the first medium is equal to v_1, in the second to v_2. Making use of Fermat's principle according to which a light ray travelling from one point to another follow a path (AMB) such that, compared with nearby paths, the time required is either a minimum or a maximum or will remain unchanged, derive the law of refraction of a light ray.

3.51. Applying Fermat's principle, derive the law of reflection of a light ray from a plane in a homogeneous medium (see Fig. 70).

3.52*. If in an electric circuit having resistance R there flows an electric current I, then the amount of heat liber-

ated per unit time is proportional to I^2R. Determine how should the current I be branched into currents $I_1, I_2, \ldots, I_n$ with the aid of n conductors whose resistances are $R_1, R_2, \ldots, R_n$, respectively, to ensure the least liberation of heat.

5. Geometrical Applications of Partial Derivatives. The *tangent plane* to a surface at its point M_0 (*point of tangency*) is the plane such that each line in the plane which passes through M_0 is tangent to the surface at M_0.

The *normal* to a surface is defined as a straight line perpendicular to the tangent plane and passing through the point of tangency.

If the equation of a surface has the form

$$F(x, y, z) = 0,$$

then the equation of the tangent plane at a point $M_0(x_0, y_0, z_0)$ is

$$F'_x(x_0, y_0, z_0)(x - x_0) + F'_y(x_0, y_0, z_0)(y - y_0) + F'_z(x_0, y_0, z_0)(z - z_0) = 0. \quad (5)$$

The equations of the normal are:

$$\frac{x - x_0}{F'_x(x_0, y_0, z_0)} = \frac{y - y_0}{F'_y(x_0, y_0, z_0)} = \frac{z - z_0}{F'_z(x_0, y_0, z_0)}. \quad (6)$$

If a surface is represented explicitly

$$z = f(x, y)$$

then the equation of the tangent plane at a point $M_0(x_0, y_0, z_0)$ has the form

$$z - z_0 = f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0),$$

and the equation of the normal:

$$\frac{x - x_0}{f'_x(x_0, y_0)} = \frac{y - y_0}{f'_y(x_0, y_0)} = \frac{z - z_0}{-1}.$$

Example 7. Find the equations of the tangent plane and the normal to the surface

$$x^2 + 2y^2 - 3z^2 + xy + yz - 2xz + 16 = 0$$

at the point M (1, 2, 3).

Designating the left-hand side of the given equation by $F(x, y, z)$, we find the partial derivatives and their values at the point M:

$$F'_x(x, y, z) = 2x + y - 2z, \qquad F'_x(1, 2, 3) = -2;$$
$$F'_y(x, y, z) = 4y + x + z, \qquad F'_y(1, 2, 3) = 12;$$
$$F'_z(x, y, z) = -6z + y - 2x, \qquad F'_z(1, 2, 3) = -18.$$

By formulas (5) and (6), we have

$$-2(x - 1) + 12(y - 2) - 18(z - 3) = 0,$$

or

$$x - 6y + 9z - 16 = 0$$

which is the equation of the tangent plane,

$$\frac{x-1}{-2} = \frac{y-2}{12} = \frac{z-3}{-18}, \quad \text{or} \quad \frac{x-1}{1} = \frac{y-2}{-6} = \frac{z-3}{9}$$

which are the equations of the normal.

Fig. 71.

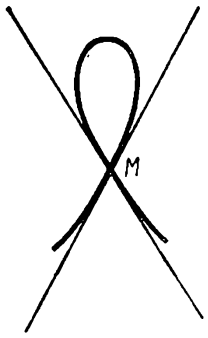

Fig. 72.

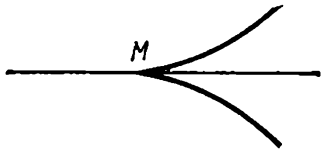

Fig. 73.

A *singular point* of a plane curve $f(x, y) = 0$ is defined as a point $M(x_0, y_0)$ whose coordinates satisfy the system of three equations:

$$f(x_0, y_0) = 0, \quad f'_x(x_0, y_0) = 0, \quad f'_y(x_0, y_0) = 0. \tag{7}$$

Let conditions (7) be fulfilled, the numbers

$$A = f''_{xx}(x_0, y_0), \quad B = f''_{xy}(x_0, y_0), \quad C = f''_{yy}(x_0, y_0)$$

be not all zero and $\Delta = AC - B^2$. Then:

(a) if $\Delta > 0$, then M is an *isolated point* (Fig. 71);

(b) if $\Delta < 0$, then M is a *node* (a *double point*) (Fig. 72);

(c) if $\Delta = 0$, then M is either a *cusp* (a *cuspidal point*) *of the first kind* (Fig. 73) or *of the second kind* (Fig. 74), or an *isolated point*, or a *point of osculation* (Fig. 75).

The slope $k = y'$ of the tangent to a curve at a singular point is found from the equation

$$A + 2Bk + Ck^2 = 0.$$

In the case of an isolated point there is no tangent at all, a node has two different tangents; there is one common tangent to two branches of a curve at a cusp or at a point of osculation.

If $\Delta = 0$, then to conclude on the type of a singular point, we have to study the arrangement of the points of a curve in some neighbourhood of this singular point.

In case of a transcendental curve we may also encounter other types of singular points: *corner points*, *end points*.

Example 8. Investigate the singular points of the conchoid

$$(x^2 + y^2)(x - a)^2 - b^2x^2 = 0 \quad (a > 0,\ b > 0).$$

Denoting the left-hand member of the given equation by $f(x, y)$, we find the partial derivatives and equate them to zero:

$$f'_x(x, y) = 2x(x - a)^2 + 2(x - a)(x^2 + y^2) - 2b^2x = 0,$$
$$f'_y(x, y) = 2y(x - a)^2 = 0.$$

The system of equations has a unique solution: $x_0 = y_0 = 0$, i.e. the curve has one singular point $O(0, 0)$.

Fig. 74.

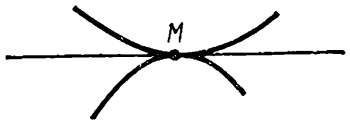

Fig. 75.

We then find the second derivatives:

$$f''_{xx}(x, y) = 2((x - a)^2 + 2x(x - a) + x^2 + y^2 + 2x(x - a) - b^2),$$
$$f''_{xy}(x, y) = 4y(x - a),$$
$$f''_{yy}(x, y) = 2(x - a)^2.$$

Evaluating them at the point O, we obtain

$$A = 2(a^2 - b^2),\quad B = 0,\quad C = 2a^2,\quad \Delta = AC - B^2 = 4a^2(a^2 - b^2).$$

If $a > b$, then $\Delta > 0$, and O is an isolated point (Fig. 76). If $a < b$, then $\Delta < 0$, and point O is a node (Fig. 77). If $a = b$, then $\Delta = 0$. We find the slope of the tangent:

$$2(a^2 - b^2) + 2a^2k^2 = 0,\quad k = \frac{b^2 - a^2}{a^2} = 0,$$

that is the tangent coincides with the x-axis.

From the equation of the curve we obtain (for $a < b$) $y = \pm \frac{x}{x - a} \times$ $\times \sqrt{2ax - x^2}$ and, consequently, the curve is symmetric about the x-axis ($0 \leqslant x < a$; $a < x \leqslant 2a$). Therefore for $a = b$ O is a cuspidal point of the first kind (Fig. 78).

The *envelope of a family of plane curves* is defined as a line (or a set of several lines) which touches all curves of a given family, each of its points being a point of tangency.

If a one-parameter family of curves $f(x, y, \alpha) = 0$ has an envelope, then its equation can be obtained from the system of equations

$$f(x, y, \alpha) = 0,\quad f'_\alpha(x, y, \alpha) = 0. \tag{8}$$

Eliminating the parameter α from system (8), we obtain an equation of the form $D(x, y) = 0$. The curve defined by this equation is called

a *discriminant curve*. This curve consists of an envelope and a set of singular points of a given family.

Example 9. The equation of the trajectory of a projectile launched from point O with initial velocity v_0 at an angle α to horizon (air resistance is neglected) is

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}.$$

Taking the angle α for a parameter, find the envelope of all trajectories of the projectile contained in one and the same vertical plane.

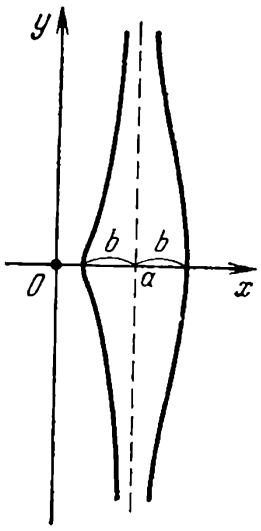

Fig. 76.

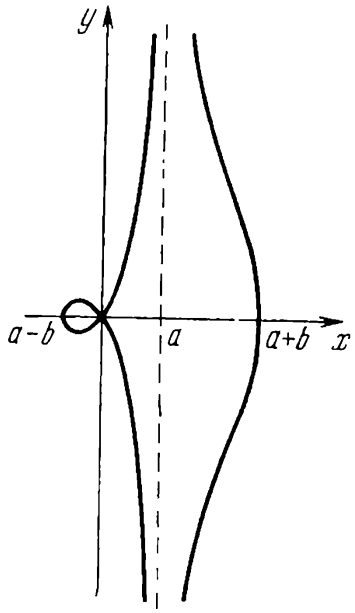

Fig. 77.

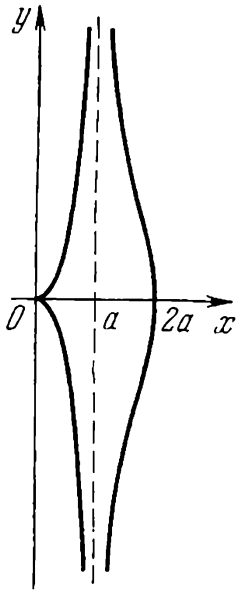

Fig. 78.

We have

$$f(x, y, \alpha) = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} - y,$$

$$f'_\alpha(x, y, \alpha) = \frac{x}{\cos^2 \alpha} - \frac{gx^2 \sin \alpha}{v_0^2 \cos^3 \alpha} = \frac{x}{\cos^2 \alpha}\left(1 - \frac{gx}{v_0^2} \tan \alpha\right).$$

Let us write a system of form (8)

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha},$$

$$\frac{x}{\cos^2 \alpha}\left(1 - \frac{gx}{v_0^2} \tan \alpha\right) = 0.$$

From the second equation we obtain: $\tan \alpha = \frac{v_0^2}{gx}$ and $\cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha} = \frac{g^2x^2}{g^2x^2 + v_0^4}$. Substituting into the first equation, we

find the equation of the envelope (*safety parabola*):

$$y = \frac{v_0^2}{g} - \frac{g^2x^2 + v_0^4}{2v_0^2g}, \text{ or } y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

3.53. Find the equations of the tangent plane and the normal to the following surfaces at the indicated points:

(a) $z = \sin x \cos y$ at the point $(\pi/4, \pi/4, 1/2)$;

(b) $z = e^{x \cos y}$ at the point $(1, \pi, 1/e)$.

3.54. Find the distance from the origin to the plane to tangent the surface $z = y \tan \frac{x}{a}$ at the point $(\frac{\pi a}{4}, a, a)$.

3.55. Find the angles formed by the normal to the surface $z = \text{arc tan } \frac{x}{y}$ at the point $\left(1, 1, \frac{\pi}{4}\right)$ with the coordinate axes.

3.56. For the surface $z = 4x - xy + y^2$ find the equation of the tangent plane parallel to the plane $4x - y + 2z + 9 = 0$.

3.57. Find the equations of the tangent plane and the normal to the following surfaces at the indicated points:

(a) $x(y + z)(xy - z) + 8 = 0$ at the point $(2, 1, 3)$;

(b) $2^{x/z} + 2^{y/z} - 8$ at the point $(2, 2, 1)$;

(c) $z^2 + 4z + x^2 = 0$ at the points of intersection with the z-axis.

3.58. For the surface $x^2 - z^2 - 2x + 6y = 4$ find the equations of the normal parallel to the straight line $\frac{x-2}{1} = \frac{y}{3} = \frac{z+1}{4}$.

3.59. On the surface $x^2 + 2y^2 + 3z^2 + 2xy + 2xz + 2yz = 8$ find the points at which the tangent planes are parallel to the coordinate planes.

3.60. Show that the tangent planes to the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ intercept on the coordinate axes segments the sum of the squares of which is constant and equals a^2.

3.61. Find the equations of the tangent plane and normal to the following surfaces represented parametrically at the indicated points:

(a) $x = r\cos\varphi$, $y = r\sin\varphi$, $z = r\cot\alpha$ at the point (r_0, φ_0);

(b) $x = u\cos v$, $y = u\sin v$, $z = av$ at the point (u_0, v_0).

3.62*. At what angle do the cylinder $x^2 + y^2 = a^2$ and hyperbolic paraboloid $bz = xy$ intersect at a common point (x_0, y_0, z_0)?

3.63*. Show that the following surfaces are pairwise orthogonal:

(a) $x^2 + y^2 + z^2 = 2ax$ and $x^2 + y^2 + z^2 = 2by$;

(b) $xyz = a^3$ and $2z^2 = x^2 + y^2 + f(x^2 - y^2)$;

(c) $xy = az^2$, $x^2 + y^2 + z^2 = b$ and $z^2 + 2x^2 = c(z^2 + 2y^2)$.

In Problems 3.64 to 3.74 investigate the given functions for singular points.

3.64. $x^2 + y^2 = x^4 + y^4$.

3.65. $y^2(a^2 + x^2) = x^2(a^2 - x^2)$. **3.66.** $x^2 + y^4 = x^6$.

3.67. $y^2 = (x - 1)^3$. **3.68.** $(y - 2x^2)^2 = x^5$.

3.69. $4y^2 = x^5 + 5x^4$. **3.70.** $y^2 = ax^2 + x^3$.

3.71. $y^2 = 1 - e^{-x^2}$. **3.72.** $y^2 = 1 - e^{-x^3}$.

3.73*. $y = \frac{x}{1 + e^{1/x}}$. **3.74*.** $y = x^x$.

3.75. Find the envelope of the family of straight lines $y = ax + a^2$.

3.76. Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = p$ (p constant, $p > 0$).

3.77. Find the envelope of the family of circles $x^2 + (y - C)^2 = R^2$ (R constant).

3.78. Find the envelope of the family of parabolas $y^2 = 2px + p^2$.

3.79. Find the envelope of the family of parabolas $y = 3a^2 + 2ax - x^2$.

3.80. Find the envelope of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{(l-a)^2} = 1$ (l constant).

3.81. Find the envelope of the family of circles passing through the origin and having centre on the parabola $y^2 = 4ax$.

3.82. Investigate the character of the discriminant curves of the family of the following lines (C variable parameter):

(a) cubical parabolas $y - 1 = (x - C)^3$;

(b) semicubical parabolas $(y - C)^2 = (x - C)^3$;

(c) Nail's parabolas $(y - 1)^3 = (x - C)^2$;

(d) strophoids $(a - x)(y - C)^2 = x^2(a + x)$.

SEC. 7.4.
APPROXIMATE NUMBERS AND OPERATIONS ON THEM

1. Absolute and Relative Errors. Let the number a be an *approximation* to the number A. For instance, $A = \sqrt{3}$ and $a = 1.7$. For $a > A$ the number a is said to be an *approximation with excess*, and for $a < A$—an *approximation with deficit*. Thus, the number 1.73 is an approximation to $\sqrt{3}$ with deficit, while the number 1.74 is an approximation to this number with excess. The *absolute error* (or, simply, *error*) of an approximation (an approximate number) a is defined by the equality

$$\Delta = |a - A|.$$

Since in most cases the exact number A is unknown, the absolute error Δ is also unknown, but we may be given the upper bound of the absolute error. The least of the upper bounds Δ_a of the absolute error is called the *limiting absolute error*. For practical purposes, one of the upper bounds is often taken for the limiting absolute error. The following inclusion takes place:

$$A \in |a - \Delta_a, a + \Delta a|,$$

which is customarily written in the form $A = a \pm \Delta_a$. For instance, $\sqrt{3} = 1.7321 \pm 0.0001$.

The *relative error* of the number a is defined by the equality

$$\delta = \frac{\Delta}{a}.$$

The *limiting relative error* is defined in a similar way:

$$\delta_a = \frac{\Delta_a}{a}.$$

For instance, for $A = \sqrt{3}$ and $a = 1.7321$

$$\delta_a = \frac{0.0001}{1.7321} = 0.00006.$$

The *significant digits* of a decimal number are all nonzero digits. A zero is regarded to be a significant digit if it is found between significant digits or stands after all significant digits.

Dropping decimals after a certain significant digit is called *rounding off* of a number. When rounding off a number the following rules should be observed:

(1) if the first digit dropped is less than 5, then the preceding digit is not changed;

(2) if the first digit dropped is greater than 5, then the preceding digit is increased by 1;

(3) if the first digit dropped is 5 and some succeeding digit is not zero, then the preceding digit is increased by 1;

(4) if the first digit dropped is 5, and all succeeding digits are zero, then the preceding digit is increased by 1 if it is odd, and is left unchanged if it is even.

If the absolute error of an approximate number a does not exceed unity of the decimal place expressed by the nth significant digit in the decimal notation of this number, then a is said to be a *number having n correct digits in a broad sense*, and if the absolute error does not exceed half the unity of the above mentioned decimal place, then the approximate number a is said to be a *number having n correct digits in a narrow sense*. For the limiting relative error δ_a the following inequalities are valid

$$\delta_a \leqslant \frac{1}{k}\left(\frac{1}{10}\right)^{n-1} \quad \text{and} \quad \delta_a \leqslant \frac{1}{2k}\left(\frac{1}{10}\right)^{n-1}$$

in the first and in the second cases, respectively; in both inequalities k denotes the first significant digit of the number a. Conversely, if the limiting relative error satisfies the inequality

$$\delta_a \leqslant \frac{1}{2(k+1)} \cdot \frac{1}{10^{n-1}},$$

then the corresponding approximate number a with the first significant digit k has n correct digits in a narrow sense.

4.1. Find the limiting absolute and relative errors of the following approximate numbers obtained in measurements:
(a) 23.015 kg; (b) 84.5 cm; (c) 25°15′.

4.2. The result obtained in measuring the length of a path is 25.2 km to within 2 m, and the result of measuring an area (air photography) is 1500 m^2 with an accuracy to 30 m^2. Compute the limiting absolute and the limiting relative errors of both results.

4.3. The length of a 10-km section of a highway was measured with an error of 10 m, and the diameter of a nut equal to 4 cm was measured with an error of 1 mm. Which of these two measurements is more accurate?

4.4. What are the limiting absolute and relative errors of the numbers obtained as a result of rounding off:
(a) 36.1; (b) 0.08.

4.5. Round off the number 29.15 and 3.25 to the first decimal.

4.6. Round off the number 5.3726 to thousandths, to hundredths, and to tenths. Find the absolute and relative errors of each of these roundings off.

4.7. Round off the numbers 0.02025, 1876672, and 599983 to three significant digits.

4.8. Determine the number of correct digits in a narrow sense and write appropriately the following approximate numbers:

(a) 413287.51 for an accuracy of 1%;

(b) 0.0794 for an accuracy of 2%.

4.9. With how many digits should the number $\sqrt{21}$ be taken so that the limiting relative error would not exceed 1%?

4.10. With how many digits should the numbers ln 40 and arc tan 2 be taken so that their limiting relative error would not exceed 0.1%?

2. Operations on Approximate Numbers. Let $u = f(x_1, x_2, \ldots, x_n)$ be a differentiable function in a considered domain. Then the limiting absolute error Δ_u of a value of the function is determined by the relation

$$\Delta_u = \sum_{k=1}^{n} \left| \frac{\partial f}{\partial x_k} \right| \Delta_{x_k}, \tag{1}$$

where Δx_k are the limiting absolute errors of the values of the appropriate arguments. For the limiting relative error the following equality takes place

$$\delta_u = \sum_{k=1}^{n} \left| \frac{1}{u} \frac{\partial f}{\partial x_k} \right| \Delta_{x_k}. \tag{2}$$

Example 1. Find the limiting absolute and relative errors of the volume of a cone of radius r and altitude h if $r = 15 \pm 0.02$ cm, $h = 19.1 \pm 0.05$ cm and $\pi = 3.14$.

We have $v = \frac{1}{3}\pi r^2 h = 4498.1$ cm^3. Taking into account that $r = 15$, $h = 19.1$, $\pi = 3.14$, $\Delta_r = 0.02$, $\Delta_h = 0.05$ and $\Delta_\pi = 0.0016$, we find $\frac{\partial v}{\partial \pi} = \frac{1}{3} r^2 h = 1432.5$, $\frac{\partial v}{\partial r} = \frac{2}{3}\pi r h = 599.74$ and

$\frac{\partial v}{\partial h}=\frac{1}{3}\pi r^2$ - 235.5. Applying formula (1), we obtain the limiting absolute error

$$\Delta_v=\left|\frac{\partial v}{\partial \pi}\right|\Delta_\pi+\left|\frac{\partial v}{\partial r}\right|\Delta_r+\left|\frac{\partial v}{\partial h}\right|\Delta_h=26.06\ \text{cm}^3.$$

The limiting relative error can be determined from the equality

$$\delta_v=\frac{26.1}{4498}=0.006.$$

Thus, $v = 4498 \pm 26.1$ cm^3.

Prove the following statements.

4.11*. The limiting absolute error of a sum is equal to the sum of the limiting absolute errors of its terms.

4.12*. The limiting relative error of a product is equal to the sum of the limiting relative errors of its factors.

4.13*. The limiting relative error of the nth power is n times the limiting relative error of the base.

4.14*. The limiting relative error of a quotient is equal to the sum of the limiting relative errors of the dividend and divisor.

4.15*. The limiting absolute error Δ_{uv} of the product uv satisfies the relation $\Delta_{uv} = \Delta_u v + \Delta_v u$.

In Problems 4.16. to 4.25 perform the indicated operations on the approximate numbers in which all the decimals are correct in a narrow sense.

4.16. 130.6 + 0.255 + 1.15224 + 41.84 + 11.8216.

4.17. 17.83 + 1.07 + $1.1\cdot 10^2$. **4.18.** 153.21 — 81.329.

4.19. 61.32 — 61.31. **4.20.** 35.2·1.748.

4.21. 65.3·78.5. **4.22.** 7.6 : 2.314. **4.23.** 170 : 5.

4.24. 40.5^3. **4.25.** $\sqrt{54.71}$.

4.26. 12 cm is the result obtained in measuring the radius of a circle with an accuracy to 0.5 cm. Find the absolute and relative errors of the area of the circle.

4.27. Determine the absolute error of the common logarithm of an approximate positive number x computed with the relative error δ.

4.28. With what limiting absolute error should the sides of a rectangle $a \approx 4$ m and $b \approx 5$ m be measured so that its area S could be calculated within 0.1 m^2?

We have: $S = ab$ and $\Delta S = 0.1$. Assuming that formula (1) consists of equal terms, we get

$$\left|\frac{\partial u}{\partial x_i}\right| \Delta_{x_i} = \frac{\Delta_u}{n}, \text{ whence } \Delta_{x_i} = \frac{\Delta_u}{n\left|\frac{\partial u}{\partial x_i}\right|}$$

(the *principle of equal contributions*). Therefore computing the partial derivatives $\frac{\partial S}{\partial a} = b = 5$ and $\frac{\partial S}{\partial b} = a = 4$, we find that

$$\Delta_a = \frac{0.1}{2 \cdot 5} = 0.01, \quad \Delta_b = \frac{0.1}{2 \cdot 4} = 0.0125.$$

Distributing the number 0.1 in the formula for Δ_s between the two terms not in equal parts, but somehow else, we shall obtain other values for Δ_a and Δ_b, which, nevertheless, ensure the same limiting absolute error.

4.29. With what absolute error should the side x of a square be measured to determine the area of this square with an accuracy to 0.001 m^2 if $2 \text{ m} < x < 3 \text{ m}$?

4.30. Compute the density of aluminium if an aluminium cylinder of diameter 2 cm and altitude 11 cm has a mass 93.4 g. The relative error of the measurement of the lengths is equal to 0.01 and the relative error of determining the mass is 0.001.

4.31. With what accuracy should the radius of the base R and the height H of a cylindrical tin be computed to determine its capacity with an accuracy to 1%?

4.32. With what accuracy should the approximate value of an angle $x \approx 25^\circ$ be taken to find the value of $\sin x$ with four correct digits in a narrow sense?

4.33. How many correct digits in a broad sense should the value of the argument $x \approx 2$ be taken with to obtain the value of the function $y = e^x$ to within 0.001?

4.34. With how many correct digits must the constant term of the equation $x^2 - 2x + \log 2 = 0$ be known to obtain the roots of this equation with four correct digits in a narrow sense?

4.35. It is required to measure to an accuracy of 1% the area of the lateral surface of the frustum of a cone the radii of the bases of which are ≈ 2 m and ≈ 1 m, and the generatrix ≈ 5 m. With what accuracy must the radii and generatrix be measured and how many digits must the number π be taken with to ensure the required accuracy?

Answers

1.1. $S = \sqrt{p(p-x)(p-y)(x+y-p)}$; $0 < x < p$, $0 < y < p$, $x + y > p$. 1.2. $V = \frac{S^2}{3\pi^2 l^3}\sqrt{\pi^2 l^4 - S^2}$; $0 < S < \pi l^2$. 1.3. $S = \frac{x+y}{4}\sqrt{4z^2 - (x-y)^2}$; $z > \frac{x-y}{2}$. 1.4. $x^2 + y^2 \leqslant R^2$. 1.5. $x^2 + y^2 \geqslant R^2$. 1.6. $x^2 + y^2 < R^2$. 1.7. $x^2 + y^2 > R^2$. 1.8. $x \neq y$. 1.9. $-1 \leqslant x^2 + y \leqslant 1$. 1.10. $x + y < 0$. 1.11. $x \leqslant x^2 + y^2 < 2x$. 1.12. Strips $-\frac{\pi}{2} + 2k\pi \leqslant x \leqslant \frac{\pi}{2} + 2k\pi$ (k is integer). 1.13. $0 < x^2 + y^2 \leqslant 1$ for $0 < a < 1$, $x^2 + y^2 \geqslant 1$ for $a > 1$. 1.14. Two obtuse vertical angles formed by the straight lines $y = 0$ and $y = -2x$, including the boundary without the common vertex (0, 0). 1.15. $4 \leqslant x^2 + y^2 \leqslant 9$. 1.16. A curvilinear triangle formed by the straight line $y = 2$ and parabolas $y^2 = \pm x$ excluding the vertex (0, 0). 1.17. $0 \leqslant \varphi \leqslant \pi$. 1.18. A portion of plane contained between the rays $\varphi = -\frac{\pi}{4}$ and $\varphi = \frac{\pi}{4}$, $\varphi = \frac{3\pi}{4}$ and $\varphi = \frac{5\pi}{4}$. 1.19. $x^2 + y^2 + z^2 \geqslant R^2$. 1.20. $0 \leqslant x^2 + y^2 \leqslant z^2$, $z \neq 0$. 1.21. $x^2 + y^2 - z^2 < 1$. 1.22. An n-dimensional cube $-1 \leqslant x_k \leqslant 1$ ($k = 1, 2, \ldots, n$). 1.23. An n-dimensional ellipsoid $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \ldots + \frac{x_n^2}{a_n^2} \leqslant 1$. 1.24. $f(2, 1) = 1/4$; $f(1, 2) = 4$; $f(3, 2) = 0$; $f(2, 3) = \infty$; $f(a, a) = -1$; $f(a, -a) = 1$. 1.25. $f(-3, 4) = -24/25$; $f(1, y/x) = f(x, y)$. 1.26. $\sqrt{1+x^2}$. 1.27. $f(x) = x^2 - x$; $z = 2y + (x-y)^2$. 1.28. $\frac{x^2(1-y)}{1+y}$. Denote $u = x + y$, $v = \frac{y}{x}$. Then $x = \frac{u}{1+v}$, $y = \frac{uv}{1+v}$, $f(u, v) = \frac{u^2}{(1+v)^2} - \frac{u^2v^2}{(1+v)^2} = \frac{u^2(1-v)}{1+v}$. It remains to substitute x and y for u and v. 1.29. (a) $x^4 - 2x^2y^2 + 2y^4$; (b) $4x^2y^2$. 1.31. (a) $\cos 2x$; (b) $\cos(x^2 - y^2)$. 1.32. -6. 1.33. 1. 1.34. 0. 1.35. e. 1.36. 1. 1.37. $\lim\limits_{\substack{x \to 0 \\ y \to 0}} z = \frac{1}{k-1}$ along the straight line $y = kx$; $\lim z = 3$ for $k = 4/3$; $\lim z = 2$ for $k = 3/2$; $\lim z = 1$ for $k = 2$; $\lim z = -2$ for $k = 1/2$. 1.40. No limit. 1.41. No limit. 1.42. *Hint.* Consider the variation of x and y along the parabola $y = x^2$. 1.44. $(1, -1)$. 1.45. (m, n), where $m, n \in \mathbb{Z}$. 1.46. The lines of discontinuity are straight lines $x = k\pi$ and $y = m\pi$, where $k, m \in \mathbb{Z}$. 1.47. The line of discontinuity is a circle: $x^2 + y^2 = 1$. 1.48. The lines of discontinuity are a straight line $x + y = 0$ and a parabola $y^2 = x$. 1.49. The lines of discontinuity are a circle $x^2 + y^2 = 1$ and a hyperbola $x^2 - y^2 = 1$. 1.50. The surfaces of discontinuity are the coordinate planes $x = 0$, $y = 0$, $z = 0$. 1.51. The surface of discontinuity is an ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} +$

$+\frac{z^2}{c^2}=1$. **1.52.** The surface of discontinuity is a cone: $x^2+y^2-z^2=0$. **1.53.** The surface of discontinuity is a hyperboloid of one sheet: $x^2+y^2-z^2=-1$. **1.54.** The surface of discontinuity is a hyperboloid of two sheets $x^2+y^2-z^2=-1$.

1.55. $\frac{\partial z}{\partial x}=5x^4-15x^2y^3$, $\frac{\partial z}{\partial y}=5y^4-15x^3y^2$, $\frac{\partial^2 z}{\partial x^2}=20x^3-30xy^3$, $\frac{\partial^2 z}{\partial x\,\partial y}=-45x^2y^2$, $\frac{\partial^2 z}{\partial y^2}=20y^3-30x^3y$. **1.56.** $\frac{\partial z}{\partial x}=y-\frac{y}{x^2}$, $\frac{\partial z}{\partial y}=x+\frac{1}{x}$, $\frac{\partial^2 z}{\partial x^2}=\frac{2y}{x^3}$ $\frac{\partial^2 z}{\partial x\,\partial y}=1-\frac{1}{x^2}$, $\frac{\partial^2 z}{\partial y^2}=0$. **1.57.** $\frac{\partial z}{\partial x}=\frac{y^3}{(x^2+y^2)^{3/2}}$, $\frac{\partial z}{\partial y}=\frac{x^3}{(x^2+y^2)^{3/2}}$, $\frac{\partial^2 z}{\partial x^2}=-\frac{3xy^3}{(x^2+y^2)^{5/2}}$, $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{3x^2y^2}{(x^2+y^2)^{5/2}}$, $\frac{\partial^2 z}{\partial y^2}=-\frac{3x^3y}{(x^2+y^2)^{5/2}}$. **1.58.** $\frac{\partial z}{\partial x}=(1-xy)\,e^{-xy}$, $\frac{\partial z}{\partial y}=-x^2e^{-xy}$, $\frac{\partial^2 z}{\partial x^2}=y\,(xy-2)\,e^{-xy}$, $\frac{\partial^2 z}{\partial x\,\partial y}=x\,(xy-2)\,e^{-xy}$, $\frac{\partial^2 z}{\partial y^2}=x^3e^{-xy}$. **1.59.** $\frac{\partial z}{\partial x}=-\frac{\cos y^2}{x^2}$, $\frac{\partial z}{\partial y}=-\frac{2y\sin y^2}{x}$, $\frac{\partial^2 z}{\partial x^2}=\frac{2\cos y^2}{x^3}$ $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2y\sin y^2}{x^2}$, $\frac{\partial^2 z}{\partial y^2}=-\frac{2\sin y^2+4y^2\cos y^2}{x}$. **1.60.** $\frac{\partial z}{\partial x}=y^x\ln y$, $\frac{\partial z}{\partial y}=xy^{x-1}$, $\frac{\partial^2 z}{\partial x^2}=y^x\ln^2 y$, $\frac{\partial^2 z}{\partial x\,\partial y}=y^{x-1}(x\ln y+1)$, $\frac{\partial^2 z}{\partial y^2}=x\,(x-1)\,y^{x-2}\,(y>0)$. **1.61.** $\frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2}$, $\frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}$, $\frac{\partial^2 z}{\partial x^2}=\frac{2(y^2-x^2)}{(x^2+y^2)^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{4xy}{(x^2+y^2)^2}$, $\frac{\partial^2 z}{\partial y^2}=\frac{2(x^2-y^2)}{(x^2+y^2)^2}$. **1.62.** $\frac{\partial z}{\partial x}=-\frac{y\operatorname{sgn}x}{x^2+y^2}$, $\frac{\partial z}{\partial y}=\frac{|x|}{x^2+y^2}$, $\frac{\partial^2 z}{\partial x^2}=\frac{2|x|y}{(x^2+y^2)^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{(y^2-x^2)\operatorname{sgn}x}{(x^2+y^2)^2}$, $\frac{\partial^2 z}{\partial y^2}=-\frac{2|x|y}{(x^2+y^2)^2}$. **1.63.** $\frac{\partial u}{\partial x}=-\frac{x}{(x^2+y^2+z^2)^{3/2}}$, $\frac{\partial^2 u}{\partial x^2}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$, $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{3xy}{(x^2+y^2+z^2)^{5/2}}$. **1.64.** $\frac{\partial u}{\partial x}=-\frac{z}{x}\left(\frac{y}{x}\right)^z$, $\frac{\partial u}{\partial y}=\frac{z}{y}\left(\frac{y}{x}\right)^z$, $\frac{\partial u}{\partial z}=\left(\frac{y}{x}\right)^z\ln\frac{y}{x}$, $\frac{\partial^2 u}{\partial x^2}=\frac{z\,(z+1)}{x^2}\left(\frac{y}{x}\right)^z$, $\frac{\partial^2 u}{y^2}=\frac{z\,(z-1)}{y^2}\left(\frac{y}{z}\right)^z$, $\frac{\partial^2 u}{\partial z^2}=\left(\frac{y}{x}\right)^z\ln^2\frac{y}{x}$, $\frac{\partial^2 u}{\partial x\,\partial y}=-\frac{z^2}{xy}\left(\frac{y}{x}\right)^z$, $\frac{\partial^2 u}{\partial x\,\partial z}=-\frac{1}{x}\left(\frac{y}{x}\right)^z\left(1+z\ln\frac{y}{x}\right)$, $\frac{\partial^2 u}{\partial y\,\partial z}=\frac{1}{y}\left(\frac{y}{x}\right)^z\left(1+z\ln\frac{y}{x}\right)$. **1.65.** $\frac{\partial u}{\partial x}=y^2z^3t^1+3$, $\frac{\partial u}{\partial y}=$

$-2xyz^3t^4-4$, $\frac{\partial u}{\partial z}=3xy^2z^2t^4+2$, $\frac{\partial u}{\partial t}=4xy^2z^3t^3-1$, $\frac{\partial^2 u}{\partial x^2}=0$, $\frac{\partial^2 u}{\partial y^2}=$ $=2xz^3t^4$, $\frac{\partial^2 u}{\partial z^2}=6xy^2zt^4$, $\frac{\partial^2 u}{\partial t^2}=12xy^2z^3t^2$, $\frac{\partial^2 u}{\partial x\,\partial y}=2yz^3t^4$, $\frac{\partial^2 u}{\partial x\,\partial z}=$ $=3y^2z^2t^4$, $\frac{\partial^2 u}{\partial x\,\partial t}=4y^2z^3t^3$, $\frac{\partial^2 u}{\partial y\,\partial z}=6xyz^2t^4$, $\frac{\partial^2 u}{\partial y\,\partial t}=8xyz^3t^3$, $\frac{\partial^2 u}{\partial z\,\partial t}=$ $=12xy^2z^2t^3$. **1.66.** $f'_x(3,\ 2)=56$, $f'_y(3,\ 2)=42$, $f''_{xx}(3,\ 2)=36$, $f''_{xy}(3,\ 2)=31$, $f''_{yy}(3,\ 2)=6$, **1.67.** $f'_x(1,\ 2)=e(2e^1-1)$, $f'_y(1,\ 2)=$ $=4e^5$, $f''_{xx}(1,\ 2)=e(6e^1-1)$, $f''_{xy}(1,\ 2)=8e^5$, $f''_{yy}(1,\ 2)=18e^5$. **1.70.** $f'''_{xxx}(0,\ 1)=0$, $f'''_{xxy}(0,\ 1)=2$, $f'''_{xyy}(0,\ 1)=0$, $f'''_{yyy}(0,\ 1)=0$. **1.71.** $\frac{\partial^4 u}{\partial x\,\partial y\,\partial\xi\,\partial\eta}=-\frac{6}{r^4}+\frac{48(x-\xi)^2(y-\eta)^2}{r^8}$, where $r=$ $=\sqrt{(x-\xi)^2+(y-\eta)^2}$. **1.72.** $\frac{\partial^6 u}{\partial x^3\,\partial y^3}=-6(\cos x+\cos y)$.

1.73. $\frac{\partial^{p+q}u}{\partial x^p\partial y^q}=p!q!$ **1.78.** $r^2\cos\theta$. **1.85.** $f'_x(0,\ 0)=f'_y(0,\ 0)=0$. *Hint.* Check to see that the function is equal to zero at all the points of the x- and y-axes, and make use of the definition of partial derivatives. **1.86.** *Hint.* Using the differentiation rules and the definition of the partial derivative, check to see that $f'_x(x,\ y)=y\left(\frac{x^2-y^2}{x^2+y^2}+\frac{4x^2y^2}{(x^2+y^2)^2}\right)$ for $x^2+y^2\neq 0$, $f'_x(0,\ 0)=0$, and, consequently, $f'_x(0,\ y)=-y$. Hence, $f''_{xy}(0,\ y)=f''_{xy}(0,\ 0)=-1$. Analogously, we find that $f''_{xy}\times$ $\times(0,\ 0)=1$. **1.87.** $\Delta z=0.33$, $dz=0.3$. **1.88.** $\Delta z=0.0187$, $dz=$ $=0.0174$.

1.89. $dz=\frac{x\,dx}{\sqrt{x^2+y^2}\,(y+\sqrt{x^2+y^2})}+\frac{dy}{\sqrt{x^2+y^2}}$. **1.90.** $dz=$ $=\frac{y}{x^2\cos^2\frac{y^2}{x}}(2x\,dy-y\,dx)$. **1.91.** $dz=\frac{1}{y^2}\tan\frac{x}{y}(x\,dy-y\,dx)$.

1.92. $du=(xy)^z\left(\frac{z}{x}dx+\frac{z}{y}dy+\ln(xy)\,dz\right)$. **1.93.** $df=(x_2-x_3)\,x_1^{x_2-x_3-1}\times$ $\times\ln x_4\,dx_1+x_1^{x_2-x_3}\ln x_1\ln x_4\,dx_2-x_1^{x_2-x_3}\ln x_1\ln x_4\,dx_3+$ $+x_1^{x_2-x_3}\frac{\partial x_4}{x_4}$. **1.94.** $df(1,\ 2,\ 1)=\frac{5dz-2(dx+2dy)}{25}$. **1.95.** 8.29. **1.96.** 2.95. **1.97.** 0.227. **1.98.** 8.2 m³. **1.99.** Reduced by 1.57 cm. **1.100.** Increased by 617.5 cm³. **1.101.** $dz=3x(x+2y)\times$ $\times dx+3(x^2-y^2)\,dy$, $d^2z=6((x+y)\,dx^2+2x\,dx\,dy-y\,dy^2)$. **1.102.** $dz=(x\,dy-y\,dx)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$, $d^2z=2\left(\frac{y}{x^3}dx^2+\left(\frac{1}{y^2}-\frac{1}{x^2}\right)\times\right.$

$\times dx\,dy - \frac{x}{y^3}\,dy^2\Big)$. **1.103.** $dz = \frac{(x+y)\,dx + x\,dy}{\sqrt{x^2+2xy}}$, $d^2z = \frac{-y^2\,dx^2 + 2xy\,dx\,dy - x^2\,dy^2}{(x^2+2xy)^{3/2}}$. **1.104.** $dz = \frac{x^2\,dy - xy\,dx}{(x^2+y^2)^{3/2}}$, $d^2z = \frac{1}{(x^2+y^2)^{5/2}}(y(2x^2-y^2)\,dx^2 + 2x(2y^2-x^2)\,dx\,dy - 3x^2y\,dy^2)$. **1.105.** $dz = e^{xy}((y^2+xy+1)\,dx + (x^2+xy+1)\,dy)$, $d^2z = e^{xy}(y(y^2+xy+2)\,dx^2 + 2(x+y)(xy+2)\,dx\,dy + x(x^2+xy+2)\,dy^2)$. **1.106.** $dz = \left(\ln\frac{y}{x} - 1\right)dx + \frac{x}{y}\,dy$; $d^2z = -\frac{1}{x}\,dx^2 + \frac{2}{y}\,dx\,dy - \frac{x}{y^2}\,dy^2$. **1.107.** $dz = \frac{1}{2x^2+2xy+y^2}(y\,dx - x\,dy)$, $d^2z = -\frac{1}{(2x^2+2xy+y^2)^2}(2y(2x+y)\,dx^2 + 2(y^2-2x^2)\,dx\,dy - 2x(x+y)\,dy^2)$. **1.108.** $du = (y+z)\,dx + (z+x)\,dy + (x+y)\times dz$, $d^2u = 2(dx\,dy + dy\,dz + dz\,dx)$. **1.109.** $du = e^{xyz}(yz\,dx + zx\,dy + xy\,dz)$, $d^2u = e^{xyz}((yz\,dx + zx\,dy + xy\,dz)^2 + 2(z\,dx\,dy + x\,dy\,dz + y\,dz\,dx))$. **1.110.** $d^3z = e^y(-\cos x\,dx^3 - 3\sin x\,dx^2\,dy + 3\cos x\,dx\,dy^2 + \sin x\,dy^3)$. **1.111.** $d^3u = 6(dx^3 + dy^3 + dz^3 - 3dx\,dy\,dz)$. **1.112.** $d^6u = -\frac{5!\,(dx+dy+dz)^6}{(x+y+z)^6}$. **1.113.** $d^mu = e^{ax+by+cz}(a\,dx + b\,dy + c\,dz)^m$.

2.1. $\frac{dz}{dt} = e^{2x-3y}(2\sec^2 t - 3(2t-1))$. **2.2.** $\frac{dz}{dt} = x^y\left(\frac{y}{xt} + \ln x\cos t\right)$. **2.3.** $\frac{dz}{dt} = \frac{2e^{2t}(x-y)}{x^2+y^2}$. **2.4.** $\frac{du}{dt} = \frac{x(z+2yt^2) - yzte^t}{tx^2}$. **2.5.** $\frac{\partial z}{\partial x} = \frac{e^x}{e^x+e^y}$, $\frac{dz}{dx} = \frac{e^x + e^y(x^2+1)}{e^x+e^y}$. **2.6.** $\frac{\partial z}{\partial x} = \frac{y}{y^2+(x+1)^2}$, $\frac{dz}{dx} = \frac{y(1-2(x+1)^2)}{y^2+(x+1)^2}$. **2.7.** $\frac{\partial z}{\partial x} = 2u\left(\frac{ux}{v} - \frac{y\ln v}{x^2}\right)$, $\frac{\partial z}{\partial y} = 2u\left(\frac{\ln v}{x} + \frac{uy}{v}\right)$. **2.8.** $dz = ((2uv-v^2)\sin y - (u^2-2uv)\,y\sin x)\,dx + ((2uv-v^2)\,x\cos y + (u^2-2uv)\cos x)\,dy$. **2.9.** $\frac{\partial z}{\partial x} = 2xf'_v(u, v) - \frac{2y}{(x+y)^2}f'_u(u, v)$, $\frac{\partial z}{\partial y} = \frac{2x}{(x+y)^2}f'_u(u, v) - 3f'_v(u, v)$. **2.10.** $\frac{\partial z}{\partial x} = \frac{2x}{x^2-y^2}f'_u(u, v) + y^2f'_v(u, v)$, $\frac{\partial z}{\partial y} = 2xy\,f'_v(u, v) - \frac{2y}{x^2-y^2}f'_u(u, v)$. **2.11.** $dz = (5x^4f'_v(u, v) - yf'_u(u, v)\sin(xy))\,dx - (x\sin(xy)\,f'_u(u, v) + 7f'_v(u, v))\,dy$. **2.12.** $dz = \frac{1}{y^2}\left(\cos\frac{x}{y}\,f'_u(u, v) + \frac{1}{2}\sqrt{\frac{y}{x}}\times f'_v(u, v)\right)(y\,(dx - x\,dy)$.

2.13. $du = (2sf'_x(x, y, z) + 2sf'_y(x, y, z) + t2f'_z(x, y, z))\,ds + (2tf'_x(x, y, z) - 2tf'_y(x, y, z) + 2sf'_z(x, y, z))\,dt$. **2.14.** $\dfrac{\partial u}{\partial x_1} = f'_{x_1}(x_1, x_2, x_3, x_4) + f'_{x_3}(x_1, x_2, x_3, x_4)\,g'_{x_1}(x_1, x_2) + f'_{x_4} \times (x_1, x_2, x_3, x_4)\,(h'_{x_1}(x_1, x_2, x_3) + h'_{x_3}(x_1, x_2, x_3)\,g'_{x_1}(x_1, x_2))$, $\dfrac{\partial u}{\partial x_2} = f'_{x_2}(x_1, x_2, x_3, x_4) + f'_{x_3}(x_1, x_2, x_3, x_4)\,g'_{x_2}(x_1, x_2) + f'_{x_4} \times (x_1, x_2, x_3, x_4)\,(h'_{x_2}(x_1, x_2, x_3) + h'_{x_3}(x_1, x_2, x_3)\,g'_{x_2}(x_1, x_2))$.

2.19. $\dfrac{\partial^2 z}{\partial x^2} = y^2 f''_{uu}(u, v) + 2f''_{uv}(u, v) + \dfrac{1}{y^2} f''_{vv}(u, v)$, $\dfrac{\partial^2 z}{\partial x\,\partial y} = xyf''_{uu}(u, v) - \dfrac{x}{y^3} f''_{vv}(u, v) + f'_u(u, v) - \dfrac{1}{y^2} f'_v(u, v)$, $\dfrac{\partial^2 z}{\partial y^2} = x^2 f''_{uu} \times (u, v) - \dfrac{2x^2}{y^2} f''_{uv}(u, v) + \dfrac{x^2}{y^4} f''_{vv}(u, v) + \dfrac{2x}{y^3} f'_v(u, v)$. **2.20.** $\dfrac{\partial^2 u}{\partial x\,\partial y} = f''_{xy} + f''_{xz}\varphi'_y + f''_{yz}\varphi'_x + f''_{zz}\varphi'_x\varphi'_y + f'_z\varphi''_{xy}$. **2.21.** $\dfrac{\partial^2 u}{\partial x^2} = f''_{11} + y^2 f''_{22} + y^2z^2 f''_{33} + 2yf''_{12} + 2yzf''_{13} + 2y^2 z f''_{23}$, $\dfrac{\partial^2 u}{\partial y^2} = x^2 f''_{22} + 2x^2 z f''_{23} + x^2 z^2 f''_{33}$, $\dfrac{\partial^2 u}{\partial z^2} = x^2 y^2 f''_{33}$, $\dfrac{\partial^2 u}{\partial x\,\partial y} = xyf''_{22} + xyz^2 f''_{33} + xf''_{12} + xzf''_{13} + 2xyzf''_{23} + f'_2 + zf'_3$, $\dfrac{\partial^2 u}{\partial x\,\partial y} = xyf''_{13} + xy^2 f''_{23} + xy^2 z f''_{33} + yf'_3$, $\dfrac{\partial^2 u}{\partial y\,\partial z} = x^2 y f''_{23} + x^2 yz f''_{33} + xf'_3$ *. **2.24.** $d^2u = 4f''(t)\cdot(x\,dx + y\,dy + z\,dz)^2 + 2f'(t)\cdot(dx^2 + dy^2 + dz^2)$. **2.25.** $d^2u = a^2 f''_{11}\,dx^2 + b^2 f''_{22}\,dy^2 + c^2 f''_{33}\,dz^2 + 2ab\,f''_{12}\,dx\,dy + 2ac\,f''_{13}\,dx\,dz + 2bc\,f''_{23}\,dy\,dz$. **2.26.** $d^2z = (\sin^2 y\cdot f''_{uu} - 2y\sin x \times \sin y\cdot f''_{uv} + y^2 \sin^2 x\,f''_{vv} - y\cos x\cdot f'_v)\,dx^2 + (x\sin 2y\cdot f''_{uu} + 2 \times (\sin y\cos x - xy\sin x\cos y)\,f''_{uv} - y\sin 2x\cdot f''_{vv} + 2(\cos y\cdot f'_u - \sin x\cdot f'_v))\,dx\,dy + (x^2\cos^2 y\cdot f''_{uu} + 2x\cos x\cos y\cdot f''_{uv} + \cos^2 x \times f''_{vv} - x\sin y\cdot f'_u)\,dy^2$.

2.27. $\dfrac{dy}{dx} = \dfrac{y^2 e^{2x} - xe^{2y}}{x^2 e^{2y} - ye^{2x}}$. **2.28.** $\dfrac{dy}{dx} = \dfrac{y\cos x + \sin(x-y)}{\sin(x-y) - \sin x}$.

2.29. $\dfrac{dy}{dx} = \dfrac{x+y-1}{x+y+1}$, $\dfrac{d^2y}{dx^2} = \dfrac{4(x+y)}{(x+y+1)^3}$. **2.30.** $\dfrac{dy}{dx} = \dfrac{1+y^2}{y^2}$, $\dfrac{d^2y}{dx^2} = -\dfrac{2(1+y^2)}{y^5}$. **2.31.** $\left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=1}} = 0$, $\left.\dfrac{d^2y}{dx^2}\right|_{\substack{x=1\\y=1}} = -\dfrac{1}{3}$, $\left.\dfrac{d^3y}{dx^3}\right|_{\substack{x=1\\y=1}} = \dfrac{1}{3}$. **2.32.** $\dfrac{\partial z}{\partial x} = 1$, $\dfrac{\partial z}{\partial y} = \dfrac{1}{2}$.

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* In the answers to Problems 2.21 and 2.25 f'_i and f''_{ij} denote the partial derivatives of the function $f(\varphi_1(x, y, z), \varphi_2(x, y, z), \varphi_3(x, y, z))$ with respect to the variables φ_i or φ_i and φ_j.

2.33. $\frac{\partial z}{\partial x}=\frac{yz(x+z)-z^3}{z^3+2xy(x+z)}$, $\frac{\partial z}{\partial y}=\frac{xz(x+z)}{z^3+2xy(x+z)}$. **2.34.** $\frac{\partial z}{\partial x}=$ $=-\frac{F'_u(u,v)+2xF'_v(u,v)}{F'_u(u,v)+2zF'_v(u,v)}$, $\frac{\partial z}{\partial y}=-\frac{F'_u(u,v)+2yF'_v(u,v)}{F'_u(u,v)+2zF'_v(u,v)}$, where $u=x+y+z$, $v=x^2+y^2+z^2$.

2.35. $\frac{\partial z}{\partial x}=-\frac{ze^{xz}f'_v(u,v)}{yf'_u(u,v)+xe^{xz}f'_v(u,v)}$, $\frac{\partial z}{\partial y}=-\frac{zf'_u(u,v)}{yf'_u(u,v)+xe^{xz}f'_v(u,v)}$, where $u=yz$, $v=e^{xz}$. **2.36.** $dz=\frac{z\,dx-z(1+x^2z^2)\,dy}{y(1+x^2z^2)-x}$. **2.37.** $dz=$ $=\frac{y^2(z+3x^2)\,dx+(3y^3+ze^{z/y})\,dy}{y(e^{z/y}-xy)}$. **2.38.** $\frac{\partial z}{\partial x}=\frac{2-x}{1+z}$, $\frac{\partial z}{\partial y}=$ $=\frac{2y}{1+z}$, $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{2y(x-2)}{(1+z)^3}$. **2.39.** $\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y^2}=$ $=-\frac{x+y+z}{(x+y+z-1)^3}$. **2.40.** $d^2z=\frac{c^4}{a^2b^2z^3}((y^2-b^2)\,dx^2-2xy\,dx\,dy+$ $+(x^2-a^2)\,dy^2)$. **2.44.** $\frac{dy}{dx}=-\frac{3}{2}$, $\frac{dz}{dx}=\frac{5}{3}$, $\frac{d^2y}{dx^2}=-\frac{3}{8}$, $\frac{d^2z}{dx^2}=$ $=\frac{5}{18}$. **2.45.** $dy=-\frac{4x}{5y}\,dx$, $dz=\frac{x}{5z}\,dx$, $d^2y=-\frac{4}{25y^3}(4x^2+5y^2)\times$ $\times dx^2$, $d^2z=\frac{1}{25z^3}(5z^2-x^2)\,dx^2$. **2.46.** $du=\frac{(y-u)\,dx+(y-v)\,dy}{x-y}$, $dv=\frac{(x-u)\,dx+(x-v)\,dy}{y-x}$, $d^2v=-d^2u=\frac{2}{(x-y)^2}((y-u)\,dx^2+$ $+(y-v+u-x)\,dx\,dy+(v-x)\,dy^2)$. **2.48.** $\frac{\partial z}{\partial x}=uv^2+u^2v$, $\frac{\partial z}{\partial y}=$ $=uv^2-u^2v$. **2.49.** $\frac{\partial z}{\partial x}=\frac{c}{a}\cos u\coth v$, $\frac{\partial z}{\partial y}=\frac{c}{b}\sin u\coth v$. **2.50.** $dz=e^{-u}((v\cos v-u\sin v)\,dx+(u\cos v+v\sin v)\,dy)$. **2.51.** $dz=$ $=-3uv\,dx+\frac{3}{2}(u+v)\,dy$. **2.52.** $\frac{d^2y}{dt^2}-y=0$. **2.53.** $\frac{d^2y}{dt^2}+y=0$. **2.54.** $\frac{d^3x}{dy^3}+\frac{d^2x}{dy^2}=0$. **2.55.** $r'^2=\frac{1-\sin 2\varphi}{\sin 2\varphi}r^2$. **2.56.** $w=r\frac{\partial u}{\partial r}$. **2.57.** $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$. **2.58.** $\frac{\partial z}{\partial v}=u\frac{\partial^2 z}{\partial u\,\partial v}$. **2.59.** $w=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}+$ $+\frac{1}{r}\frac{\partial u}{\partial r}$. **2.60.** $w=\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}+\frac{2}{\rho}\frac{\partial u}{\partial\rho}+$ $+\frac{\cot\theta}{\rho^2}\frac{\partial u}{\partial\theta}$. **2.61.** $\frac{\partial w}{\partial v}=0$. **2.62.** $\frac{\partial^2 w}{\partial u^2}=0$. **2.63.** $\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial u\,dv}-$ $=2w$.

3.1. $f(x+h, y+k) = xy^2 + y^2h + 2xyk + 2yhk + xk^2 + hk^2$. **3.2.** $\Delta f(x, y) = -h^2 + 2hk + 3k^2$. **3.3.** $f(x, y) = 12 + 15(x-2) + 6(x-2)^2 + 3(x-2)(y-1) - 6(y-1)^2 + (x-2)^3 - 2(y-1)^3$. **3.4.** $f(x+h, y+k, z+l) = f(x, y, z) + h(2x+y+3) + k(x+4y-2z-1) + l(6z-2y-4) + h^2 + 2k^2 + 3l^2 + hk - 2kl$. **3.5.** $f(x, y, z) = 8 - 8(y+1) + 4(z-2) + (x-1)^2 + (y+1)^2 + (z-2)^2 - 2(x-1)(y+1) - 2(x-1)(z-2) - 2(y+1)(z-2)$. **3.6.** $f(x, y) = 1 + y + \frac{1}{2!}(y^2 - x^2) + \frac{1}{3!}(y^3 - 3x^2y) + o(\rho^3)$, where $\rho = \sqrt{x^2+y^2}$. **3.7.** $f(x, y) = xy + \frac{1}{3!} \times (xy^3 - x^3y) + o(\rho^4)$, where $\rho = \sqrt{x^2+y^2}$. **3.8.** $f(x, y) = 1 - (x-1) + (y-1) + (x-1)^2 - (x-1)(y-1) - (x-1)^3 + (x-1)^2(y-1) + o(\rho^3)$, where $\rho = \sqrt{(x-1)^2 + (y-1)^2}$. **3.9.** $f(x, y, z) = (x-1) + (y-1) - \frac{1}{2}(x-1)^2 - \frac{1}{2}(y-1)^2 + z^2 + o(\rho^2)$, where $\rho = \sqrt{(x-1)^2 + (y-1)^2 + z^2}$. **3.10.** $z = 1 + \frac{2}{3}(x-1) - \frac{1}{2}(y-1) - \frac{2}{9}(x-1)^2 - \frac{1}{8}(y-1)^2 + o(\rho^2)$, where $\rho = \sqrt{(x-1)^2 + (y-1)^2}$. **3.11.** $z_{\min} = -9$ for $x = 0$, $y = 3$. **3.12.** $z_{\max} = 1/64$ for $x = 1/4$, $y = 1/2$. **3.13.** $z_{\min} = -4/3$ for $x = 0$, $y = -2/3$. No extremum at the stationary point $(2, -2/3)$. **3.14.** $z_{\min} = 30$ for $x = 5$, $y = 2$. **3.15.** $z_{\min} = 10 - 18 \ln 3$ for $x = 1$, $y = 3$. **3.16.** $z_{\min} = -28$ for $x = 2$, $y = 1$; $z_{\max} = 28$ for $x = -2$, $y = -1$. No extrema at the stationary points $(1, 2)$, $(-1, -2)$. **3.17.** $z_{\min} = 0$ for $x = y = 0$. No extrema at the stationary points $(-5/3, 0)$, $(1, 4)$, $(1, -4)$. **3.18.** $z_{\min} = 0$ for $x = y = 0$; $z_{\max} = 2e^{-1}$ for $x = \pm 1$, $y = 0$. No extrema at the stationary points $(0, \pm 1)$. **3.19.** $z_{\max} = 2$ for $x = y = 0$. **3.20.** $u_{\min} = -14$ for $x = 2$, $y = -3$, $z = 1$. **3.21.** $u_{\max} = 1/7^7$ for $x = y = z = 1/7$. **3.22.** $u_{\min} = 2^{9/4}$ for $x = 2^{1/4}$, $y = 2^{1/2}$, $z = 2^{3/4}$. **3.23.** The equation defines two functions one of which has a maximum ($z_{\max} = 6$) for $x = -2$, $y = 1$, the other a minimum ($z_{\min} = -2$) for $x = -2$, $y = 1$; at the points of the circle $(x+2)^2 + (y-1)^2 = 16$ either of these functions has a boundary extremum $z = 2$. *Hint.* The indicated functions are defined explicitly by the equality $z = 2 \pm \sqrt{16 - (x+2)^2 - (y-1)^2}$ only inside the circle $(x+2)^2 + (y-1)^2 = 16$ and its circumference at whose points both functions attain the value $z = 2$. This value is the least for one function and the greatest for the other. **3.24.** The equation defines two functions one of which has a minimum ($z_{\min} = 1$) for $x = 0$, $y = -2$ and the other a maximum ($z_{\max} = -8/7$) for $x = 0$, $y = 16/7$. **3.25.** $z_{\min} = -19/4$ for $x = y = -3/2$. **3.26.** $z_{\min} = 2$ for $x = y = 1$. **3.27.** $z_{\min} = -1 - 2\sqrt{2}$ for $x = -1/\sqrt{2}$, $y = -1/\sqrt{2}$; $z_{\max} = 1 - 2\sqrt{2}$ for $x = 1/\sqrt{2}$, $y = -1/\sqrt{2}$. **3.28.** $z_{\min} = 0$ for $x = 1$, $y = 0$; $z_{\max} = 1/27$ for $x = y = 1/3$. **3.29.** $z_{\min} = -\sqrt{5}$ for $x = -2/\sqrt{5}$, $y = -1/\sqrt{5}$; $z_{\max} = \sqrt{5}$ for $x = 2/\sqrt{5}$, $y = 1/\sqrt{5}$. **3.30.** $u_{\min} = -18$ for $x = -4$, $y = -2$, $z = 4$; $u_{\max} = 18$ for $x = 4$, $y = 2$, $z = -4$. **3.31.** $u_{\min} = 4$ for $x = y = 0$,

$z = \pm 2$; $u_{max} = 16$ for $x = \pm 4$, $y = z = 0$; for $x = z = 0$, $y = \pm 3$, no extremum. **3.32.** $u_{max} = 2^6$ for $x = y = z = 2$. **3.33.** $u_{max} = 2$ at points (2, 1, 1), (1, 2, 1), (1, 1, 2); $u_{min} = 50/27$ at points (2/3, 5/3, 5/3), (5/3, 2/3, 5/3), (5/3, 5/3, 2/3). **3.34.** *Hint.* Look for a minimum of the function $u = (x^3 + y^3 + z^3)/3$ for $x + y + z = s$. **3.35.** (a) $z_{greatest} = 6$ for $x = 1$, $y = 0$; (b) $z_{greatest} = 5$ for $x = y = 0$. **3.36.** $z_{greatest} = 6$ for $x = 3$, $y = 0$ and for $x = 0$, $y = 3$; $z_{least} = -1$ for $x = y = 1$. **3.37.** $z_{greatest} = 1/2$ for $x = -y = \pm 1/\sqrt{2}$; $z_{least} = -1/2$ for $x = -y = \pm 1/\sqrt{2}$. **3.38.** $z_{greatest} = 2/(3\sqrt{3})$ for $x = 1/\sqrt{3}$, $y = \pm\sqrt{2/3}$; $z_{least} = -2/(3\sqrt{3})$ for $x = -1/\sqrt{3}$, $y = \pm\sqrt{2/3}$. **3.39.** $a = \sqrt[4]{a}\cdot\sqrt[4]{a}\cdot\sqrt[4]{a}\cdot\sqrt[4]{a}$. **3.40.** A cube with edge of length a. **3.41.** A cube with edge of length $d/\sqrt{3}$. **3.42.** The coordinates of the desired point are equal to the arithmetic means of the coordinates of the vertices. **3.43.** The lengths of the sides of the inscribed parallelepiped: $2R/\sqrt{3}$, $2R/\sqrt{3}$, $R/\sqrt{3}$. **3.44.** The lengths of the sides of the inscribed parallelepiped: $\frac{2\sqrt{2}}{3}R$, $\frac{2\sqrt{2}}{3}R$, $\frac{H}{3}$. **3.45.** An isosceles triangle with the lateral side $a/(2\sin\alpha/2)$. **3.46.** $(-12/5, -3/5)$, $(12/5, 3/5)$. *Hint.* Replace the sufficient conditions for an extremum by geometrical considerations. **3.47.** $C\left(\frac{\sqrt{3}-1}{\sqrt{2}}, -\frac{1+\sqrt{3}}{2\sqrt{2}}\right)$. *Hint.* Take advantage of the expression of the area of a triangle in terms of the coordinates of its vertices.

3.48. $x = y = z = \sqrt[3]{V} + 2\delta$. **3.49.** $x = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n}$, $y = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n}$. **3.50.** $\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}$. *Hint.* Obviously, the point M at which the light ray passes from one medium to the other must lie between A_1 and B_1, and $AM = \frac{a}{\cos\alpha}$, $BM = \frac{b}{\cos\beta}$, $A_1M = a\tan\alpha$, $B_1M = b\tan\beta$. The duration of motion of the ray is equal to $\frac{a}{v_1\cos\alpha} + \frac{b}{v_2\cos\beta}$. The problem is reduced to finding the minimum of the function $f(\alpha, \beta) = \frac{a}{v_1\cos\alpha} + \frac{b}{v_2\cos\beta}$ on the condition that $a\tan\alpha + b\tan\beta = c$. **3.51.** $\alpha = \beta$. **3.52.** $I_1 : I_2 : \ldots : I_n = \frac{1}{R_1} : \frac{1}{R_2} : \ldots : \frac{1}{R_n}$. *Hint.* Find the minimum of the function $f(I_1, I_2, \ldots, I_n) = I_1^2R_1 + I_2^2R_2 + \ldots + I_n^2R_n$ for $I_1 + I_2 + \ldots + I_n = I$.

3.53. (a) $x - y - 2z + 1 = 0$, $\frac{x - \frac{\pi}{4}}{1} = \frac{y - \frac{\pi}{4}}{-1} = \frac{z - \frac{1}{2}}{-2}$; (b) $x + ez -$

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$-2=0$, $\frac{x-1}{1}=\frac{y-\pi}{0}=\frac{z-\frac{1}{e}}{e}$. **3.54.** $\frac{\pi a}{2\sqrt{6}}$. **3.55.** $\cos\alpha=$ $=\frac{1}{\sqrt{6}}$, $\cos\beta=-\frac{1}{\sqrt{6}}$, $\cos\gamma=-\frac{2}{\sqrt{6}}$. **3.56.** $4x+y+2z-$ $-78=0$. **3.57.** (a) $2x+7y-5z+4=0$, $\frac{x-2}{2}=\frac{y-1}{7}=\frac{z-3}{-5}$; (b) $x+y-$ $-4z=0$, $\frac{x-2}{1}=\frac{y-2}{1}=\frac{z-1}{-4}$; (c) $z=0$, $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$ (at point $(0, 0, 0)$); $z=-4$, $\frac{x}{0}=\frac{y}{0}=\frac{z+4}{1}$ (at point $(0, 0, -4)$).

3.58. $\frac{x-2}{1}=\frac{y-\frac{10}{3}}{3}=\frac{z+4}{4}$.

3.59. At points $(0, \pm 2\sqrt{2}, \mp 2\sqrt{2})$ the tangent planes are parallel to the xy-plane, at the points $(\pm 2, \mp 4, \pm 2)$—to the xz-plane, at the points $(\pm 4, \mp 2, 0)$—to the yz-plane.

3.61. (a) $x\cos\varphi_0+y\sin\varphi_0-z\tan\alpha=0$, $\frac{x-r_0\cos\varphi_0}{\cos\varphi_0}=\frac{y-r_0\sin\varphi_0}{\sin\varphi_0}=$ $=\frac{z-r_0\cot\alpha}{-\tan\alpha}$; (b) $ax\sin v_0-ay\cos v_0+u_0z=au_0v_0$, $\frac{x-u_0\cos v_0}{a\sin v_0}=$ $=\frac{y-u_0\sin v_0}{-a\cos v_0}=\frac{z-av_0}{u_0}$. **3.62.** $\cos\varphi=\frac{2bz_0}{a\sqrt{a^2+b^2}}$.

Hint. The angle between two surfaces at the point of their intersection is defined as the angle between the tangent planes drawn to these surfaces at a given point. **3.63.** *Hint.* Surfaces are said to be orthogonal if they intersect at right angles at each point of the line of their intersection. **3.64.** An isolated point $(0, 0)$. **3.65.** Node $(0, 0)$. **3.66.** Isolated point $(0, 0)$. **3.67.** Cuspidal point of the first kind $(1, 0)$. **3.68.** Cuspidal point of the second kind $(0, 0)$. **3.69.** Point of osculation $(0, 0)$. **3.70.** $(0, 0)$ is an isolated point if $a<0$; a node if $a>0$; a cuspidal point of the first kind if $a=0$. **3.71.** Node $(0, 0)$. **3.72.** Cusp of the first kind $(0, 0)$. **3.73.** Corner point $(0, 0)$. *Hint.* Show that $\lim\limits_{x\to+0} y'=$ $=0$, $\lim\limits_{x\to-0} y'=1$. **3.74.** End point $(0, 1)$. *Hint.* Show that $\lim\limits_{x\to+0} y=1$. **3.75.** $y=-x^2/4$. **3.76.** $x^2+y^2=p^2$. **3.77.** $x=\pm R$.

3.78. No envelope. **3.79.** $y=-\frac{4}{3}x^2$. **3.80.** $x^{2/3}+y^{2/3}=l^{2/3}$.

3.81. $y^2=-\frac{x^2}{x+2a}$. **3.82.** (a) The discriminant curve $y=1$ is an envelope and the set of inflection points of the given family; (b) the discriminant curve decomposes into two straight lines: $y=x$ (envelope) $-\frac{4}{27}$

and $y = x$ (set of cuspidal points of the first kind); (c) the discriminant curve $y = 1$ is a set of cuspidal points of the first kind and is not an envelope; (d) the discriminant curve decomposes into two straight lines: $x = -a$ (envelope) and $x = 0$ (set of nodes).

4.1. (a) 1 g, 0.0043%; (b) 1 mm, 0.12%; (c) 1′, 0.066%. **4.2.** (1) $\Delta = 0.002$ km, $\delta = 0.008\%$; (2) $\Delta = 30\,m^2$, $\delta = 2\%$. **4.3.** The first one. **4.4.** (a) 0.05, 0.14%; (b) 0.005, 6.25%. **4.5.** 29.2 and 3.2. **4.6.** (1) 5.373, 0.0004, 0.0074%; (2) 5.73, 0.0026, 0.048%; (3) 5.4, 0.0274, 0.51%. **4.7.** $202 \cdot 10^{-1}$, $188 \cdot 10^{1}$, $600 \cdot 10^{3}$. **4.8.** (a) Two, $41 \cdot 10^{1}$; (b) one, $8 \cdot 10^{-2}$. **4.9.** Not less than with two digits. **4.10.** Not less than with three digits. **4.11.**–**4.15.** *Hint.* Take advantage of formula (1) (Sec. 7.4). **4.16.** 185.7. **4.17.** $1.3 \cdot 10^{2}$. **4.18.** 71.88. **4.19.** Subtraction cannot be performed. **4.20.** 61.6. **4.21.** $512 \cdot 10$. **4.22.** 3.3. **4.23.** $3 \cdot 10$. **4.24.** $66 \cdot 10^{3}$. **4.25.** 7.397. **4.26.** $\leqslant 12\pi$ cm^2, $\leqslant 8.3\%$. **4.27.** $\approx 0.43\delta$. **4.29.** $\leqslant 0.17$ mm. **4.30.** (2.7 ± 0.1) g/cm^3. **4.31.** According to the principle of equal contributions, R should be measured with a relative error 0.25%, and the altitude H with a relative error 0.5%. **4.32.** 12″. **4.33.** 4. **4.34.** 4. **4.35.** According to the principle of equal influences, π may be taken with three correct digits in the narrow sense, the radii should be measured with an accuracy to 0.8 cm, and the generatrix with an accuracy to 1.25 cm.